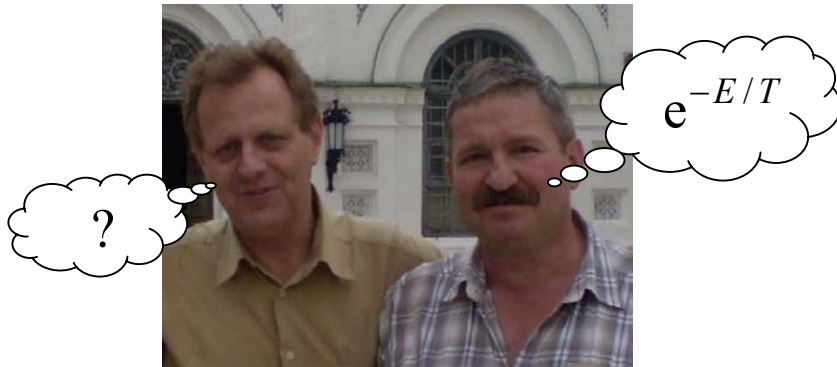


Instabilities Driven Equilibration of the Quark-Gluon Plasma

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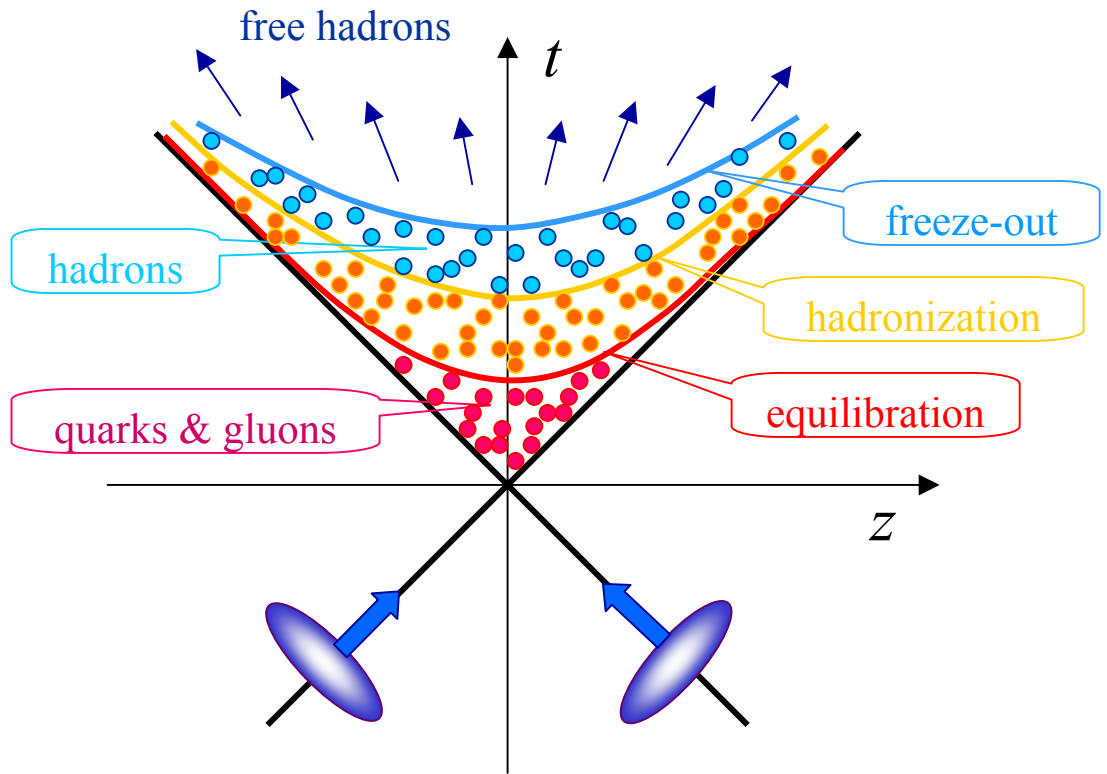
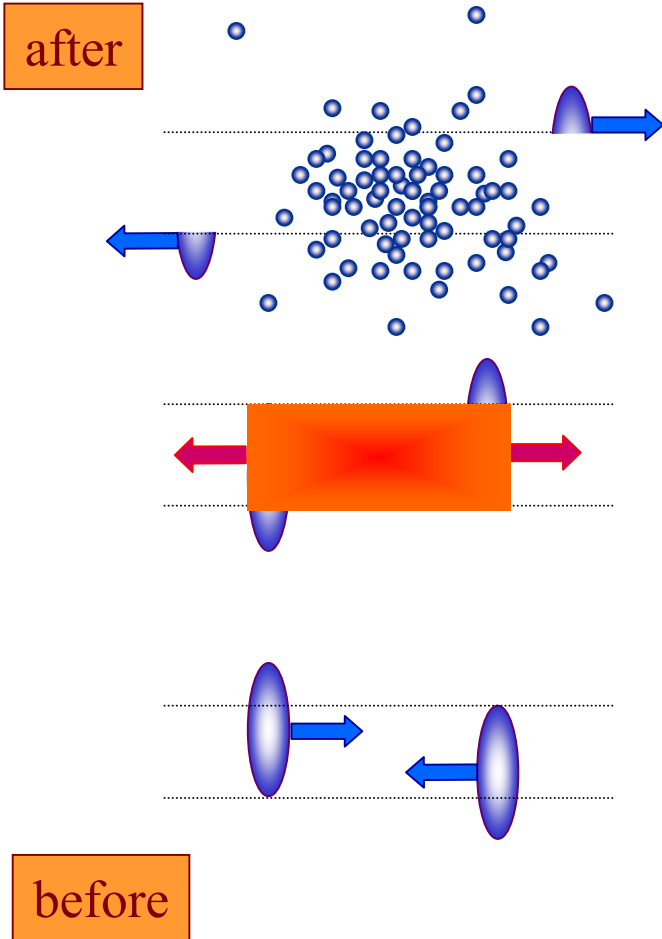
- why Mark's favorite formula $e^{-E/T}$ is so good -



Happy Birthday Mark!

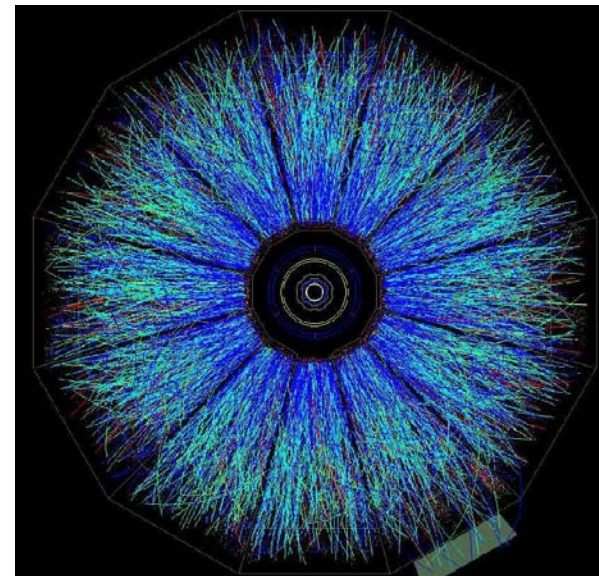
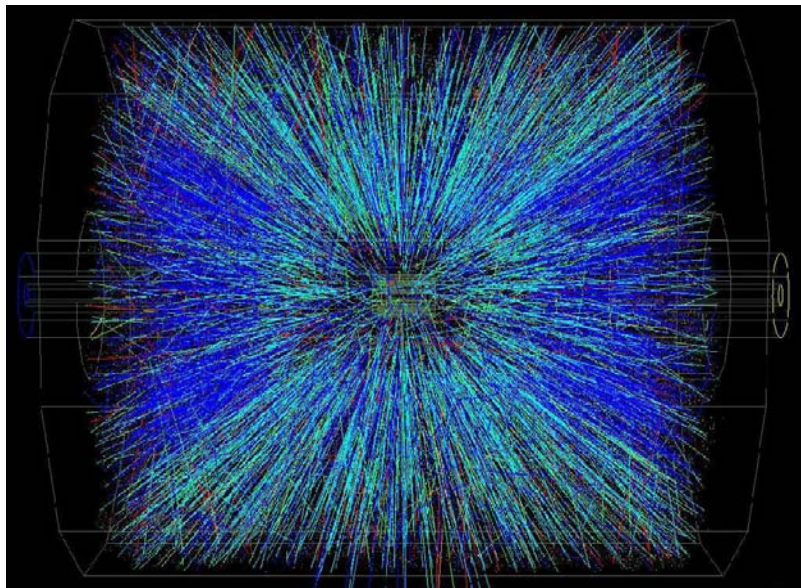


