Stochastic Processes

Mid-term exam, 28/11/2023
10:15 — 12:00

Problem 1:

In this problem you will derive a relation between the autocorrelation function (of a stationary process) and the spectral density function. The spectral density function is defined as

$$\langle X^2 \rangle = \int_{-\infty}^{\infty} S(\omega) \mathrm{d}\omega, \tag{1}$$

and tells us how the variance is distributed among different Fourier modes of the process.
To this end, we consider a stationary stochastic process $X(t)$ with $\langle X(t) \rangle = 0$. Imagine that we observe the process for finite time T such that $t \in [0, T]$. Define its Fourier transform,

$$A_T(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t X_T(t) e^{-i\omega t} = \int_0^T \mathrm{d}t X_T(t) e^{-i\omega t}, \qquad \text{where} \qquad X_T(t) = \begin{cases} X(t), & t \in [0, T], \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The inverse transform is then

$$X_T(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega A_T(\omega) e^{i\omega t}. \tag{3}$$

- Consider variance $\langle X_T^2 \rangle$. Express it in terms of $A_T(\omega)$ using the following form of the Parseval's identity: $\int_{-\infty}^{\infty} |X_T(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |A_T(\omega)|^2 \mathrm{d}\omega$.
- Show that this implies that the following relation for the spectral density function

$$S(\omega) = \lim_{T \to \infty} \frac{1}{T} \langle |A_T(\omega)|^2 \rangle. \tag{4}$$

- Finally, using the above relation find a formula for $S(\omega)$ in terms of the autocorrelation function $\kappa(t, t') \equiv \langle X(t) X(t') \rangle$. Note that for a stationary process $\kappa(t, t') = \kappa(|t - t'|)$.

Problem 2:

Two containers of volumes V_1 and V_2 filled with a gas are in thermal equilibrium. The total number of gas particles is N. The containers are connected by a narrow opening allowing for a slow exchange of particles.

- Write down the master equation for the probability p_n that n particles are in the left container. Assume that processes in which particles are moving from one container to the other are independent from each other and the jump probabilities per unit time for a single particle are α and γ, as indicated in the figure.
- Find the stationary solution for this situation.
- Find the relation between α and γ (remember that the gas is in thermal equilibrium).