On three-dimensional Weyl structures with reduced holonomy[*]

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Abstract
Cartan's list of three-dimensional Weyl structures with reduced holonomy
is revisited. We show that the only Einstein–Weyl structures on this list
correspond to the structures generated by the solutions of the dKP equation.

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1. Introduction

In [1], Cartan gave a complete list of three-dimensional Weyl geometries with reduced
holonomy. Cartan did not study the Einstein–Weyl equations for the geometries from his list.
On the other hand, in recent years, various authors [2–4, 7] have been studying the Einstein–
Weyl equations in three dimensions, mainly due to their relations with twistor theory and
integrable systems theory. In particular, Dunajski *et al* [2] characterized all three-dimensional
Einstein–Weyl spaces which admit a covariantly constant weighted vector field, as being
generated by solutions to the dispersionless Kadomtsev–Petviashvili (dKP) equation. Their
analysis is very much in the spirit of the reduced holonomy ideas, since the existence of
such a vector field reduces the holonomy of the considered Weyl geometry. However, it
is not clear from their analysis if all the three-dimensional Weyl geometries with reduced
holonomy may be obtained by means of an assumption of the existence of a covariantly
constant weighted vector field. Quick inspection of the Cartan list of [1] (look also at table 1
of the present paper) shows that such an assumption is very strong and that it excludes a large
class of Weyl geometries with reduced holonomy. A natural question, if the geometries from
this class may be Einstein, is addressed in the present paper. Here, we first simplify and
rephrase in modern language Cartan's classification of three-dimensional Weyl geometries
with reduced holonomy. This is done by inspecting all possible subalgebras of $\mathfrak{co}(2, 1)$ and
$\mathfrak{co}(3)$. Then, by means of the integration of the first structure equations, we determine which of
them may appear as the Weyl holonomy algebras. The integration procedure enables us to give
canonical representatives of the metric and the Weyl potential for each algebra representing

the reduced holonomy. We also specify the geometric object that reduces the holonomy. It is either a covariantly constant vector field or a covariantly constant null direction. This second possibility corresponds to the class of Weyl geometries present in the Cartan list, but not considered by Dunajski *et al*. The last part of our paper imposes the Einstein condition on all the geometries from the Cartan list. The result is included in propositions 3.1 and 3.2 which strengthen the results of [2] to the following statement:

All three-dimensional Einstein–Weyl geometries with reduced holonomy are either flat or are generated by the solutions of the dispersionless Kadomtsev–Petviashvili equation.

2. Weyl structures

A Weyl structure on a real n-dimensional manifold $\mathcal{M}$ consists of a conformal class of metrics $[g]$ of signature (p, q) and a torsion-free covariant derivative ∇, such that for each representative g of $[g]$ there exists a 1-form ν satisfying

$$\nabla g = -2\nu \otimes g. \tag{1}$$

When g changes as $g \to \mathrm{e}^{2\phi} g$, then ν changes as $\nu \to \nu - \mathrm{d}\phi$, so as to leave (1) invariant. The class of pairs $[g, \nu]$ considered modulo this gauge, uniquely defines the Weyl structure. If ν is closed, then the Weyl structure can be locally reduced to a metric structure by an appropriate gauge; thus we assume $\mathrm{d}\nu \neq 0$.

Let

$$CO(p, q) = \mathbb{R}_+ \times O(p, q) = \{M \in \mathrm{Matr}_{n \times n}(\mathbb{R}) \mid M^T g M = \lambda g, \lambda \in \mathbb{R}_+\}$$

denote the Lie group preserving the conformal class $[g]$. Then $[g]$ defines the bundle

$$CO(p, q) \to \mathcal{P} \to \mathcal{M},$$

a reduction of the bundle of linear frames on $\mathcal{M}$. Any Weyl structure on $\mathcal{M}$ is alternatively defined by a linear $\mathfrak{co}(p, q)$-valued torsion-free connection on $\mathcal{P}$. This enables us to apply all the results of the theory of connections [5] to this case. In particular, the notion of holonomy is well defined and one can study Weyl structures with reduced holonomy. By means of the reduction theorem ([5], p 83), a Weyl structure has its holonomy reduced to some subgroup $H \subset CO(p, q)$, if and only if the Weyl connection is reducible to an $\mathfrak{h}$-valued connection on the holonomy bundle $H \to \mathcal{P}' \to \mathcal{M}$ of $\mathcal{P}$.

