CONFORMAL STRUCTURES WITH EXPLICIT AMBIENT METRICS AND CONFORMAL $G_2$ HOLONOMY

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Abstract. Given a generic 2-plane field on a 5-dimensional manifold we consider its $(3, 2)$-signature conformal metric $[g]$ as defined in [1]. Every conformal class $[g]$ obtained in this way has very special conformal holonomy: it must be contained in the split-real-form of the exceptional group $G_2$. In this note we show that for special 2-plane fields on 5-manifolds the conformal classes $[g]$ have the Fefferman-Graham ambient metrics which, contrary to the general Fefferman-Graham metrics given as a formal power series [2], can be written in an explicit form. We propose to study the relations between the conformal $G_2$-holonomy of metrics $[g]$ and the possible pseudo-Riemannian $G_2$-holonomy of the corresponding ambient metrics.

1. The $(3, 2)$-signature conformal metrics

Consider an equation
\begin{equation}
\varepsilon' = F(x, y, y', y'', z) \quad \text{with} \quad F_{y'y''} \neq 0,
\end{equation}
for two real functions $y = y(x)$, $z = z(x)$ of one real variable $x$. To simplify notation introduce new symbols $p = y'$ and $q = y''$. Equation (1.1) is totally encoded in the system of three 1-forms:
\begin{align*}
\omega^1 &= dz - F(x, y, p, q, z)dx \\
\omega^2 &= dy - p dx \\
\omega^3 &= dp - q dx,
\end{align*}

living on a 5-dimensional manifold $J$ parametrized by $(x, y, p, q, z)$. In particular, every solution to (1.1) is a curve $\gamma(t) = (x(t), y(t), p(t), q(t), z(t)) \subset J$ on which all the forms $\omega^1, \omega^2, \omega^3$ identically vanish.

We introduce an equivalence relation between equations (1.1) which identifies the equations having the same set of solutions. This leads to the following definition:

Definition 1.1. Two equations $\varepsilon' = F(x, y, y', y'', z)$ and $\tilde{\varepsilon}' = \tilde{F}(\tilde{x}, \tilde{y}, \tilde{y}', \tilde{y}'', \tilde{z})$, defined on spaces $J$ and $\tilde{J}$ parametrized, respectively, by $(x, y, p = y', q = y'', z)$ and $(\tilde{x}, \tilde{y}, \tilde{p} = \tilde{y}', \tilde{q} = \tilde{y}'', \tilde{z})$, are said to be (locally) equivalent, iff there exists a (local) diffeomorphism $\phi : J \rightarrow \tilde{J}$ transforming the corresponding forms
\begin{align*}
\omega^1 &= dz - F(x, y, p, q, z)dx & \tilde{\omega}^1 &= d\tilde{z} - \tilde{F}(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q}, \tilde{z})d\tilde{x} \\
\omega^2 &= dy - p dx & \tilde{\omega}^2 &= d\tilde{y} - \tilde{p} d\tilde{x} \\
\omega^3 &= dp - q dx & \tilde{\omega}^3 &= d\tilde{p} - \tilde{q} d\tilde{x}
\end{align*}

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via:

$$\phi^*(\mathcal{L}^1) = \alpha\omega^1 + \beta\omega^2 + \gamma\omega^3$$
$$\phi^*(\mathcal{L}^2) = \delta\omega^1 + \epsilon\omega^2 + \lambda\omega^3$$

$$\phi^*(\mathcal{L}^3) = \kappa\omega^1 + \mu\omega^2 + \nu\omega^3$$

where functions $\alpha, \beta, \gamma, \delta, \epsilon, \lambda, \kappa, \nu$ on $J$ such that

$$\det\begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \lambda \\ \kappa & \mu & \nu \end{pmatrix} \neq 0.$$

It follows that equation (1.1) considered modulo equivalence relation of Definition 1.1 uniquely defines a conformal class of $(3, 2)$-signature metrics $[g_F]$ on the space $J$. In coordinates $(x, y, p, q, z)$ this class may be described as follows. Let

$$D = \partial_x + p\partial_y + q\partial_q + F\partial_z$$

be a total differential associated with equation (1.1) on $J$. Then a representative $g_F$ of the conformal class $[g_F]$ may be written as

