CONFORMAL STRUCTURES WITH EXPLICIT AMBIENT METRICS AND CONFORMAL G_2 HOLONOMY

PAWEŁ NUROWSKI

Abstract. Given a generic 2-plane field on a 5-dimensional manifold we consider its $(3,2)$-signature conformal metric $[g]$ as defined in [7]. Every conformal class $[g]$ obtained in this way has very special conformal holonomy: it must be contained in the split-real-form of the exceptional group G_2. In this note we show that for special 2-plane fields on 5-manifolds the conformal classes $[g]$ have the Fefferman-Graham ambient metrics which, contrary to the general Fefferman-Graham metrics given as a formal power series [2], can be written in an explicit form. We propose to study the relations between the conformal G_2-holonomy of metrics $[g]$ and the possible pseudo-Riemannian G_2-holonomy of the corresponding ambient metrics.

1. The $(3,2)$-signature conformal metrics

Consider an equation

$$z' = F(x,y,y',y'',z) \qquad \text{with} \qquad F_{y''y''} \neq 0, \tag{1.1}$$

for two real functions $y = y(x)$, $z = z(x)$ of one real variable x. To simplify notation introduce new symbols $p = y'$ and $q = y''$. Equation (1.1) is totally encoded in the system of three 1-forms:

$$\begin{aligned} \omega^1 &= \mathrm{d}z - F(x,y,p,q,z)\mathrm{d}x \\ \omega^2 &= \mathrm{d}y - p\mathrm{d}x \\ \omega^3 &= \mathrm{d}p - q\mathrm{d}x, \end{aligned} \tag{1.2}$$

living on a 5-dimensional manifold J parametrized by (x,y,p,q,z). In particular, every solution to (1.1) is a curve $\gamma(t) = (x(t),y(t),p(t),q(t),z(t)) \subset J$ on which all the forms $\omega^1,\omega^2,\omega^3$ identically vanish.

We introduce an equivalence relation between equations (1.1) which identifies the equations having the same set of solutions. This leads to the following definition:

Definition 1.1. Two equations $z' = F(x,y,y',y'',z)$ and $\bar{z}' = \bar{F}(\bar{x},\bar{y},\bar{y}',\bar{y}'',\bar{z})$, defined on spaces J and $\bar{J}$ parametrized, respectively, by $(x,y,p=y',q=y'',z)$ and $(\bar{x},\bar{y},\bar{p}=\bar{y}',\bar{q}=\bar{y}'',\bar{z})$, are said to be *(locally) equivalent*, iff there exists a (local) diffeomorphism $\phi : J \to \bar{J}$ transforming the corresponding forms

$$\begin{aligned} \omega^1 &= \mathrm{d}z - F(x,y,p,q,z)\mathrm{d}x \\ \omega^2 &= \mathrm{d}y - p\mathrm{d}x \\ \omega^3 &= \mathrm{d}p - q\mathrm{d}x \end{aligned} \qquad \text{and} \qquad \begin{aligned} \bar{\omega}^1 &= \mathrm{d}\bar{z} - \bar{F}(\bar{x},\bar{y},\bar{p},\bar{q},\bar{z})\mathrm{d}\bar{x} \\ \bar{\omega}^2 &= \mathrm{d}\bar{y} - \bar{p}\mathrm{d}\bar{x} \\ \bar{\omega}^3 &= \mathrm{d}\bar{p} - \bar{q}\mathrm{d}\bar{x} \end{aligned}$$

Date: January 31, 2007.