Let (e_i) be a frame on $\mathcal{M}$, such that the dual coframe (θ^i) is orthonormal for some representative g of $[g]$, i.e. $g = g_{ij} \theta^i \theta^j$ with all the coefficients g_{ij} being constant. By a Weyl connection 1-form Γ we understand the pullback of the Weyl connection from $CO(p, q) \to \mathcal{P} \to \mathcal{M}$ to $\mathcal{M}$, through the frame (e_i) considered as a section of $\mathcal{P}$. The Weyl connection 1-forms Γ^i_j are uniquely defined by the relations

$$\mathrm{d}\theta^i + \Gamma^i_{j \wedge} \theta^j = 0, \tag{2}$$

$$\Gamma_{(ij)} = g_{ij} \nu, \qquad \text{where} \quad \Gamma_{ij} = g_{jk} \Gamma^k_j. \tag{3}$$

A Weyl structure has its holonomy reduced to H if the matrix (Γ^i_j) takes values in the Lie algebra $\mathfrak{h}$ of H. Due to

$$\mathfrak{co}(p, q) = \mathbb{R} \oplus \mathfrak{o}(p, q)$$

Γ decomposes into the $\mathbb{R}$-valued part ν and the $\mathfrak{o}(p, q)$-valued part $\widetilde{\Gamma}$ so that

$$\Gamma = \nu \cdot \mathrm{id} + \widetilde{\Gamma}. \tag{4}$$

Hence the subgroups $H \subset O(p, q) \subset CO(p, q)$ do not appear as holonomy groups for Weyl structures.

A tensor field of weight m on $\mathcal{M}$ is a tensor object T transforming as $T \rightarrow e^{m\phi} T$ when $g \rightarrow e^{2\phi} g$. The weighted covariant derivative of a (k, l)-tensor field of weight m

$$\widehat{\nabla} T = \nabla T + m\nu \otimes T$$

is a $(k + 1, l)$-tensor field of weight m. If $\widehat{\nabla} T = 0$, then T is said to be covariantly constant. A direction $\mathcal{K}$ spanned by a vector field K is said to be covariantly constant, when $\nabla_X K \in \mathcal{K}$ for an arbitrary vector field X. A non-null direction $\mathcal{K}$ is covariantly constant iff there is a covariantly constant vector field $K \in \mathcal{K}$ of weight -1. The existence of a covariantly constant null direction is a weaker property than the existence of a covariantly constant weighted vector in this direction.

The curvature 2-form Ω, the Ricci tensor Ric and the Ricci scalar R of a Weyl structure are defined by

$$\Omega^i_j = d\Gamma^i_j + \Gamma^i_k {\wedge} \Gamma^k_j, \qquad \Omega^i_j = \tfrac{1}{2}\Omega^i_{jkl}\theta^k {\wedge} \theta^l,$$
$$\mathrm{Ric}_{ij} = \Omega^k_{ikj}, \qquad R = \mathrm{Ric}_{ij} g^{ij}.$$

Ω and Ric have weights 0 whereas R has weight -2. Einstein–Weyl (E–W) structures are, by definition, those Weyl structures for which the symmetric trace-free part of the Ricci tensor vanishes,

$$\mathrm{Ric}_{(ij)} - \frac{1}{n} R \cdot g_{ij} = 0.$$

A Weyl structure is flat, i.e. $\Omega = 0$, iff it has a (local) representative $(g = \eta, \nu = 0)$, where η is the flat metric.

3. Three-dimensional Weyl structures with reduced holonomy

In order to find all possible 3D Weyl structures with reduced holonomy, we integrate equations (2) for each subalgebra of $\mathfrak{co}(2, 1)$ or $\mathfrak{co}(3)$. These subalgebras are classified in [6] up to adjoint transformations. We use this classification in the following.