$$g_F = | DF_{qq} F_{zz} + 6DF_q DF_{qq} F_{zz} | F_{qq} F_{zz} -$$
$$3DF_{qq} F_{zz} + 9DF_{qq} F_{qq} - 9DF_{qq} F_{zz} +$$
$$9DF_{qq} F_{zz} - 18DF_{qq} F_{zz} + 3DF_{zz} F_{zz} -$$
$$6DF_{qq} F_{qq} + 6DF_{qq} F_{zz} - 8DF_q DF_{qq} F_{qq} +$$
$$8DF_q DF_{qz} + 3DF_{qq} F_{zz} - 3DF_{qz} F_{zz} -$$
$$3DF_{qz} F_{zz} + 4DF_{qz} F_{zz} - 8DF_q F_{qz} -$$
$$3DF_q F_{qz} + 12DF_q F_{qz} + 3DF_{qq} F_{qz} -$$
$$6DF_{qq} F_{qz} + 4DF_{qz} F_{qz} + 6DF_q F_{qz} +$$

(1.3)

$$8DF_q F_{qz} - 4DF_q F_{qz} F_{qz} -$$
$$9DF_q F_{qz} + 2DF_q F_{qz} F_{qz} - 8DF_q F_{qz} F_{qz} +$$
$$8DF_q F_{qz} + 6DF_q F_{qz} F_{qz} - 6DF_q F_{qz} F_{qz} +$$
$$18DF_q F_{qz} + 6DF_q F_{qz} F_{qz} + 3DF_{qq} F_{z} -$$
$$2F_{qz} F_{qz} + 4F_{qz} F_{qz} + 4F_{qz} F_{qz} -$$
$$3F_{qz} F_{qz} - 9F_{qz} F_{qz} F_{z} | (\omega)$$

$$\{ 6DF_{qq} F_{zz} - 6DF_{qz} F_{zz} + 8DF_q F_{qz} F_{qz} -$$
$$8DF_q F_{qz} F_{qz} - 8DF_q F_{qz} F_{qz} - 6DF_q F_{qz} F_{qz} +$$
$$6DF_q F_{qz} F_{qz} - 6DF_q F_{qz} F_{qz} + 6DF_q F_{qz} F_{qz} +$$
$$2F_{qz} F_{qz} - 8DF_q F_{qz} F_{qz} + 6DF_q F_{qz} F_{qz} F_{z} | \omega_1 \omega_2 +$$

$$\{ 10DF_q F_{qz} - 10DF_q F_{qz} F_{qz} + 10DF_q F_{qz} F_{qz} -$$
$$10DF_q F_{qz} F_{qz} + 10DF_q F_{qz} F_{qz} F_{z} | \omega_1 \omega_2 +$$
\[
30F_{\psi}^4 \tilde{\omega}^1 \tilde{\omega}^4 + \left[30DF_{\psi}F_{\psi}^3 - 30F_{\psi}F_{\psi}^3 - 30F_{\psi}F_{\psi}^3 F_{\psi} \right] \tilde{\omega}^1 \tilde{\omega}^5 + \\
\left[4F_{\psi\psi}^2 - 3F_{\psi\psi\psi\psi} \right] \tilde{\omega}^2 \tilde{\omega}^3 - 10F_{\psi\psi}^2 F_{\psi\psi\psi} \tilde{\omega}^2 \tilde{\omega}^3 + 30F_{\psi} \tilde{\omega}^2 \tilde{\omega}^5 - 20F_{\psi} \tilde{\omega}^3 \tilde{\omega}^3 \tilde{\omega}^3^2
\]

where

\[
\tilde{\omega}^1 = dy - p dx \\
\tilde{\omega}^2 = dz - F dx - F_y (dp - q dx) \\
\tilde{\omega}^3 = dp - q dx \\
\tilde{\omega}^4 = dq \\
\tilde{\omega}^5 = dx.
\]

(1.4)

It follows from the construction described in Ref. [7] that when the equation (1.1) undergoes a diffeomorphism \( \phi \) of Definition [11] the above metric \( g_F \) transforms conformally.

The conformal class of metrics \([g_F]\) is very special among all the \((3, 2)\)-signature conformal metrics in dimension 5: the Cartan normal conformal connection for this class, instead of having values in full \( so(4, 3) \) Lie algebra, has values in its certain 14-dimensional subalgebra. This subalgebra turns out to be isomorphic to the split real form of the exceptional Lie algebra \( g_2 \subset so(4, 3) \). Thus, conformal metrics \([g_F]\) provide an abundance of examples of metrics with an exceptional conformal holonomy. This holonomy is always a subgroup of the noncompact form of the exceptional Lie group \( G_2 \). We strongly believe that randomly chosen function \( F \), such that \( F_{\psi} \neq 0 \), give rise to conformal metrics \([g_F]\) with conformal holonomy equal to \( G_2 \).