Relativistic heavy-ion collision – *Little Bang*



Relativistic heavy-ion collisions

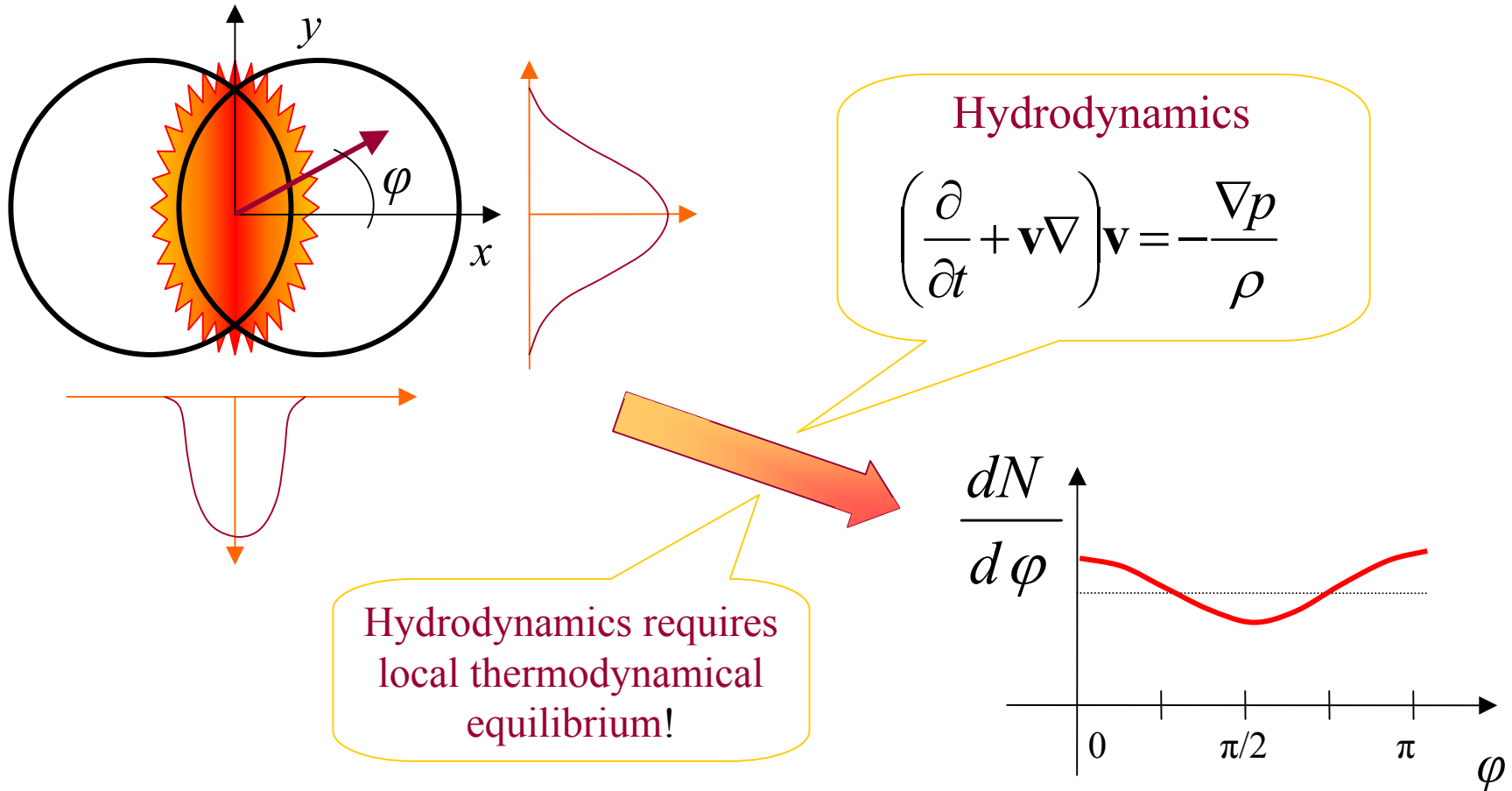
Au–Au collisions @ $\sqrt{s} = 100 + 100$ GeV/NN



 STAR experiment @ RHIC

Evidence of the early stage equilibration

Success of hydrodynamic models in describing elliptic flow



Equilibration is fast

$$v_2 \sim \varepsilon = \left\langle \frac{x^2 - y^2}{x^2 + y^2} \right\rangle$$

Eccentricity decays due to the free streaming!

$$\varepsilon \searrow \Rightarrow v_2 \searrow$$



$$t_{\text{eq}} \leq 1 \text{ fm}/c$$

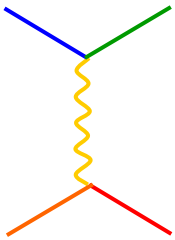
time of equilibration

Collisions are too slow

Time scale of hard parton-parton scattering

$$t_{\text{hard}} \sim \frac{1}{g^4 \ln(1/g) T}$$

dominated by



hard scattering \sim momentum transfer of order of T

either single hard scattering or multiple soft scatterings

g – QCD coupling constant

$$t_{\text{eq}} \approx t_{\text{hard}} \geq 2.6 \text{ fm}/c$$

Instabilities

stationary state

$$A(t) = A_0 + \delta A(t)$$

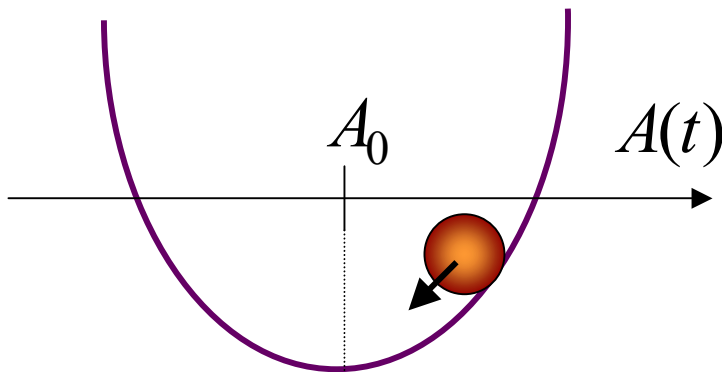
fluctuation

Instability

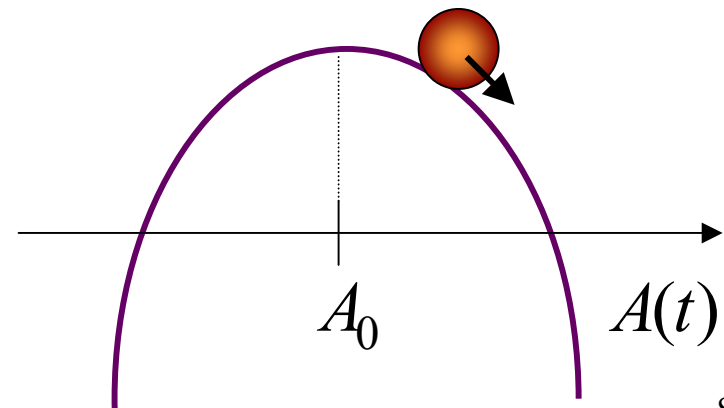
$$\delta A(t) \propto e^{\gamma t}$$

$$\gamma > 0$$

stable configuration



unstable configuration



Terminology

Plasma instabilities – interplay of particles and classical fields

Quantum Field Theory – no particles, no classical fields

$$p_{\text{hard}} \sim T$$

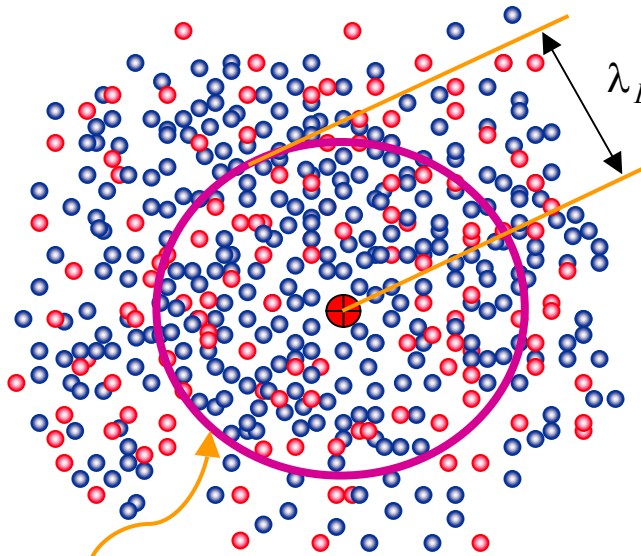
- particles – hard excitations, hard modes