We begin with the more complicated Lorentzian case. Let us choose a coframe $(\theta^1, \theta^2, \theta^3)$ such that

$$g = (\theta^2)^2 - 2\theta^1\theta^3.$$

The algebra $\mathfrak{co}(2, 1)$ now reads

$$\begin{pmatrix} p + a & b & 0 \\ c & p & b \\ 0 & c & p - a \end{pmatrix}.$$

The subalgebras with $p \neq 0$ are the following:

$$
\begin{aligned}
&A\text{:} \quad c = 0, \\
&B_q\text{:} \quad c = 0, a = -(q + 1)p, q \in \mathbb{R}, \\
&C\text{:} \quad c = 0, b = 0, \\
&D_q\text{:} \quad c = 0, b = 0, a = (q + 1)p, q > -1, \\
&E\text{:} \quad a = 0, c = -b, \\
&F\text{:} \quad c = 0, a = 0, b = \pm p, \\
&G_q\text{:} \quad a = 0, b = qp, c = -qp, q \in \mathbb{R}.
\end{aligned}
$$

Obviously, A contains B, C, D, F, and E contains G.

Let us integrate the system (2) for the subalgebra A. In this case the system reads

$$d\theta^1 + (\nu + \alpha)_\wedge\theta^1 + \beta_\wedge\theta^2 = 0, \qquad d\theta^2 + \nu_\wedge\theta^2 + \beta_\wedge\theta^3 = 0, \qquad d\theta^3 + (\nu - \alpha)_\wedge\theta^3 = 0, \qquad (5)$$

where $\alpha = \frac{1}{2}(\Gamma^1_1 - \Gamma^2_2)$, $\beta = \Gamma^1_2 = \Gamma^2_3$, $\gamma = \Gamma^2_1 = \Gamma^2_2$, $\nu = \Gamma^2_2 = \frac{1}{2}(\Gamma^1_1 + \Gamma^2_2)$. We have a three-parameter family of transformations preserving this system; this is the Lie group G_A of the algebra A. The coframe transformation $\theta^i \rightarrow M^i_j\theta^j$ with $G_A \ni M = \exp\left(\begin{smallmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -t \end{smallmatrix}\right)$

sends $\alpha \rightarrow \alpha - dt$, $\beta \rightarrow e^t\beta$ and $\nu \rightarrow \nu$. Similarly $M = \exp\left(\begin{smallmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{smallmatrix}\right)$ sends $\alpha \rightarrow \alpha$, $\beta \rightarrow \beta$,

$\nu \rightarrow \nu - dt$ and $M = \exp\left(\begin{smallmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{smallmatrix}\right)$ transforms $\beta \rightarrow \beta - t\alpha - dt$, leaving α and ν invariant. Exploiting this gauge freedom we easily achieve

$$\beta_\wedge\theta^2_\wedge\theta^3 = 0, \qquad (\nu + \alpha)_\wedge\theta^1\theta^3 = 0, \qquad (\nu - \alpha)_\wedge\theta^3 = 0. \qquad (6)$$

Now $d\theta^3 = 0$, which enables us to make $\theta^3 = dx$. Moreover, $d\theta^2_\wedge\theta^2 = 0$ and $d\theta^1_\wedge\theta^3 = 0$, so that $\theta^2 = a\,dz$ and $\theta^1 = dy + b\,dx$. In addition, $0 \neq \theta^1_\wedge\theta^2_\wedge\theta^3 = ab\,dx_\wedge dy_\wedge dz$; hence (x, y, z) is a coordinate system on $\mathcal{M}$. In this coordinate system, using (5), (6), it is easy to get $\nu = -(\log a)_y\,dy + c\,dx$ and $g = a^2dz^2 - 2dx\,dy - 2b\,dx^2$. This, when rescaled via $g \rightarrow a^{-2}g$, $\nu \rightarrow \nu - \frac{1}{2a}\,da$, after an appropriate redefinition of a, b, c, reads

$$g = dz^2 + 2H(x, y, z)\,dx\,dy + K(x, y, z)\,dx^2, \qquad \nu = L(x, y, z)\,dx - \frac{1}{2}\frac{H_z}{H}\,dz, \qquad (7)$$

where H, K are sufficiently smooth arbitrary functions of the coordinates (x, y, z). In the above gauge the remaining connection 1-forms α, β are

$$\alpha = \left(\frac{1}{2}\frac{K_y}{H} - \frac{H_x}{H} - L\right)dx - \frac{1}{2}\frac{H_z}{H}\,dz, \qquad \beta = \frac{1}{2}\left(\frac{H_z}{H}K - K_z\right)dx + L\,dz. \qquad (8)$$