This work was supported in part by the Polish Ministerstwo Nauki i Informatyzacji grant nr: 1 P03B 07529 and the US Institute for Mathematics and Its Applications in Minneapolis.

via:

$$\begin{array}{rcl}\phi^*(\bar\omega^1) &=& \alpha\omega^1+\beta\omega^2+\gamma\omega^3\\ \phi^*(\bar\omega^2) &=& \delta\omega^1+\epsilon\omega^2+\lambda\omega^3,\\ \phi^*(\bar\omega^3) &=& \kappa\omega^1+\mu\omega^2+\nu\omega^3\end{array}\quad \text{with functions } \alpha,\beta,\gamma,\delta,\epsilon,\lambda,\kappa,\nu \text{ on } J \text{ such that}$$

$$\det\begin{pmatrix}\alpha & \beta & \gamma\\ \delta & \epsilon & \lambda\\ \kappa & \mu & \nu\end{pmatrix}\neq 0.$$

It follows that equation (1.1) considered modulo equivalence relation of Definition 1.1 uniquely defines a conformal class of $(3,2)$-signature metrics $[g_F]$ on the space J. In coordinates (x,y,p,q,z) this class may be described as follows. Let

$$D=\partial_x+p\partial_y+q\partial_p+F\partial_z$$

be a total differential associated with equation (1.1) on J. Then a representative g_F of the conformal class $[g_F]$ may be written as

$$\begin{aligned}
&g_F=[\ DF_{qq}^2F_{qq}^2+6DF_qDF_{qqq}F_{qq}^2-6DF_{qqq}F_pF_{qq}^2-\\
&3DDF_{qq}F_{qq}^3+9DF_{qp}F_{qq}^3-9F_{pp}F_{qq}^3+\\
&9DF_{qz}F_qF_{qq}^3-18F_{pz}F_qF_{qq}^3+3DF_zF_{qq}^4-\\
&6DF_qF_{qq}^2F_{qqp}+6F_pF_{qq}^2F_{qqp}-8DF_qDF_{qq}F_{qq}F_{qqq}+\\
&8DF_{qq}F_pF_{qq}F_{qqq}+3DDF_qF_{qq}^2F_{qqq}-3DF_pF_{qq}^2F_{qqq}-\\
&3DF_zF_qF_{qq}^2F_{qqq}+4(DF_q)^2F_{qqq}^2-8DF_qF_pF_{qqq}^2-\\
&3(DF_q)^2F_{qq}F_{qqqq}+4F_p^2F_{qqq}^2+6DF_qF_pF_{qq}F_{qqqq}-\\
&3F_p^2F_{qq}F_{qqqq}-6DF_qF_qF_{qq}^2F_{qqz}+6F_pF_qF_{qq}^2F_{qqz}-\\
&3DF_qF_{qq}^3F_{qz}+12F_pF_{qq}^3F_{qz}+3F_{qq}^2F_{qqq}F_y-\\
&6DF_{qqq}F_qF_{qq}^2F_z+4DF_{qq}F_{qq}^3F_z+6F_qF_{qq}^2F_{qqp}F_z+\\
&8DF_{qq}F_qF_{qq}F_{qqq}F_z-4DF_qF_{qq}^2F_{qqq}F_z- \qquad (1.3)\\
&9F_{qp}F_{qq}^3F_z+F_pF_{qq}^2F_{qqq}F_z-8DF_qF_qF_{qqq}^2F_z+\\
&8F_pF_qF_{qqq}^2F_z+6DF_qF_qF_{qq}F_{qqqq}F_z-6F_pF_qF_{qq}F_{qqqq}F_z+\\
&18F_{qq}^3F_{qy}+6F_q^2F_{qq}^2F_{qqz}F_z+3F_qF_{qq}^3F_{qz}F_z-\\
&2F_{qq}^4F_z^2+F_qF_{qq}^2F_{qqq}F_z^2+4F_q^2F_{qqq}^2F_z^2-\\
&3F_q^2F_{qq}F_{qqqq}F_z^2-9F_q^2F_{qq}^3F_{zz}\ ]\ (\tilde\omega^1)^2+\\
\\
&[\ 6DF_{qqq}F_{qq}^2-6F_{qq}^2F_{qqp}-8DF_{qq}F_{qq}F_{qqq}+\\
&8DF_qF_{qqq}^2-8F_pF_{qqq}^2-6DF_qF_{qq}F_{qqqq}+\\
&6F_pF_{qq}F_{qqqq}-6F_qF_{qq}^2F_{qqz}+6F_{qq}^3F_{qz}+\\
&2F_{qq}^2F_{qqq}F_z-8F_qF_{qqq}^2F_z+6F_qF_{qq}F_{qqqq}F_z\ ]\ \tilde\omega^1\tilde\omega^2+\\
\\
&[\ 10DF_{qq}F_{qq}^3-10DF_qF_{qq}^2F_{qqq}+10F_pF_{qq}^2F_{qqq}-\\
&10F_{qq}^4F_z+10F_qF_{qq}^2F_{qqq}F_z\ ]\ \tilde\omega^1\tilde\omega^3+
\end{aligned}$$