- classical fields – highly populated soft excitations, soft modes

$$\sim 1/g^2$$

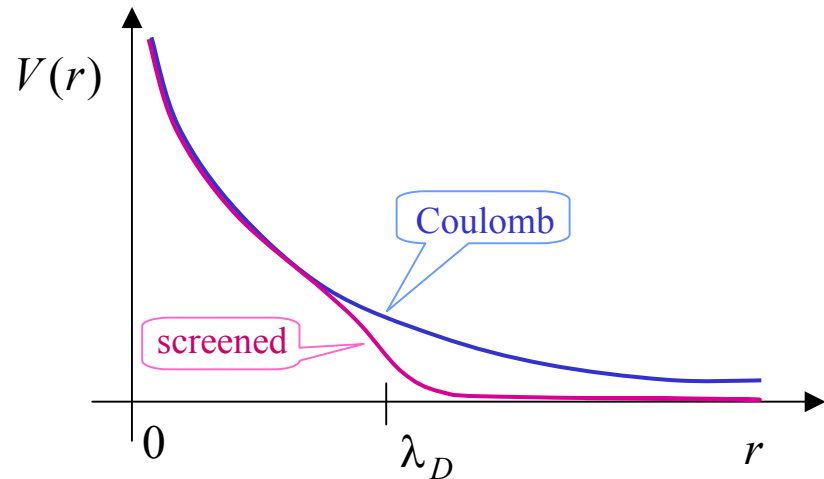
$$p_{\text{soft}} \sim gT$$

Plasma manifests collective behavior



Debye sphere

$$\lambda_D = \frac{1}{m_D} \sim \frac{1}{gT} \quad \text{screening length} \quad V(r) \sim \frac{e^{-\frac{r}{\lambda_D}}}{r}$$



$$V_D = \frac{4}{3} \pi \lambda_D^3 \sim \frac{1}{g^3 T^3}, \quad n \sim T^3, \quad n V_D \sim \frac{1}{g^3} \gg 1 \quad \text{if } g \ll 1$$

In a weakly coupled plasma, there are many particles in a Debye sphere!

Plasma instabilities

▶ instabilities in configuration space – **hydrodynamic instabilities**

▶ instabilities in momentum space – **kinetic instabilities**

instabilities due to non-equilibrium
momentum distribution

$$f(\mathbf{p}) \text{ is not } \sim \exp\left(-\frac{E}{T}\right)$$

Kinetic instabilities

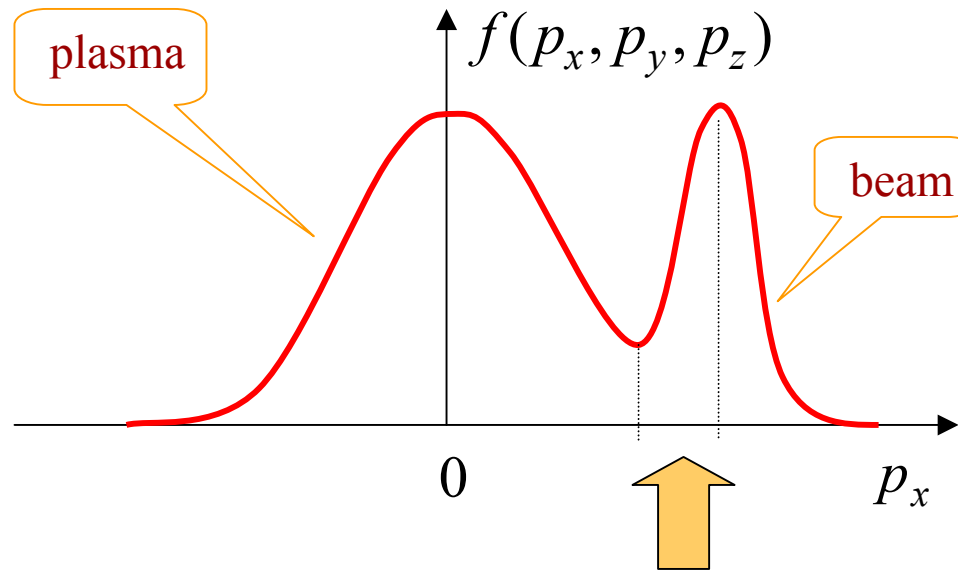
▶ **longitudinal modes** – $\mathbf{k} \parallel \mathbf{E}$, $\delta\rho \sim e^{-i(\omega t - \mathbf{k}\mathbf{r})}$

▶ **transverse modes** – $\mathbf{k} \perp \mathbf{E}$, $\delta\mathbf{j} \sim e^{-i(\omega t - \mathbf{k}\mathbf{r})}$

\mathbf{E} – electric field, \mathbf{k} – wave vector, ρ – charge density, \mathbf{j} – current

Logitudinal modes

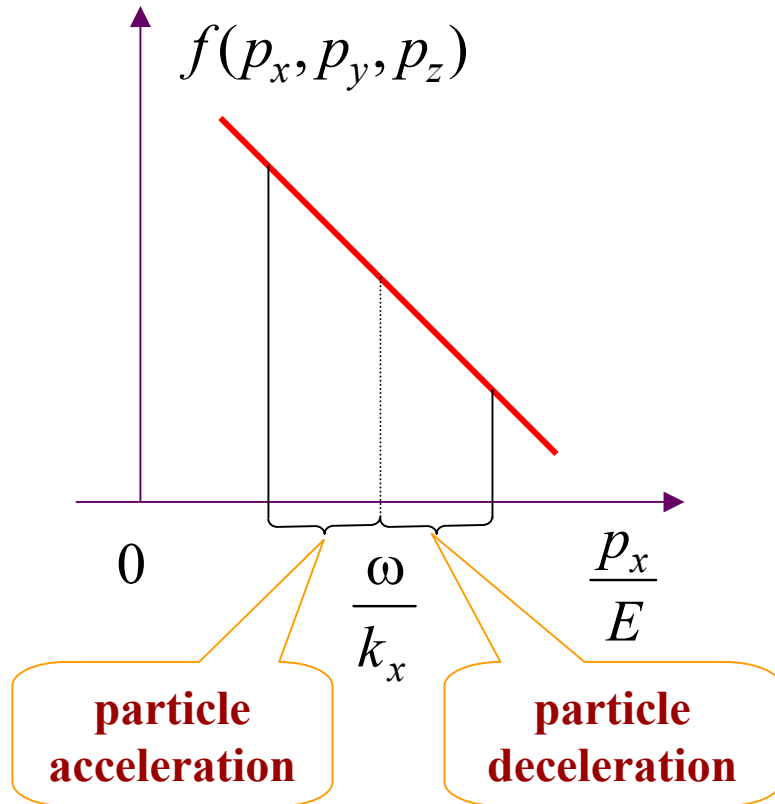
unstable configuration



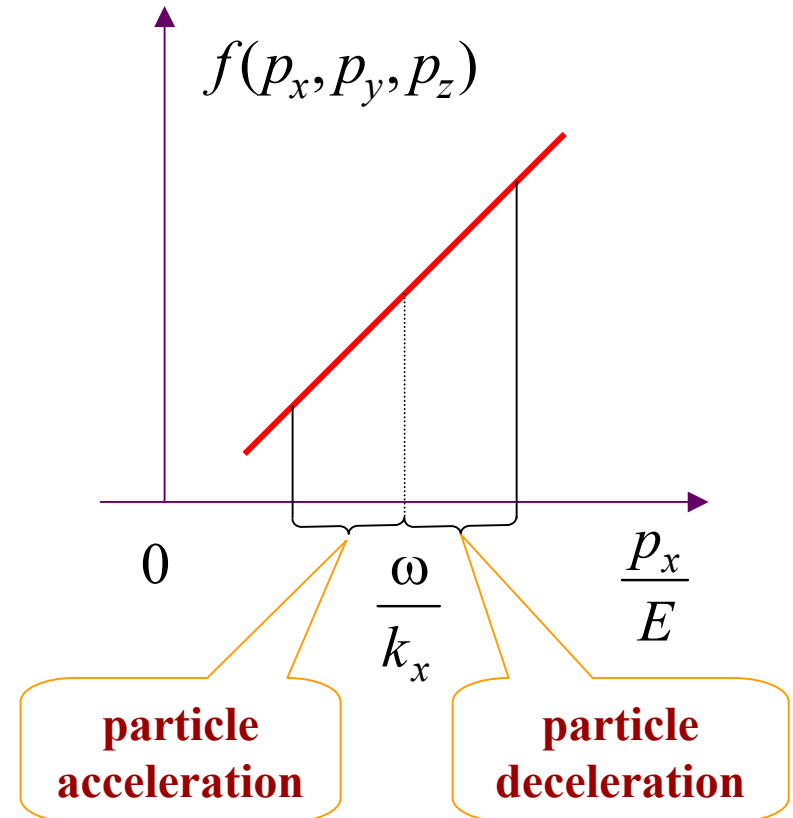
Energy is transferred from particles to fields