Since the subgroup G_A of $CO(2, 1)$ preserves a null direction, these Weyl structures have a covariantly constant null direction. It is generated by the vector field ∂_y.

Let us pass to the structures with holonomy B_q. Since B_q is contained in A, we can use (7)–(8) together with the condition of further reduction of holonomy. This is reduced from A to B_q iff $\alpha = -(q + 1)\nu$, which restricts the possible H, K and L by

$$(q + 2)H_z = 0, \qquad \text{and} \qquad qL = \frac{H_x}{H} - \frac{1}{2}\frac{K_y}{H}.$$

For $q = -2$ we have

$$g = dz^2 + 2H(x, y, z)\,dx\,dy + K(x, y, z)\,dx^2, \qquad \nu = \left(\frac{K_y}{4H} - \frac{H_x}{2H}\right)dx - \frac{H_z}{2H}\,dz. \qquad (9)$$

For $q \neq -2$, $H = H(x, y)$ and it may be gauged to $H = 1$ by means of the transformation $H \rightarrow Y_y(x, y)$, $K \rightarrow 2Y_x + 2K(x, y, z)$ followed by the change of coordinates $y \rightarrow Y$. In this gauge $2qL = -K_y$. Thus, for $q \neq 0, -2$ we have

$$g = dz^2 + 2dx\,dy + K(x, y, z)\,dx^2, \qquad \nu = -\frac{1}{2q}K_y dx,$$

and for $q = 0$

$$g = dz^2 + 2dx\,dy + K(x, z)\,dx^2, \qquad \nu = L(x, y, z)\,dx. \qquad (10)$$

Table 1. Three-dimensional Weyl structures with reduced holonomy.

Type	Structure	Covariantly constant object	Holonomy algebra
A	$g = dz^2 + 2H(x, y, z)\,dx\,dy + K(x, y, z)\,dx^2$ $\nu = L(x, y, z)\,dx - \frac{1}{2H}H_z dz$	Null direction of ∂_y	$\mathfrak{a}_1 \oplus \mathbb{R}$
B_0	$g = dz^2 + 2dx\,dy + K(x, z)\,dx^2$ $\nu = L(x, y, z)\,dx$	Null vector ∂_y	$\mathfrak{a}_1$
B_{-2}	$g = dz^2 + 2H(x, y, z)\,dx\,dy + K(x, y, z)\,dx^2$ $\nu = \frac{1}{4H}(K_y - 2H_x)\,dx - \frac{1}{2H}H_z dz$	Null 1-form dx, Null vector ∂_y of weight -2	$\mathfrak{a}_1$
$B_{q, q \neq 0, -2}$	$g = dz^2 + 2dx\,dy + K(x, y, z)\,dx^2$ $\nu = -\frac{1}{2q}K_y dx$	Null vector ∂_y of weight q	$\mathfrak{a}_1$ for $q \neq -1$ $\mathbb{R}^2$ for $q = -1$
C	$g = dz^2 + 2H(x, y, z)\,dx\,dy$ $\nu = -\frac{1}{2H}H_z dz$	Spatial direction of ∂_z	$\mathbb{R}^2$
D	$g = dz^2 + 2H(y, z)\,dx\,dy$ $\nu = -\frac{1}{2H}H_z dz$	Spatial direction of ∂_z, Null vector ∂_x	$\mathbb{R}$
E	$g = K(x, y, z)(dx^2 + dy^2) \pm dz^2$ $\nu = -\frac{1}{2K}K_z dz$	(Timelike) direction of ∂_z	$\mathbb{R}^2$

All the structures with holonomy B_q have a covariantly constant null vector field of weight q. In the above coordinates it is given by ∂_y. In particular, in (10) $q = 0$, thus we have a covariantly constant null vector field ∂_y there; in (9) $q = -2$ and we have also a covariantly constant null 1-form dx in this case.