It is interesting to study the conformal classes \([g_F]\) from the point of view of the Fefferman-Graham ambient metric construction [2]. Since for each \( F \) defining equation (1.1) we have a conformal class of metrics \([g_F]\) in dimension five, then since five is odd, Fefferman-Graham guarantees [2] that there is a unique formal power series of a Ricci-flat metric of signature \((4, 3)\) corresponding to \([g_F]\). Moreover, since given \( F \) the metric \( g_F \) is explicitly determined by formula (1.3), we see that starting with real analytic \( F \), the metric \( g_F \) is real analytic. Thus, every analytic \( F \) of (1.1) leads to analytic \( g_F \) and then, in turn, via Fefferman-Graham, leads to a unique real analytic ambient metric \( \tilde{g}_F \) of signature \((4, 3)\). Since both the Levi-Civita connection for \( \tilde{g}_F \) and the Cartan normal conformal connection for the corresponding 5-dimensional metric \( g_F \) have values in (possibly subalgebras of) the same Lie algebra \( so(4, 3) \), it is interesting to ask about the relations between them.

We discuss these relations on examples.

2. The strategy for constructing explicit examples of ambient metrics

We start with the Fefferman-Graham result [2] adapted to the 5-dimensional situation of conformal metrics \([g_F]\).

Let \( g_F \) be a representative of the conformal class \([g_F]\) defined on \( J \) by (1.3). Consider a manifold \( J \times \mathbb{R}_+ \times \mathbb{R} \). Introduce coordinates \((0 < t, u)\) on \( \mathbb{R}_+ \times \mathbb{R} \) in

\[1\]Note that formula for \( g_F \) differs from the one given in Ref. [2] by tilde signs over all \( \omega \).

In Ref. [2], when copying the calculated metric \( g_F \), by mistake, we forgot to put these tilde signs over the \( \omega \). Hence, in Ref. [2], formula for \( g_F \) is true, provided that one puts the tilde signs over the \( \omega \) and supplements it by the definitions (1.1) of the tilded \( \omega \).
$J \times \mathbb{R}_+ \times \mathbb{R}$. We have a natural projection $\pi : J \times \mathbb{R}_+ \times \mathbb{R} \to J$, which enables us to pullback forms from $J$ to $J \times \mathbb{R}_+ \times \mathbb{R}$. Omitting the pullback sign in the expressions like $\pi^*(\varrho_F)$ we define a formal power series

$$\varrho_F = -2\partial_t^2 u + t^2 \varrho_F - ut \alpha + u^2 \beta + u^3 t^1 \gamma + \sum_{k=4}^{\infty} a_k t^k \mu_k.$$  

Here $\alpha, \beta, \gamma, \mu_k, k = 4, 5, 6, \ldots$, are pullbacks of symmetric bilinear forms $\alpha, \beta, \gamma, \mu_k$ from $J$ to $J \times \mathbb{R}_+ \times \mathbb{R}$. Thus $\varrho_F$ is a formal bilinear form on $J \times \mathbb{R}_+ \times \mathbb{R}$. This formal bilinear form has signature $(4,3)$ in some neighbourhood of $u = 0$. The following theorem is due to Fefferman and Graham \cite{3}.

**Theorem 2.1.** Among all the bilinear forms $\tilde{\varrho}_F$ which, via (2.1), are associated with metric $\varrho_F$ of (1.2) there is precisely one, say $\tilde{\varrho}_F$, satisfying the Ricci flatness condition $\text{Ric}(\tilde{\varrho}_F) = 0$.

Given $\varrho_F$, all the bilinear forms $\alpha, \beta, \gamma, \mu_k$ in $\tilde{\varrho}_F$ are totally determined. Another issue is to calculate them explicitly. For example, it is quite difficult to find the general formulas for the higher order forms $\mu_k$. Nevertheless the explicit expressions for the forms $\alpha, \beta, \gamma$ are known \cite{4} \cite{5}. We write them below in the form obtained by C R Graham. We define the coefficients $\alpha_{ij}, \beta_{ij}$ and $\gamma_{ij}$ by

$$\alpha = \alpha_{ij} dx^i dx^j, \quad \beta = \beta_{ij} dx^i dx^j, \quad \gamma = \gamma_{ij} dx^i dx^j,$n