Logitudinal modes

Electric field decays - **damping**



Electric field grows - **instability**



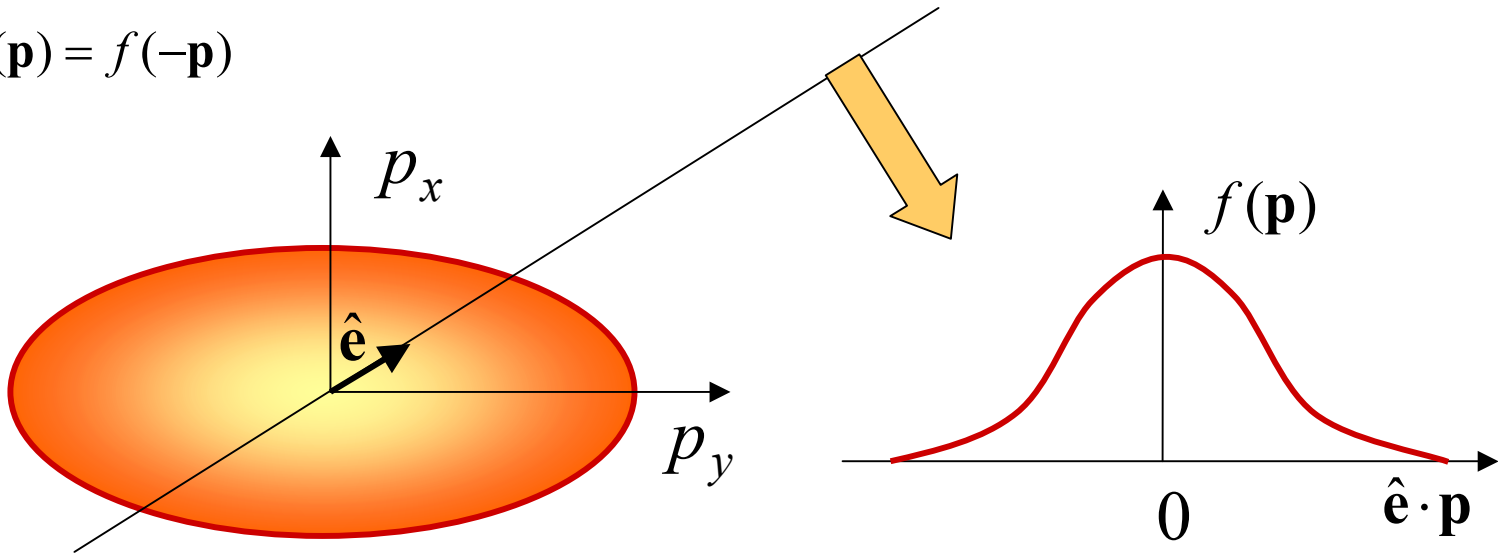
$\frac{\omega}{k_x}$ - phase velocity of the electric field wave,

$\frac{p_x}{E}$ - particle's velocity

Transverse modes

Unstable modes occur due to anisotropy of the momentum distribution

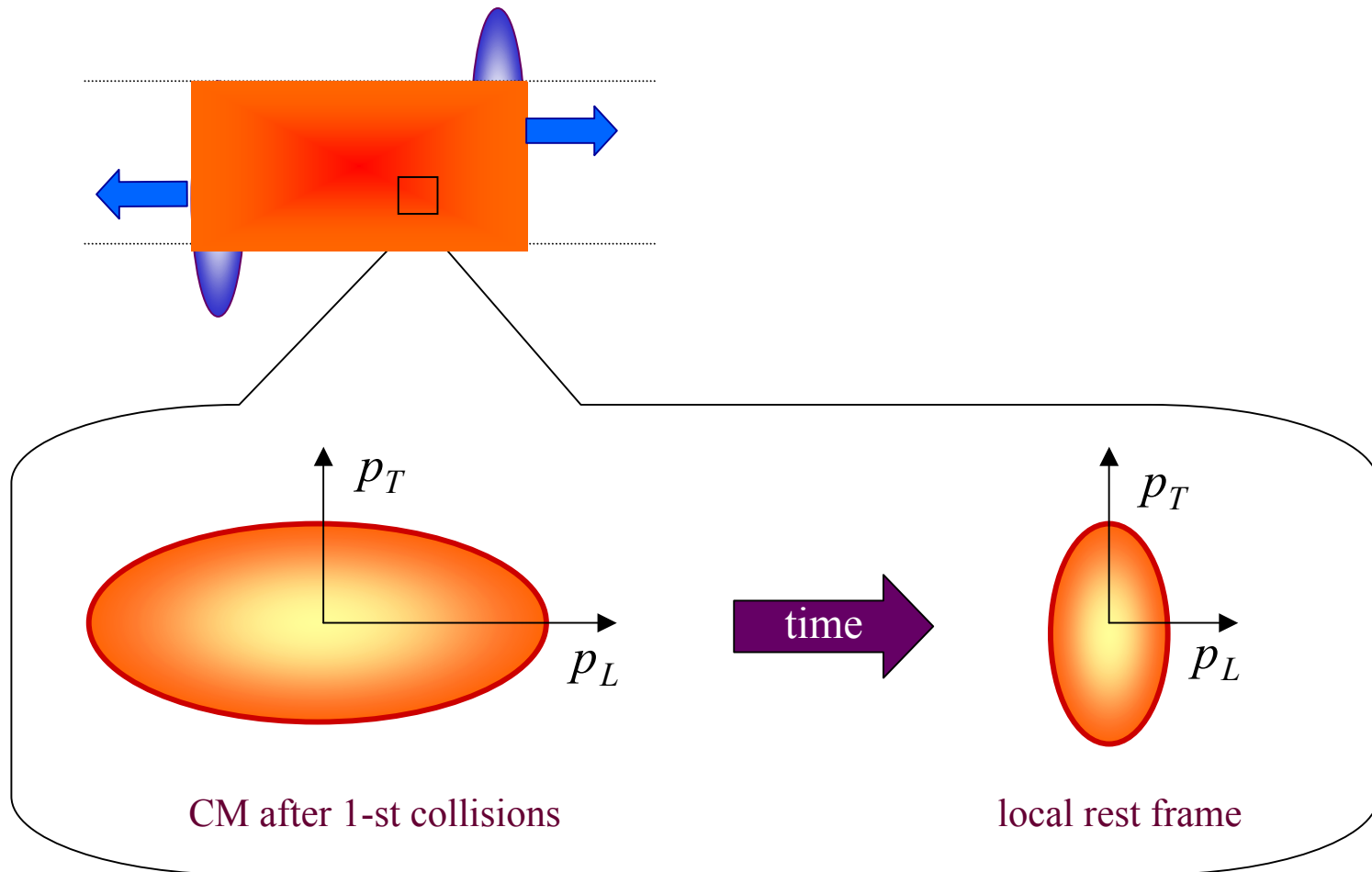
$$f(\mathbf{p}) = f(-\mathbf{p})$$



Momentum distribution distribution can monotonously decrease in every direction

Transverse modes are relevant for relativistic nuclear collisions!