$$30F_{qq}^4\ \tilde\omega^1\tilde\omega^4+[\ 30DF_qF_{qq}^3-30F_pF_{qq}^3-30F_qF_{qq}^3F_z\ ]\ \tilde\omega^1\tilde\omega^5+$$
$$[\ 4F_{qqq}^2-3F_{qq}F_{qqqq}\ ]\ (\tilde\omega^2)^2-10F_{qq}^2F_{qqq}\ \tilde\omega^2\tilde\omega^3+30F_{qq}^3\ \tilde\omega^2\tilde\omega^5-20F_{qq}^4\ (\tilde\omega^3)^2$$

where[1]

$$\begin{aligned}\tilde\omega^1&=\mathrm{d}y-p\mathrm{d}x\\ \tilde\omega^2&=\mathrm{d}z-F\mathrm{d}x-F_q(\mathrm{d}p-q\mathrm{d}x)\\ \tilde\omega^3&=\mathrm{d}p-q\mathrm{d}x\\ \tilde\omega^4&=\mathrm{d}q\\ \tilde\omega^5&=\mathrm{d}x.\end{aligned}\tag{1.4}$$

It follows from the construction described in Ref. [7] that when the equation (1.1) undergoes a diffeomorphism ϕ of Definition 1.1, the above metric g_F transforms conformally.

The conformal class of metrics $[g_F]$ is very special among all the $(3,2)$-signature conformal metrics in dimension 5: the Cartan normal conformal connection for this class, instead of having values in full $\mathfrak{so}(4,3)$ Lie algebra, has values in its certain 14-dimensional subalgebra. This subalgebra turns out to be isomorphic to the split real form of the exceptional Lie algebra $\mathfrak{g}_2\subset\mathfrak{so}(4,3)$. Thus, conformal metrics $[g_F]$ provide an abundance of examples of metrics with an *exceptional* conformal *holonomy.* This holonomy is always a subgroup of the noncompact form of the exceptional Lie group G_2. We strongly believe that randomly chosen function F, such that $F_{qq}\neq 0$, give rise to conformal metrics $[g_F]$ with conformal holonomy *equal* to G_2.

It is interesting to study the conformal classes $[g_F]$ from the point of view of the Fefferman-Graham ambient metric construction [2]. Since for each F defining equation (1.1) we have a conformal class of metrics $[g_F]$ in dimension five, then since five is *odd*, Fefferman-Graham guarantees [2] that there is a *unique* formal power series of a *Ricci-flat metric* of signature $(4,3)$ corresponding to $[g_F]$. Moreover, since given F the metric g_F is explicitely determined by formula (1.3), we see that starting with *real analytic* F, the metric g_F is *real analytic*. Thus, every analytic F of (1.1) leads to analytic g_F and then, in turn, via Fefferman-Graham, leads to a unique *real analytic* ambient metric $\tilde g_F$ of signature $(4,3)$. Since both the Levi-Civita connection for $\tilde g_F$ and the Cartan normal conformal connection for the corresponding 5-dimensional metric g_F have values in (possibly subalgebras of) the same Lie algebra $\mathfrak{so}(4,3)$, it is interesting to ask about the relations between them. We discuss these relations on examples.

2. The strategy for constructing explicit examples of ambient metrics

We start with the Fefferman-Graham result [2] adapted to the 5-dimensional situation of conformal metrics $[g_F]$.