where $(x^i) = (x, y, p, q, z)$ are coordinates on $J$. Then Graham’s expressions for $\alpha_{ij}, \beta_{ij}$ and $\gamma_{ij}$ are \cite{3}:

$$\alpha_{ij} = 2P_{ij},$$

$$\beta_{ij} = -B_{ij} + P_{ij} \mu_{kj},$$

$$\gamma_{ij} = 3\gamma_{ijkl} B_{ijkl} + 4P_{ki} B_{ij}^k - 4P_{kij} B_{j}^k + 4P_k B_{j}^k C_{(ij)kl} - 2C_{i}^{kj} C_{jkl} + C_{i}^{kl} C_{jkl} + 2P_{kij} P_{kl} P_{mnl},$$

where

$$P_{ij} = \frac{1}{3}(R_{ij} - \frac{1}{8}Rg_{Fij}),$$

is the Schouten tensor for the metric $\varrho_F = g_{Fij} dx^i dx^j$,

$$W_{ijkl} = R_{ijkl} - 2(P_{ijkl} g_{Fij} - P_{ijkl} g_{Fij})$$

is its Weyl tensor,

$$C_{ijk} = P_{ijk} - P_{ikj}$$

is the Cotton tensor, and

$$B_{ij} = C_{ij}^k - P_{ijkl} W_{ki}^l$$

is the Bach tensor.

Of course all the above quantities can be explicitly calculated once $F$, and in turn the metric $\varrho_F$, is chosen.

In the rest of the paper we will choose particular functions $F = F(x, y, p, q, z)$, and we will calculate the corresponding forms $\alpha, \beta, \gamma$ for them. We will give examples of $F$’s for which the bilinear form $\gamma$ is identically vanishing.

$$\gamma \equiv 0.$$  

Given such $F$’s we will consider

$$\tilde{\varrho}_F = -2\partial_t^2 u + t^2 \varrho_F - ut \alpha + u^2 \beta.$$
Note that $\tilde{g}_F$ coincides with the ambient metric $\hat{g}_F$ up to the terms quadratic in the ambient coordinates $t, u$. If by chance the bilinear form $\tilde{g}_F$ satisfies the Ricci flatness condition

$$Ric(\tilde{g}_F) \equiv 0,$$

then by the uniqueness of the ambient metric $\tilde{g}_F$ stated in Theorem 2.1 it will coincide with the ambient metric $\hat{g}_F$:

$$\tilde{g}_F \equiv \hat{g}_F.$$

The uniqueness result of Theorem 2.1 together with the Ricci flatness of $\tilde{g}_F$, is powerful enough to guarantee that not only the coefficient $\gamma$ in the ambient metric $\hat{g}_F$ identically vanishes, but that all the coefficients $\mu_k$, $k = 4, 5, 6, \ldots$, vanish too!

Thus the strategy of finding explicit ambient metrics $\hat{g}_F$ for $g_F$ is as follows:

- find $F = F(x, y, p, q, z)$ for which the corresponding metric $g_F$ has identically vanishing form $\gamma$ of (2.2);
- calculate the approximate ambient metric $\tilde{g}_F$ for such $F$;
- check if the Ricci tensor $Ric(\tilde{g}_F)$ of $\tilde{g}_F$ is identically vanishing;
- if you have $F$ with the above properties then the approximate metric $\tilde{g}_F$ is the ambient metric $\hat{g}_F$ for $g_F$.

3. Conformally Einstein Example

As the first example, following Ref. [7], we calculate $g_F$ and its approximate ambient metric $\tilde{g}_F$ for a very simple equation:

$$z' = F(y''), \quad \text{with} \quad F_{y'y''} \neq 0.$$  

It was shown in Ref. [7] that the conformal class $[g_F]$ may be represented by

- $-15(F''')^{10/3} g_F =
- 30(F''')^4 \left[ dqdy \mu dxdz \right] + \left[ 4(F')^2 - 3(F'')^2 \right] dz^2 +
- 2 [-5(F''')^2 F^{(3)} - 4F'F''F^{(3)} + 3F''F''^2 F^{(4)}] dxdz +
- 2 [15(F''')^3 + 5q(F''')^2 F^{(3)} - 4F'F''F^{(3)} + 4qF''F''^2 F^{(4)} + 3F''F''^2 F^{(4)} -
- 3qF'F''F^{(4)}] dx dz +
\nonumber
- 3q(F''')^3 F^{(4)} dx dz +
\nonumber
- [20(F'')^4 + 10F'^2 F'' F^{(3)} + 4F'^2 F''^2 F^{(3)} - 3(F')^2 F'' F^{(4)}] dp^2 +
- 2 [-15F'F''(F''')^3 + 20qF''F''^2 F^{(3)} - 10qF''(F''')^2 F^{(3)} +
- 4FF''F'' F^{(3)} - 4qF''^2 F^{(3)} + 3FF''F'' F^{(4)} + 3q(F')^2 F'' F^{(4)} -
\nonumber
- 30qF''(F''')^3 + 30qF''(F''')^2 - 20q^2 (F''')^4 -
- 10qF(F''')^2 F^{(3)} + 10q^2 F'(F''')^2 F^{(3)} + 4F''F'' F^{(3)} +
- 8qFF''F'' F^{(3)} + 4q^2 (F')^2 F^{(3)} - 3FF'' F^{(4)} +
- 6qFF'' F^{(4)} - 3q^2 (F')^2 F'' F^{(4)}] dx^2.

(3.1)

As noted in Ref. [7] this metric is conformal to a Ricci flat metric $\hat{g}_F = e^{2\Upsilon} g_F$ with a conformal scale $\Upsilon = \Upsilon(q)$ satisfying second order ODE:

$$90F''(\Upsilon' - \Upsilon'^2) - 60F'' F^{(3)} \Upsilon' + 3FF'' F^{(4)} - 4(F'')^2 = 0.$$
Thus, since for each $F = F(q)$ the conformal class $[g_F]$ contains a Ricci flat metric, its conformal holonomy must be a proper subgroup of the noncompact form of $G_2$. An interesting feature of this conformal class is that it is very special among all the conformal classes associated with equation (1.1). Not only has $g_F$ very special conformal holonomy, making it very similar to the Lorentzian 4-dimensional Brinkman metrics; moreover, since its Weyl tensor has essentially only one nonvanishing component (see Ref. [2] for details) it is not weakly generic (see Ref. [3] for definition). This makes $[g_F]$ analogous to the Lorentzian type $N$ metrics in 4-dimensions, such as for example, Fefferman metrics.

Having $g_F$ of (3.1) we used the symbolic computer calculation program Mathematica to calculate its associated form $\gamma$ of (2.2). We checked that this form identically vanishes. We further used Mathematica to calculate the corresponding approximate ambient metric $\tilde{g}_F$. On doing so we observed that, surprisingly, the bilinear form $\beta$ is also identically vanishing. The explicit formula for the approximate ambient metric is given below:

\begin{equation}
\tilde{g}_F = t^2 g_F - 2 dt du - 2tu F''F/3 P d\sigma^2,
\end{equation}

with

$$P = \frac{4F^{(3)|2} - 3 F''F^{(4)}}{90(F'')^{10/3}},$$

and $g_F$ given by (3.1). The metric $\tilde{g}_F$ is defined locally on $J \times \mathbb{R}_+ \times \mathbb{R}$ with coordinates $(x,y,p,q,z,t,u)$. It obviously has signature $(4,3)$. We also checked, again using Mathematica, that $Ric(\tilde{g}_F) = 0$. Thus, we fulfilled the strategy outlined in Section 2. This enables us to conclude that $\tilde{g}_F$ of (3.2) coincides with the ambient metric $\tilde{g}_F$ for $g_F$. To give expressions for the Cartan normal conformal connection for $g_F$ and the Levi-Civita connection for $g_F = \tilde{g}_F$ we first introduce a nonholonomic coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ on $J$ given by

$\theta^1 = dy - px \, dx$
$\theta^2 = dz - Fdx - F'(dy - q dx)$
$\theta^3 = \frac{2}{\sqrt{3}}(F'')^{1/3}(dp - q dx)$
$30(F'')^{10/3} \theta^4 = (3F''F''F^{(4)} - 4F'F^{(3)|2} - 10(F'')^2F^{(3)})(dp - q dx) + (4F^{(3)|2} - 3F''F^{(4)}) (dz - Fdx) + 30(F'')^3 dx$
$\theta^5 = -(F'')^{-1/3} d\sigma$.