Momentum Space Anisotropy in Nuclear Collisions

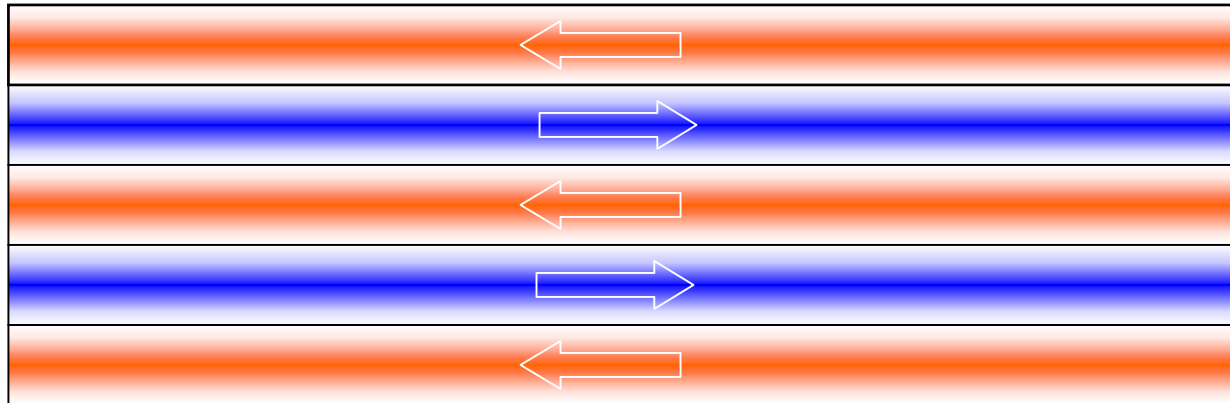


Seeds of instability

$\langle j_a^\mu(x) \rangle = 0$ but current fluctuations are finite

$$\langle j_a^\mu(x_1) j_b^\nu(x_2) \rangle = \frac{1}{8} \delta^{ab} \int \frac{d^3 p}{(2\pi)^3} \frac{p^\mu p^\nu}{E_p^2} f(\mathbf{p}) \delta^{(3)}(\mathbf{x} - \mathbf{v}t) \neq 0$$

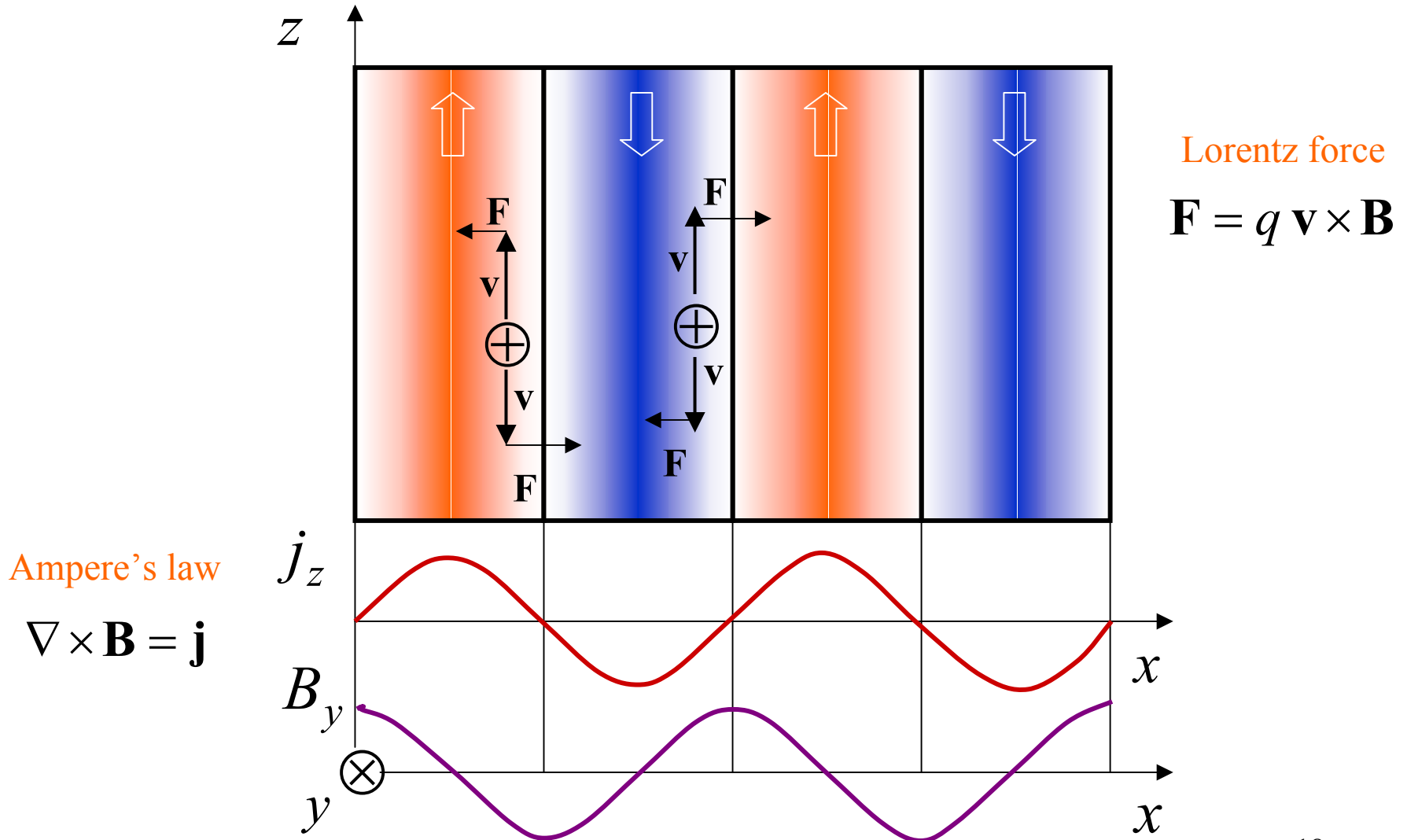
$$x_1 = (t_1, \mathbf{x}_1), \quad x_2 = (t_2, \mathbf{x}_2), \quad x = (t_1 - t_2, \mathbf{x}_1 - \mathbf{x}_2)$$



Direction of the momentum surplus



Mechanism of filamentation

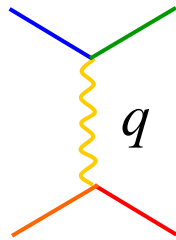


Instabilities vs. collisions

Time scale of parton-parton scattering

$$t_{\text{hard}} \sim \frac{1}{g^4 \ln(1/g) T}$$

$$t_{\text{soft}} \sim \frac{1}{g^2 \ln(1/g) T}$$



hard scattering: $q \sim T$

soft scattering: $q \sim gT$

Time scale of collective phenomena

$$t_{\text{collec}} \sim \frac{1}{g T}$$

$$g^2 \ll 1 \Rightarrow t_{\text{hard}} \gg t_{\text{soft}} \gg t_{\text{collec}}$$

The instabilities are fast!

Dispersion equation

Equation of motion of chromodynamic field A^μ in momentum space

$$[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] A_\nu(k) = 0$$

gluon self-energy

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

Instabilities – solutions with $\text{Im}\omega > 0$ $\Rightarrow A^\mu(x) \sim e^{\text{Im}\omega t}$

Dynamical information is hidden in $\Pi^{\mu\nu}(k)$. How to get it?

Transport theory

fundamental

$$p_\mu D^\mu Q - \frac{g}{2} p^\mu \{F_{\mu\nu}(x), \partial_p^\nu Q\} = C$$

quarks

$$p_\mu D^\mu \bar{Q} + \frac{g}{2} p^\mu \{F_{\mu\nu}(x), \partial_p^\nu \bar{Q}\} = \bar{C}$$

antiquarks

adjoint

$$p_\mu \mathcal{D}^\mu G - \frac{g}{2} p^\mu \{F_{\mu\nu}(x), \partial_p^\nu G\} = C_g$$

gluons

free streaming

mean-field force

collisions

$$D^\mu \equiv \partial^\mu - ig[A^\mu, \dots], \quad F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$$

$$D_\mu F^{\mu\nu} = j^\nu [Q, \bar{Q}, G]$$

mean-field generation

$$\text{collisionless limit: } C = \bar{C} = C_g = 0$$

Diagrammatic Hard Loop approach

$$\Pi^{\mu\nu}(k) = \left(\begin{array}{c} \text{Diagram 1: } \text{Wavy line } k \text{ enters from left, } k \text{ exits to right, loop with } p \text{ and } p+k \text{ } \\ \text{Diagram 2: } \text{Wavy line } k \text{ enters from left, } k \text{ exits to right, loop with } p \text{ and } p+k \text{ } \\ \text{Diagram 3: } \text{Wavy line } k \text{ enters from bottom, } k \text{ exits to bottom, loop with } p \text{ and } p+k \text{ } \end{array} \right)$$