Let g_F be a representative of the conformal class $[g_F]$ defined on J by (1.3). Consider a manifold $J\times\mathbb{R}_+\times\mathbb{R}$. Introduce coordinates $(0<t,u)$ on $\mathbb{R}_+\times\mathbb{R}$ in

[1]Note that formula for g_F differs from the one given in Ref. [7] by tilde signs over the all omegas. In Ref. [7], when copying the calculated metric g_F, by mistake, we forgot to put these tilde signs over the omegas. Hence, in Ref. [7], formula for g_F is true, provided that one puts the tilde signs over the omegas and supplements it by the definitions (1.4) of the tilded omegas.

$J\times\mathbb{R}_+\times\mathbb{R}$. We have a natural projection $\pi: J\times\mathbb{R}_+\times\mathbb{R}\to J$, which enables us to pullback forms from J to $J\times\mathbb{R}_+\times\mathbb{R}$. Ommiting the pulback sign in the expressions like $\pi^*(g_F)$ we define a formal power series

$$\check{g}_F=-2\mathrm{d}t\mathrm{d}u+t^2g_F-ut\alpha+u^2\beta+u^3t^{-1}\gamma+\sum_{k=4}^{\infty}u^kt^{2-k}\mu_k. \tag{2.1}$$

Here $\alpha,\beta,\gamma,\mu_k$, $k=4,5,6,....$, are pullbacks of symmetric bilinear forms $\alpha,\beta,\gamma,\mu_k$ from J to $J\times\mathbb{R}_+\times\mathbb{R}$. Thus $\check{g}_F$ is a formal *bilinear form* on $J\times\mathbb{R}_+\times\mathbb{R}$. This formal bilinear form has signature $(4,3)$ in some neighbourhood of $u=0$. The following theorem is due to Fefferman and Graham [2].

Theorem 2.1. *Among all the bilinear forms $\check{g}_F$ which, via (2.1), are associated with metric g_F of (1.3) there is* precisely one*, say $\tilde{g}_F$, satisfying the Ricci flatness condition $Ric(\tilde{g}_F)=0$.*

Given g_F, all the bilinear forms $\alpha,\beta,\gamma,\mu_k$ in $\tilde{g}_F$ are totally determined. Another issue is to calculate them explicitely. For example, it is quite difficult to find the general formulas for the higher order forms μ_k. Nevertherless the explicit expressions for the forms α,β,γ are known [4, 5]. We write them below in the form obtained by C R Graham. We define the coefficients α_{ij}, β_{ij} and γ_{ij} by $\alpha=\alpha_{ij}\mathrm{d}x^i\mathrm{d}x^j$, $\beta=\beta_{ij}\mathrm{d}x^i\mathrm{d}x^j$, $\gamma=\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$, where $(x^i)=(x,y,p,q,z)$ are coordinates on J. Then Graham's expressions for α_{ij}, β_{ij} and γ_{ij} are [4]:

$$\begin{aligned}
&\alpha_{ij}=2\mathsf{P}_{ij},\\
&\beta_{ij}=-B_{ij}+\mathsf{P}_i{}^k\mathsf{P}_{jk},\\
&3\gamma_{ij}=B_{ij;k}{}^k-2W_{kijl}B^{kl}+4\mathsf{P}_{k(i}B_{j)}{}^k-4\mathsf{P}_k{}^kB_{ij}+4\mathsf{P}^{kl}C_{(ij)k;l}-\\
&2C^k{}_i{}^lC_{ljk}+C_i{}^{kl}C_{jkl}+2\mathsf{P}^k{}_{k;l}C_{(ij)}{}^l-2W_{kijl}\mathsf{P}^k{}_m\mathsf{P}^{ml},
\end{aligned}\tag{2.2}$$

where

$$\mathsf{P}_{ij}=\tfrac{1}{3}(R_{ij}-\tfrac{1}{8}Rg_{Fij}),$$

is the Schouten tensor for the metric $g_F=g_{Fij}\mathrm{d}x^i\mathrm{d}x^j$,

$$W_{ijkl}=R_{ijkl}-2(\mathsf{P}_{i[k}g_{Fl]j}-\mathsf{P}_{j[k}g_{Fl]i})$$

is its Weyl tensor,

$$C_{ijk}=\mathsf{P}_{ij;k}-\mathsf{P}_{ik;j}$$

is the Cotton tensor, and

$$B_{ij}=C_{ijk;}{}^k-\mathsf{P}^{kl}W_{kijl}$$

is the Bach tensor.