We find structures with holonomy C and D for $q = 0$ in an analogous way. We show that if the holonomy is reduced to type D for $q \neq 0$ or F, then the corresponding Weyl structures are necessarily flat. In the nontrivial cases of Weyl structures with holonomies of types C and D with $q = 0$ we have a covariantly constant spatial direction. The case D with $q = 0$ admits also a covariantly constant null vector. In a similar way, we get a family of Weyl structures with holonomy of type E

$$g = K(x, y, z)(dx^2 + dy^2) - dz^2, \qquad \nu = -\frac{K_z}{2K}\,dz. \tag{11}$$

They admit a covariantly constant timelike direction generated by ∂_z. We close the discussion of the Lorentzian case by mentioning that the structures with holonomy of type G do not exist.

The Euclidean case is much simpler due to the structure of $\mathfrak{co}(3)$. It has only two proper subalgebras up to adjoint automorphisms. They constitute the counterparts of types E and G from the Lorentzian case. Structures of type G do not exist, and structures of type E have a form similar to (11), differing from it merely by the sign standing by the dz^2 term.

All the structures with reduced holonomy, together with their geometric characterization, are given in table 1. Types A–D have Lorentzian signature and type E may have both Lorentzian and Euclidean signature. In this table $\mathfrak{a}_1$ denotes the unique two-dimensional non-commutative Lie algebra.

3.1. Three-dimensional E–W structures with reduced holonomy

We calculate E–W equations in three dimensions

$$\text{Ric}_{(ij)} - \tfrac{1}{3}R \cdot g_{ij} = 0$$

for the structures in table 1. It appears, as was observed in [2], that E–W structures of types B_q for $q \neq -\frac{1}{2}$, C, D and E are flat ($\Omega = 0$). Type $B_{-1/2}$ is more interesting. Here the E–W equations reduce to the dispersionless Kadomtsev–Petviashvili (dKP) equation

$$(K K_y - 2K_x)_y = K_{zz}.$$

The structures of type A were not considered in [2]. The E–W system for them consists of four PDEs for the functions H, K, L. One of these equations is $H_{yz}H - H_z H_y = 0$ with the general solution $H = H_1(x, z)H_2(x, y)$. We absorb $H_2(x, y)$ by a redefinition $y = y(x, Y)$, $H_2 y_Y = 1$ of the y-coordinate. Hence, without loss of generality, we take $H = H(x, z)$. After the substitution $H = \exp(-F(x, z))$, $K = G(x, y, z) \exp(-F(x, z))$, two of the remaining three E–W equations read

$$L_y = G_{yy}, \qquad L_z = G_{yz} + \tfrac{1}{2}F_{xz}.$$

They can be easily solved. Now, the Weyl structure reads

$$g = \mathrm{e}^F \mathrm{d}z^2 - 2\mathrm{d}x\,\mathrm{d}y + G\mathrm{d}x^2, \qquad \nu = (G_y + f'(x))\,\mathrm{d}x.$$

It appears that this structure admits a covariantly constant null vector field $X = \exp\left(\frac{3}{4}F - \frac{1}{2}f(x)\right)\partial_y$ of weight $-1/2$; so the holonomy is of type $B_{-1/2}$. Hence, if we impose the last of the E–W equations, the structure will reduce to the one generated by the solutions of the dKP equation. Thus, type A, although more general than $B_{-1/2}$, provides no essential generalization of the dKP equation. We may summarize this section with the following two propositions.

Proposition 3.1. *Every three-dimensional Euclidean Einstein–Weyl geometry with reduced holonomy is flat.*

Proposition 3.2. *Every three-dimensional Lorentzian Einstein–Weyl geometry with reduced holonomy is flat or has a covariantly constant null vector field of weight $-\frac{1}{2}$. In the latter case E–W equations reduce to the dKP equation in some coordinate system.*

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