Hard loop approximation: $k^\mu \ll p^\mu$

$$\Pi^{\mu\nu}(k) = \frac{g^2}{2} \int \frac{d^3 p}{(2\pi)^3} \frac{p^\mu}{E} \left[g^{\mu\lambda} - \frac{p^\nu k^\lambda}{p^\sigma k_\sigma + i0^+} \right] \frac{\partial f(\mathbf{p})}{\partial p^\lambda}$$

$$\Pi^{\mu\nu}(k) = \Pi^{\nu\mu}(k), \quad k_\mu \Pi^{\mu\nu}(k) = 0$$

Chromo-hydrodynamic approach

$$D_{\mu} n^{\mu}(x) = 0$$

$$D_{\mu} T^{\mu\nu}(x) - \frac{g}{2} \{F^{\mu\nu}, n_{\mu}(x)\} = 0$$

Postulated form of $n^{\mu}(x)$ and $T^{\mu\nu}(x)$:

$$n^{\mu}(x) = n(x) u^{\mu}(x)$$

$$T^{\mu\nu}(x) = \frac{1}{2} (\varepsilon(x) + p(x)) \{u^{\mu}(x), u^{\nu}(x)\} - p(x) g^{\mu\nu}$$

$$n(x), \varepsilon(x), p(x), u^{\mu}(x) \text{ matrices!} \quad u^{\mu}(x) u_{\mu}(x) = 1$$

To close the system of equations:

$$\nabla p = 0 \quad \text{or} \quad \varepsilon = 3p \quad \Leftarrow \quad T_{\mu}^{\mu} = 0$$

Dispersion equation

Dispersion equation

$$\det[k^2 g^{\mu\nu} - k^\mu k^\nu - \Pi^{\mu\nu}(k)] = 0$$

$$k_\mu \Pi^{\mu\nu}(k) = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} - \frac{1}{\omega^2} \Pi^{ij}(k) \quad \text{chromodielectric tensor}$$

$$k^\mu \equiv (\omega, \mathbf{k})$$

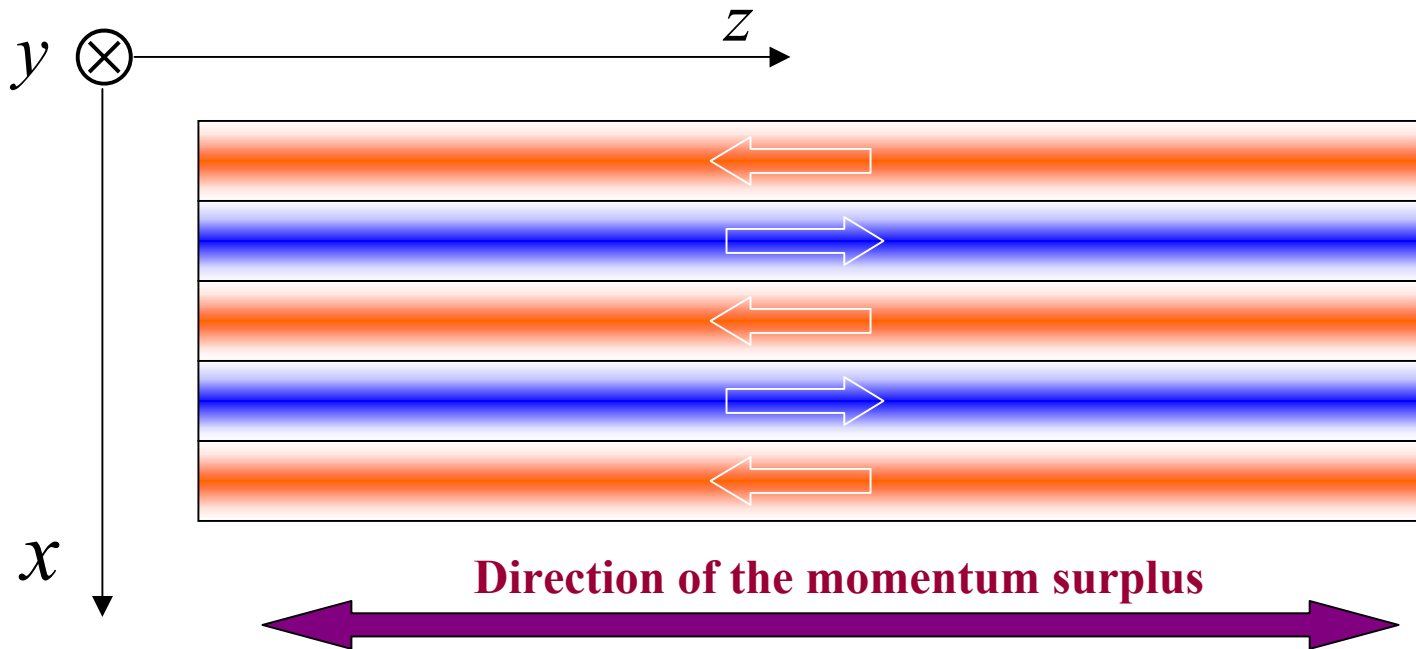
Dispersion equation

$$\det[\mathbf{k}^2 \delta^{ij} - k^i k^j - \omega^2 \varepsilon^{ij}(k)] = 0$$

$$\varepsilon^{ij}(k) = \delta^{ij} + \frac{g^2}{2\omega} \int \frac{d^3 p}{(2\pi)^3} \frac{v^i}{\omega - \mathbf{k}\mathbf{v} + i0^+} \frac{\partial f(\mathbf{p})}{\partial p^l} \left[\left(1 - \frac{\mathbf{k}\mathbf{v}}{\omega}\right) \delta^{lj} + \frac{k^l v^j}{\omega} \right]$$

$$\mathbf{v} \equiv \mathbf{p} / E \quad 24$$

Dispersion equation – configuration of interest



$$\mathbf{j} = (0, 0, j), \quad \mathbf{E} = (0, 0, E), \quad \mathbf{k} = (k, 0, 0)$$

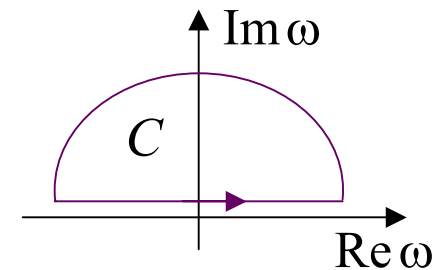
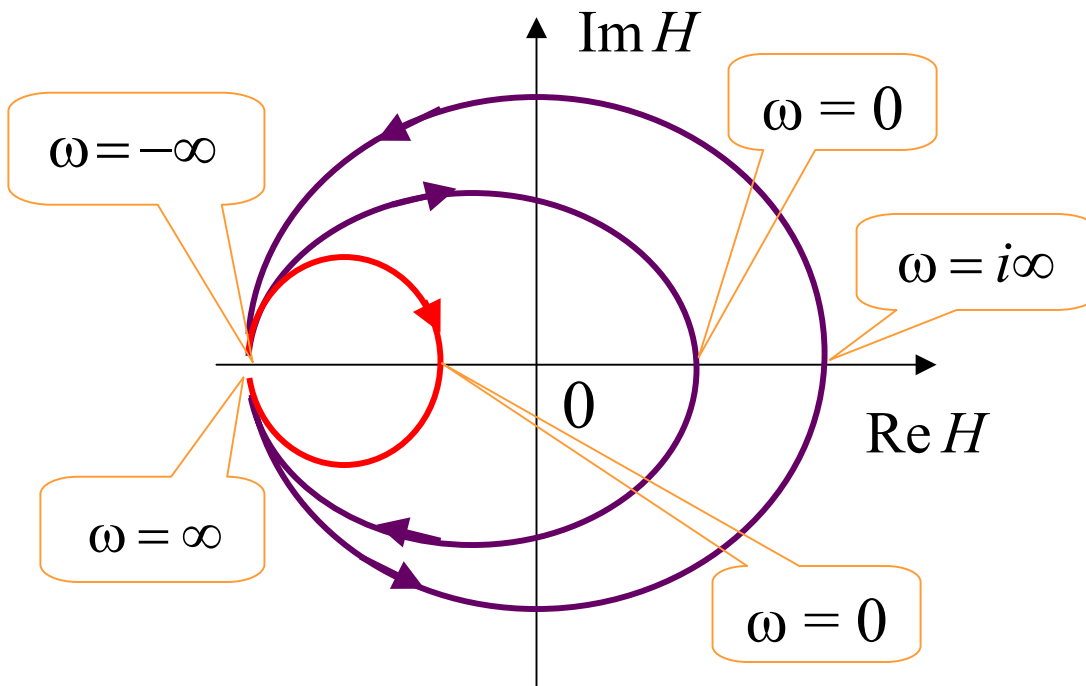
Dispersion equation

$$k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0$$

Existence of unstable modes – Penrose criterion

$$H(\omega) \equiv k^2 - \omega^2 \varepsilon^{zz}(\omega, k)$$

$$\oint_C \frac{d\omega}{2\pi i} \frac{1}{H(\omega)} \frac{dH(\omega)}{d\omega} = \begin{cases} \oint_C \frac{d\omega}{2\pi i} \frac{d \ln H(\omega)}{d\omega} = \ln H(\omega) \Big|_{\phi=\pi^+}^{\phi=\pi^-} \\ \text{number of zeros of } H(\omega) \text{ in } C \end{cases}$$



There are unstable modes if

$$H(\omega = 0) < 0$$

Anisotropy!

