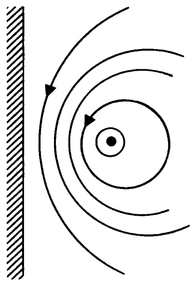
Hydrodynamics and Elasticity 2023/2024

Sheet 9

Problem 2 An inviscid and incompressible fluid occupies the region $x \geqslant 0$ bounded by a planar rigid wall at $x = 0$. At a point (d, y_0) there is a linear vortex of circulation Γ. Assuming that the vortex moves at the local flow velocity due to everything other than itself, use the method of images to show that the vortex moves downwards with a velocity

$$\frac{dy_0}{dt} = -\frac{\Gamma}{4\pi d}$$



Next, find the total force which the motion of the vortex exerts on the wall. Remember that - because the vortex is moving - the flow in the laboratory frame is not stationary. What would the force be if the vortex were fixed at $(d, 0)$?

Solution An image vortex with circulation $-\Gamma$ inserted at $x = -d$ results in zero normal velocity on the wall. Thus, using the method of images, at $t = 0$, the complex potential is

$$\omega = \frac{-i\Gamma}{2\pi}\log(z-d) + \frac{i\Gamma}{2\pi}\log(z+d) = \omega_0 + \omega_i,$$

where ω_0 is associated with the original vortex and ω_i with its image.
To determine the motion of the vortex, we need to consider only the velocity field of the image vortex, which is given by

$$v_x - iv_y = \frac{d\omega_i}{dz} = \frac{i\Gamma}{2\pi(z+d)}.$$

At the position of the original vortex, this yields

$$v_y = -\frac{\Gamma}{4\pi d}e_y,$$

which is also the velocity with which the original vortex moves. As observed, the vortex moves downwards with a constant velocity.
Now, let us examine the velocity field related to both vortices at the wall ($x = 0$):

$$v_x - iv_y = -\frac{i\Gamma}{2\pi(iy-d)} + \frac{i\Gamma}{2\pi(iy+d)} = \frac{i\Gamma d}{\pi(d^2+y^2)}.$$

Thus,

$$v_y(x=0) = -\frac{\Gamma d}{\pi(d^2+y^2)}.$$

Next, we calculate the time derivative of the velocity potential. We have

$$\omega(t) = \frac{-i\Gamma}{2\pi}\log(z-d-iy_0(t)) + \frac{i\Gamma}{2\pi}\log(z+d-iy_0(t)),$$

where $y_0(t)$ is the y-coordinate of the vortex, with $\frac{dy_0}{dt} = -\frac{\Gamma}{4\pi d}$. Then,

$$\frac{\partial \omega}{\partial t} = \frac{\Gamma^2}{8\pi d}\left(\frac{1}{z - d - iy_0(t)} - \frac{1}{z + d - iy_0(t)}\right).$$

We are interested in the value of this quantity on the wall, i.e., $z = iy$, at the moment $t = 0$ when $y_0 = 0$. Thus,

$$\frac{\partial \phi}{\partial t}(y) = \mathrm{Re}\left(\frac{\partial \omega}{\partial t}(y)\right) = -\frac{\Gamma^2}{4\pi(y^2 + d^2)}.$$

The Bernoulli theorem for non-stationary irrotational flow states

$$\frac{d\phi}{dt} + \frac{p}{\rho} + \frac{1}{2}u^2 = 0,$$

or

$$-\frac{\Gamma^2}{4\pi(y^2 + d^2)} + \frac{p}{\rho} + \frac{1}{2}\frac{\Gamma^2 d^2}{\pi^2(y^2 + d^2)^2} = 0.$$

From the above, we can calculate the pressure and then, by integration, the total force on the wall, F, using the following integrals:

$$\int_{y=-\infty}^{\infty} \frac{1}{y^2 + d^2} = \frac{\pi}{d}$$

and

$$\int_{y=-\infty}^{\infty} \frac{1}{(y^2 + d^2)^2} = \frac{\pi}{2d^3},$$

which ultimately gives $F = 0$. If we fix the vortex, then we eliminate the time derivative term in the Bernoulli theorem, and $F = \frac{\rho\Gamma^2}{4\pi d}$.