In this coframe the metric $g_F$ is simply:

$$g_F = 2\theta^1 \theta^5 - 2\theta^2 \theta^4 + (\theta^3)^2.$$

By means of the canonical projection

$$\pi(x,y,p,q,z,t,u) = (x,y,p,q,z)$$

the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ can be pulled back to five linearly independent forms $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ on $J \times \mathbb{R}_+ \times \mathbb{R}$. They can be supplemented by

$$\theta^0 = dt \quad \text{and} \quad \theta^6 = du$$

to form a coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6)$ on the ambient space $J \times \mathbb{R}_+ \times \mathbb{R}$.
The Cartan normal conformal connection, when written on $J$ in the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ reads:

$$\omega_{\mathcal{G}_2} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -P\theta^5 & 0 \\
\frac{3}{\sqrt{3}}\theta^1 & 0 & \frac{1}{\sqrt{3}}\theta^1 & -\frac{1}{\sqrt{3}}\theta^3 & 0 & -P\theta^5 \\
0 & 0 & 0 & \frac{1}{\sqrt{3}}\theta^5 & 0 & -\frac{1}{\sqrt{3}}\theta^3 & 0 \\
\frac{1}{\sqrt{3}}\theta^3 & 0 & -2\sqrt{3}P\theta^5 & 0 & \frac{1}{\sqrt{3}}\theta^5 & -\frac{1}{\sqrt{3}}\theta^4 & 0 \\
0 & 0 & 0 & -2\sqrt{3}P\theta^5 & 0 & Q\theta^2 + \frac{9}{2}\sqrt{3}P\theta^3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \theta^5 & -\theta^4 & \theta^3 & -\theta^2 & \theta^1 & 0
\end{pmatrix}$$

Here:

$$Q = \frac{40F^{(3)}F^{(4)}F^{(5)} - 45F^{(3)}F^{(4)}F^{(5)} + 9F^{(2)}F^{(5)}}{90F^{(5)}}.$$ 

Now we use coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ to write down the Levi-Civita connection for $\tilde{g}_F$. We have

$$\tilde{g}_F = g_{ij}\tilde{\theta}^i\tilde{\theta}^j,$$

with the indices range: $i, j = 0, 1, 2, ... 6$, and the matrix $g_{ij}$ given by

$$g_{ij} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & t^2 & 0 \\
0 & 0 & 0 & -t^2 & 0 & 0 & 0 \\
0 & 0 & t^2 & 0 & 0 & 0 & 0 \\
0 & -t^2 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

The Levi-Civita connection for $\tilde{g}_F$ on $J \times \mathbb{R}_+ \times \mathbb{R}$, when written in the coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6)$ reads:

$$\omega_{\text{LC}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -tP\theta^5 & 0 \\
\frac{1}{\sqrt{3}}\theta^1 + \frac{u}{Pp^5} & \frac{1}{\sqrt{3}}\theta^0 & \frac{9}{2}\sqrt{3}P\theta^3 & \frac{1}{\sqrt{3}}\theta^1 & -\frac{1}{\sqrt{3}}\theta^3 & 0 & -\frac{1}{\sqrt{3}}\theta^3 \\
\frac{1}{\sqrt{3}}\theta^2 & 0 & \frac{1}{\sqrt{3}}\theta^2 & -\frac{1}{\sqrt{3}}\theta^3 & 0 & -\frac{1}{\sqrt{3}}\theta^3 & 0 \\
\frac{1}{\sqrt{3}}\theta^3 & 0 & -2\sqrt{3}P\theta^5 & \frac{1}{\sqrt{3}}\theta^3 & 0 & -\frac{1}{\sqrt{3}}\theta^4 & 0 \\
\frac{1}{\sqrt{3}}\theta^4 & 0 & 0 & -2\sqrt{3}P\theta^5 & \frac{1}{\sqrt{3}}\theta^4 & Q\theta^2 + \frac{9}{2}\sqrt{3}P\theta^3 & 0 \\
\frac{1}{\sqrt{3}}\theta^5 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}}\theta^4 & 0 \\
0 & t\theta^5 & -t\theta^4 & t\theta^3 & -t\theta^2 & t\theta^1 - uP\theta^5 & 0
\end{pmatrix}.$$
Note that on $\Sigma = \{(x, y, p, q, z, t, u) : u = 0, t = 1\}$ we trivially have $\theta^0 \equiv 0 \equiv \theta^1$. Thus, restricting the formula for $\omega_{LC}$ to $\Sigma$, we see that $\omega_{G_2} = \omega_{LC}|_{\Sigma}$. Off this set the two connections: $\omega_{LC}$ and the pullbacked-by-$\pi$-connection $\omega_{G_2}$, differ significantly. To see this it is enough to observe that contrary to $\omega_{LC}$, the connection $\pi^*(\omega_{G_2})$ has torsion. Indeed writing the first Cartan structure equations for the $\pi^*(\omega_{G_2})$ in the coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ we find that the torsion is:

$$\begin{pmatrix}
0 \\
-\theta^2 \wedge \theta^1 - P\theta^5 \wedge \theta^6 \\
-\theta^0 \wedge \theta^2 \\
-\theta^0 \wedge \theta^3 \\
-\theta^0 \wedge \theta^4 \\
-\theta^0 \wedge \theta^5 \\
0
\end{pmatrix}.$$