Unstable solutions

$$f(\mathbf{p}) = \frac{2^{1/2}}{\pi^{3/2}} \frac{\rho \sigma_{\perp}^4}{\sigma_{\parallel}} \frac{1}{(p_{\perp}^2 + \sigma_{\perp}^2)^3} e^{-\frac{p_{\parallel}^2}{2\sigma_{\parallel}^2}}$$

$$\rho = 6 \text{ fm}^{-3}$$

$$\alpha_s = g^2 / 4\pi = 0.3$$

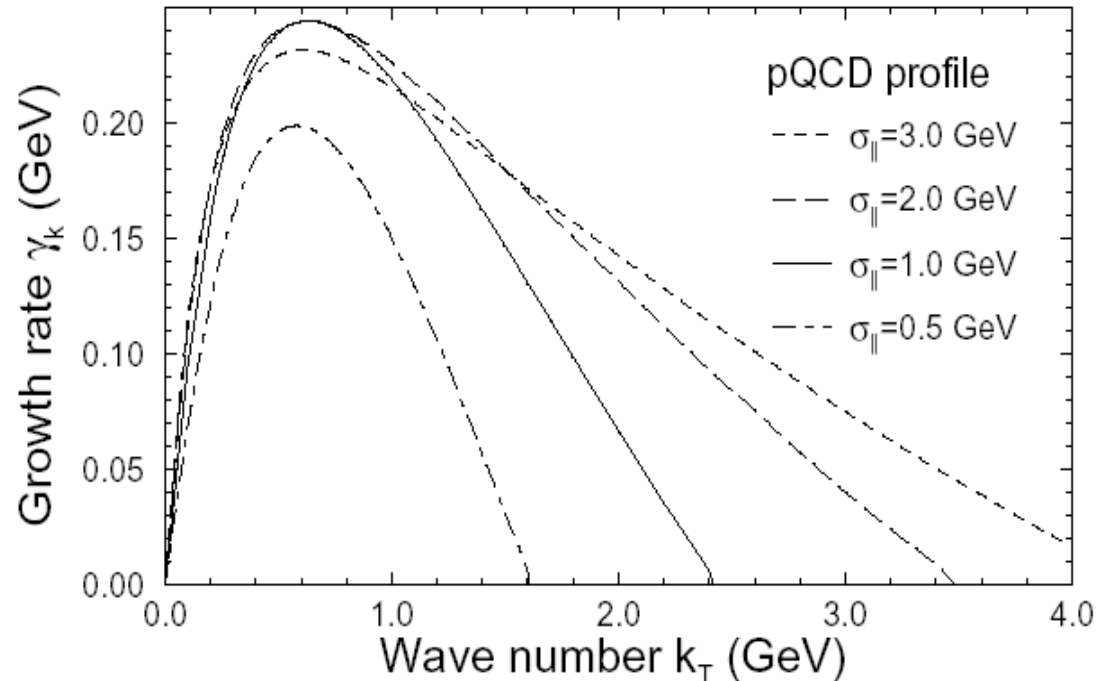
$$\sigma_{\perp} = 0.3 \text{ GeV}$$

$$k^2 - \omega^2 \varepsilon^{zz}(\omega, k) = 0$$

solution

$$\omega(k) = \pm i \gamma_k$$

$$0 < \gamma_k \in \mathfrak{R}$$



Hard-Loop dynamics

Soft fields in the passive background of hard particles

Braaten-Pisarski action generalized to anisotropic momentum distribution:

$$L_{\text{eff}} = \frac{g^2}{2} \int \frac{d^3 p}{(2\pi)^3} \left[f(\mathbf{p}) F_{\mu\nu}^a(x) \left(\frac{p^\nu p^\rho}{(p \cdot D)^2} \right)_{ab} F_{\rho}^{b\mu}(x) \right. \\ \left. + i \frac{C_F}{3} \tilde{f}(\mathbf{p}) \psi(x) \frac{p \cdot \gamma}{p \cdot D} \psi(x) \right]$$

$$k_\mu \Pi^{\mu\nu}(k) = 0, \quad k_\mu \Lambda^\mu(p, q, k) = \Sigma(p) + \Sigma(q)$$

Growth of instabilities – 1+1 numerical simulations

SU(2) Hard Loop Dynamics

1+1 dimensions

$$A_a^\mu = A_a^\mu(t, z)$$

Scaled
field energy
density

Anisotropic particle's
momentum distribution

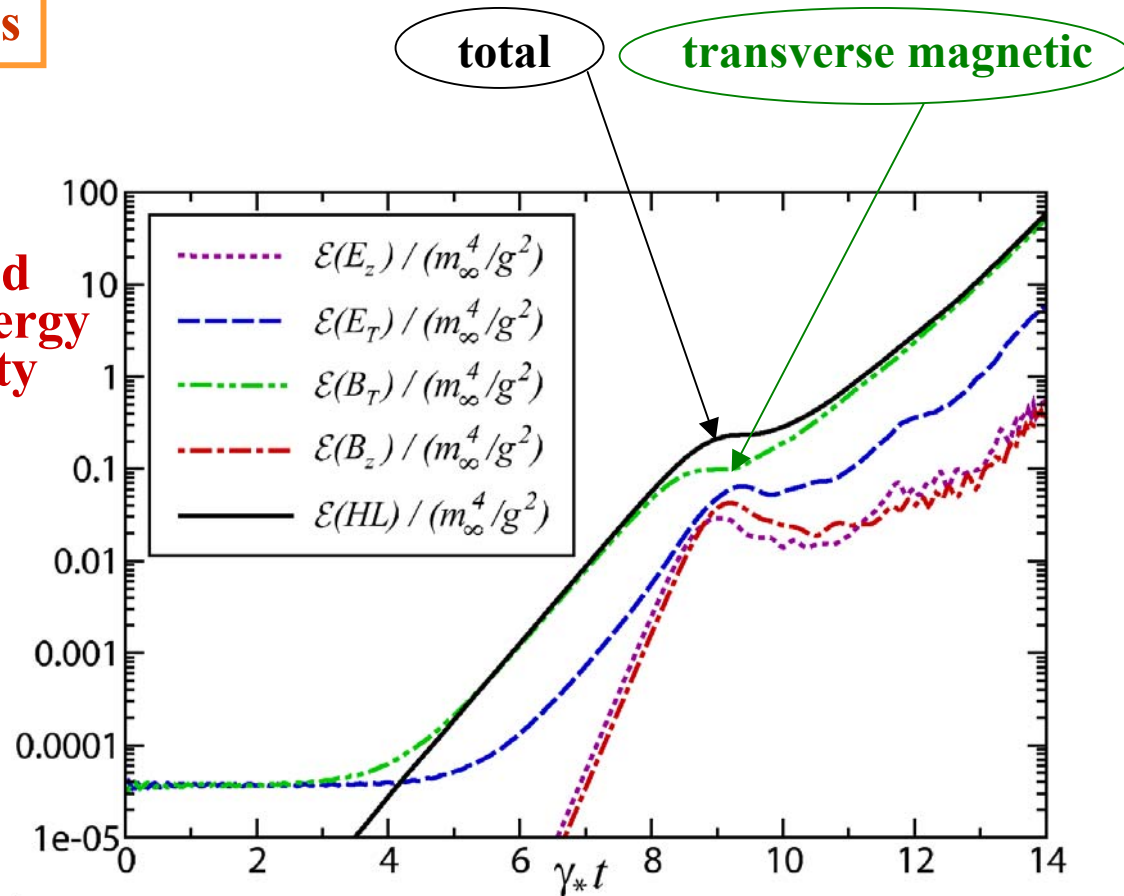
$$f(\mathbf{p}) = f_{\text{iso}}(|\mathbf{p}| + \zeta p_z)$$

$$m_D^2 = -\frac{\alpha_s}{\pi} \int_0^\infty dp p^2 \frac{df_{\text{iso}}(p)}{dp}$$

