Hydrodynamics and Elasticity 2023/2024

Sheet 10

One of the problems will be handed in and marked.

Problem 1 A rotating sphere in Stokes flow A rigid sphere of radius a is immersed in an infinite expanse of viscous fluid. The sphere rotates with an angular velocity Ω. The Reynolds number $\mathrm{Re} = \Omega a^2/\nu$ is small, so that the slow flow (Stokes) equations

$$\nabla p = -\mu \nabla \times (\nabla \times \boldsymbol{v}) = 0, \qquad \nabla \cdot \boldsymbol{v} = 0,$$

apply. Using spherical polar coordinates (r, θ, ϕ) with $\theta = 0$ as the rotation axis, show that a purely rotary flow $\boldsymbol{v} = v_\phi(r,\theta)\boldsymbol{e}_\phi$ is possible provided that

$$\frac{\partial^2}{\partial r^2}(r v_\phi) + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi \sin\theta)\right] = 0.$$

Write down the boundary conditions which v_ϕ must satisfy at $r = a$ and as $r \to \infty$, and hence solve the equation above to find

$$v_\phi = \frac{\Omega a^3}{r^2}\sin\theta.$$

Show that the ϕ-component of stress on $r = a$ is $t_\phi = -3\mu \sin\theta$, and deduce that the torque needed to maintain the rotation of the sphere is

$$\tau = 8\pi\mu a^3 \Omega,$$

which is linear in Ω just as the force on a translating sphere is linear in its translational velocity.

Problem 2 A viscous, incompressible fluid is located between two planes at $z = 0$ and $z = h$. The lower plane is stationary, while the upper plane rotates with a constant angular velocity Ω around the z-axis. The Reynolds number, $Re = \frac{\Omega h^2}{\nu}$, is small, so the flow satisfies the Stokes equations. Find the flow of the fluid in this system, assuming that it is purely rotational $\mathbf{u} = u_\theta(r,z)e_\theta$. Show that this leads to the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2}\right)u_\theta = 0,$$

which u_θ must satisfy. Look for solutions of this equation in the form $u_\theta = rf(z)$. After finding the velocity field, calculate the force exerted by the fluid on the unit surface of the upper plane.

Problem 3 Consider again the system described in Problem 2, but this time do not assume that the Reynolds number is is zero. Show that in this case a steady solution of the full Navier-Stokes equations of the form $\mathbf{u} = u_\theta(r,z)e_\theta$ is not possible, so that any rotary motion $u_\theta(r,z)$ must be accompanied by a secondary flow $(u_r, u_z \neq 0)$.

(*) Problem 4 Determine the velocity of a spherical drop of viscosity η_p moving under the action of force F in a viscous fluid of viscosity η, assuming that the Reynolds numbers are small. (Rybczyński, 1911)

Maciej Lisicki & Piotr Szymczak