The vanishing of this torsion on the initial hypersurface $\Sigma$ confirms our earlier statement that the two connections $\omega_{G_2}$ and $\omega_{LC}$ coincide there.

It is interesting to note that the curvature $\omega_{LC} + \omega_{LC} \wedge \omega_{LC}$ does not depend on $t, u$ and is annihilated by $\partial_t$ and $\partial_u$. Thus it can be considered to be a 2-form on $\Sigma$. As such it is precisely equal to the curvature $\omega_{G_2} + \omega_{G_2} \wedge \omega_{G_2}$ of the connection $\omega_{G_2}$:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \theta^2 \wedge \theta^5,$$

where $A_5 = \frac{-224 F^{(3)} F^{(4)} - 336 F^{(2)} F^{(4)} - 51 F^{(2)} F^{(4)} - 80 F^{(2)} F^{(5)} + 10 F^{(6)} F^{(5)}}{100 F^{(2)} F^{(3)}}$.

4. **Non-conformally Einstein example**

To get quite different example of $[g_F]$ we consider equation (1.1) in the form:

$$z' = y'^2 + a_0 y^6 + a_5 y^5 + a_4 y^4 + a_3 y^3 + a_2 y^2 + a_1 y + a_0 + b z,$$

where $a_i, i = 0, 1, \ldots, 6$, and $b$ are real constants. This equation has the defining function

$$F = q^2 + a_0 p^6 + a_5 p^5 + a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0 + b z$$

and, via (1.3), leads to a conformal class $[g_F]$ represented by a metric

$$15(2) \frac{2}{2} g_F = \left[ 9 a_2 + 2 b^2 + 27 a_3 + 54 a_4 p^2 + 90 a_5 p^3 + 135 a_6 p^4 \right] dy^2 +$$

$$+ \left[ 15 a_0 + 2 (b^2 - 3 a_2) p^2 - 3 a_3 p^3 + 9 a_4 p^4 + 30 a_5 p^5 + 60 a_6 p^6 \right] -$$

$$- \left[ 20 b p q + 5 q^2 + 15 b z \right] dx^2 +$$

$$+ \left[ 15 a_1 + 4 (3 a_2 - b^2) p - 9 a_3 p^2 - 48 a_4 p^3 - 105 a_5 p^4 - 180 a_6 p^5 +$$

$3$We use the letter $A_5$ to denote the nonvanishing component of the curvature to be in accordance with [2] and Cartan’s paper [3]. Note however that in order to avoid collision of notations between the present and the next sections we use capital $A_5$ instead of $a_5$ of paper [2].
\[20bq|dx|dy + 20dy^2 - \\
10(bp + q)dpdx + 10bdydy - 30dxdy - 15dxdz + 30pydqdx.\]

This metric is not conformal to an Einstein metric. The quickest way to check this is the calculation of the Cotton, \(C_{ijk}\), and the Weyl, \(W_{ijkl}\), tensors for \(g_F\). Once these tensors are calculated, it is easy to observe that they do not admit a vector field \(K^i\) such that \(C_{ijk} + K^lW_{ijkl} = 0\). As a consequence the metric is not a conformal C-space metric. This proves our statement since every conformally Einstein metric is necessarily a conformal C-space metric (see e.g. Ref. [2]).

Recall that \(g_F\) of (1.1), as a member of the family of metrics (1.3), defines a conformal class \([g_F]\) with conformal holonomy \(H\) reduced to the noncompact group \(G_2\) or to one of its subgroups. But since the metric (1.1) is not conformal to an Einstein metric, we do not have an immediate reason to conclude that \(H \neq G_2\).

We conjecture that \(H = G_2\) here and try to prove it in a subsequent paper [3].