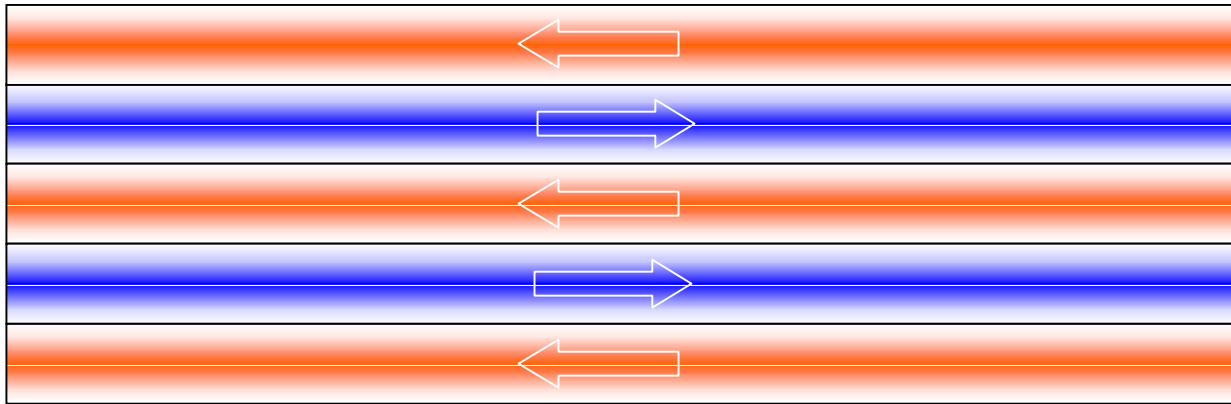
(m_D, ζ)

Strong anisotropy $\zeta = 10$

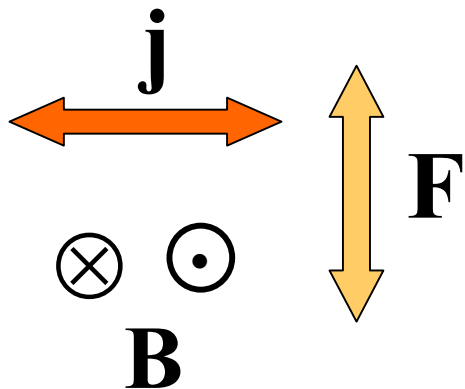
γ_* - maximal growth rate



Isotropization - particles

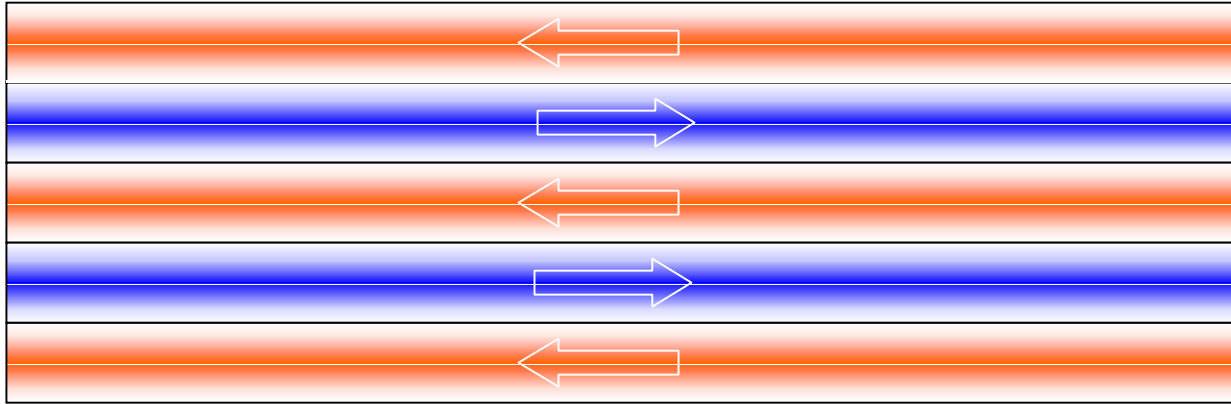


Direction of the momentum surplus

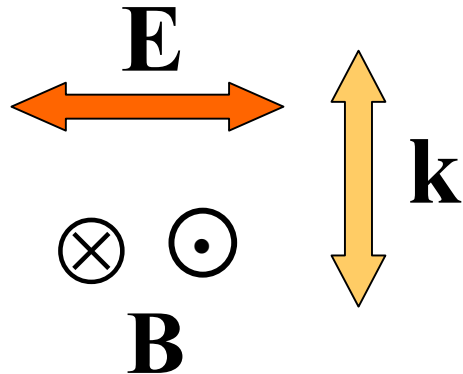


$$\Delta \mathbf{p} = \int dt \mathbf{F}$$

Isotropization - fields



Direction of the momentum surplus



$$\mathbf{P}_{\text{fields}} \sim \mathbf{B}^a \times \mathbf{E}^a \sim \mathbf{k}$$

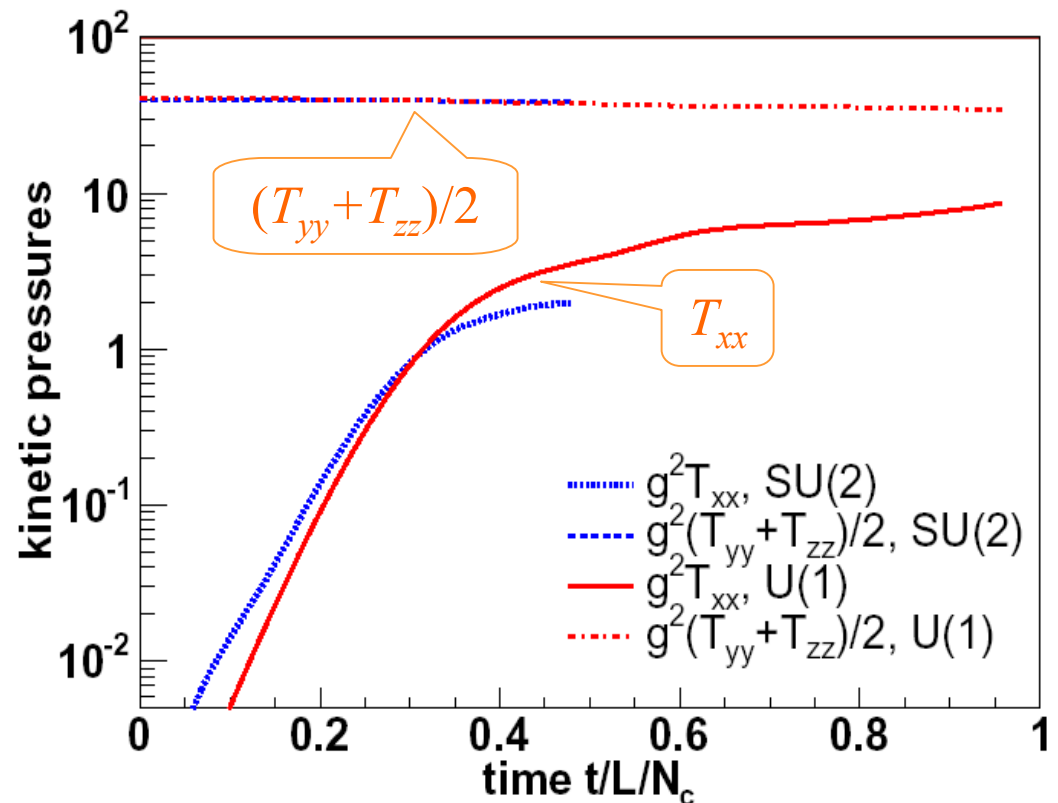
Isotropization – numerical simulation

Classical system of colored particles & fields

$$T_{ij} = \int \frac{d^3 p}{(2\pi)^3} \frac{p_i p_j}{E} f(\mathbf{p})$$

Isotropy:

$$T_{xx} = (T_{yy} + T_{zz}) / 2$$



Conclusion

The scenario of instabilities driven equilibration provides a plausible solution of the fast equilibration problem