It is remarkable that the ambient metric \(\hat{g}_F\) for \(g_F\) of (1.1) assumes a very compact form:

\[\hat{g}_F = t^2 g_F - 2 \, dt du - \\
2 \, tu \left[ \frac{1}{2} (-2a_2 + 4b^2 + 3a_3p + 6a_4p^2 - 20a_5p^3 - 120a_6p^4)dx^2 - \\
\frac{9}{2}(a_3 - 10a_5p^3 - 40a_6p^3)dy^2 - \frac{9}{2}(a_4 + 5a_5p + 15a_5p^2)d\theta^2 \right] + \\
u^2 \left[ \frac{3}{2v^2 \gamma^2} (a_4 - 10a_5p + 60a_6p^2)dx^2 + \frac{9}{4\gamma^2} (a_5 - 12a_6p)dy^2 + \frac{81}{4\gamma^2} a_0 dy^2 \right].\]

This is checked by applying our strategy described in Section 2 to the metric (1.1). As in the previous example, using Mathematica, we calculated the bilinear form \(\gamma\) for (1.1). It turned out to be equal to zero, \(\gamma = 0\). Then we calculated \(\hat{g}_F\), and checked that it is Ricci flat. Thus we concluded that \(\hat{g}_F\) coincides with the ambient metric for \(\hat{g}_F\). The above given formula for \(\hat{g}_F\) is therefore just \(\hat{g}_F\), which we calculated using [2].

We find this example as a sort of miracle. Apriori there is no reason for \(g_F\) to have the ambient metric truncated at the second order in terms of the ambient parameters \(t\) and \(u\). We are intrigued by this fact.

Now, following the general procedure outlined in [2], we introduce a special coframe for \(g_F\) given by:

\[\theta^1 = dy - pdx \]
\[\theta^2 = dz - Fdx - 2q(dp - qdx) \]
\[\theta^3 = -\frac{2v^2 \gamma^2}{\gamma^3}(dp - qdx) \]
\[\theta^4 = 2 \frac{1}{v^2 \gamma^2}dx \]
\[15(2)^{1/3} g^5 = (9a_2 + 2b^2 + 27a_3p + 54a_4p^2 + 135a_5p^3 + 90a_6p^4)(dy - pdx) + \\
10b(dp - qdx) - 30dq + \\
15(a_1 + 2a_2p + 3a_3p^2 + 4a_4p^3 + 5a_5p^4 + 6a_6p^5 + 2bq)d\theta.\]

In this coframe the metric \(g_F\) is:

\[g_F = 2\theta^1 g^5 - 2\theta^2 \theta^4 + (\theta^3)^2.\]

As in the previous section, we use the canonical projection

\[\pi(x, y, p, q, z, t, u) = (x, y, p, q, z)\]
to pullback the coframe \((\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)\) to five linearly independent forms \((\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)\)
on \(J \times \mathbb{R}_+ \times \mathbb{R}\), which are further supplemented by
\[
\theta^0 = dt \quad \text{and} \quad \theta^i = du
\]
to form a coframe \((\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5)\) on the ambient space \(J \times \mathbb{R}_+ \times \mathbb{R}\).

It turns out that if \(b = 0\) the coframes on \(J\) and \(J \times \mathbb{R}_+ \times \mathbb{R}\) defined in this
way are suitable to analyze the relations between the Cartan normal conformal connection \(\omega_{G_2}\) for \([g_R]\) and the Levi-Civita connection \(\omega_{LC}\) for \(g_R\). If \(b \neq 0\)
the connection \(\omega_{G_2}\) in the coframe \((\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)\) and the connection \(\omega_{LC}\) in the
coframe \((\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5)\) do not coincide on \(t = 1, u = 0\). We will not
analyze this case here.

Restricting to the
\[b = 0\]
case we find the following:

- the connections \(\omega_{G_2}\) in the coframe \((\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)\) and the connection \(\omega_{LC}\) in the coframe \((\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5)\) coincide on \(t = 1, u = 0\).
- the torsion of \(\pi^*(\omega_{G_2})\) in the coframe \((\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5)\) is nonvanishing off the set \(t = 1, u = 0\).
- unlike the example of the previous section the curvature \(d\omega_{LC} + \omega_{LC} \wedge \omega_{LC}\) significantly depends on \(t\) and \(u\).
- even on \(t = 1, u = 0\), the curvature \(d\omega_{G_2} + \omega_{G_2} \wedge \omega_{G_2}\) and the restriction of \(d\omega_{LC} + \omega_{LC} \wedge \omega_{LC}\) do not coincide.

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REFERENCES


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