Of course all the above quantities can be explicitely calculated once F, and in turn the metric g_F, is chosen.

In the rest of the paper we will chose particular functions $F=F(x,y,p,q,z)$, and we will calculate the corresponding forms α,β,γ for them. We will give examples of F's for which the bilinear form γ is identically vanishing,

$$\gamma\equiv 0. \tag{2.3}$$

Given such F's we will consider

$$\bar{g}_F=-2\mathrm{d}t\mathrm{d}u+t^2g_F-ut\alpha+u^2\beta.$$

Note that $\bar{g}_F$ coincides with the ambient metric $\tilde{g}_F$ up to the terms *quadratic* in the ambient coordinates t, u. If by *chance* the bilinear form $\bar{g}_F$ satisfies the Ricci flatness condition

$$Ric(\bar{g}_F) \equiv 0,$$

then by the *uniqueness* of the ambient metric $\tilde{g}_F$ stated in Theorem 2.1, it will *coicide* with the ambient metric $\tilde{g}_F$:

$$\bar{g}_F \equiv \tilde{g}_F.$$

The uniqueness result of Theorem 2.1, together with the Ricci flatness of $\bar{g}_F$, is powerfull enough to guarantee that not only the coefficient γ in the ambient metric $\tilde{g}_F$ identically vanishes, but that *all* the coefficients μ_k, $k = 4, 5, 6, \ldots.$, vanish too!

Thus the strategy of finding explicit ambient metrics $\tilde{g}_F$ for g_F is as follows:

- find $F = F(x, y, p, q, z)$ for which the corresponding metric g_F has identically vanishing form γ of (2.2);
- calculate the approximate ambient metric $\bar{g}_F$ for such F;
- check if the Ricci tensor $Ric(\bar{g}_F)$ of $\bar{g}_F$ is identically vanishing;
- if you have F with the above properties then the approximate metric $\bar{g}_F$ is the ambient metric $\tilde{g}_F$ for g_F.

3. Conformally Einstein example

As the first example, following Ref. [7], we calculate g_F and its approximate ambient metric $\bar{g}_F$ for a very simple equation:

$$z' = F(y''), \qquad \text{with} \qquad F_{y''y''} \neq 0.$$

It was shown in Ref. [7] that the conformal class $[g_F]$ may be represented by[2]

$$\begin{aligned}
&-15(F'')^{10/3} g_F = \\
&30(F'')^4 \,[\, \mathrm{d}q\mathrm{d}y - p\mathrm{d}q\mathrm{d}x \,] \;+\; [\, 4F^{(3)2} - 3F''F^{(4)} \,]\, \mathrm{d}z^2 + \\
&2\,[-5(F'')^2F^{(3)} - 4F'F^{(3)2} + 3F'F''F^{(4)} \,]\, \mathrm{d}p\mathrm{d}z + \\
&2\,[15(F'')^3 + 5q(F'')^2F^{(3)} - 4FF^{(3)2} + 4qF'F^{(3)2} + 3FF''F^{(4)} - \\
&3qF'F''F^{(4)} \,]\, \mathrm{d}x\mathrm{d}z + \\
&[-20(F'')^4 + 10F'(F'')^2F^{(3)} + 4(F')^2F^{(3)2} - 3(F')^2F''F^{(4)} \,]\, \mathrm{d}p^2 + \\
&2\,[-15F'(F'')^3 + 20q(F'')^4 + 5F(F'')^2F^{(3)} - 10qF'(F'')^2F^{(3)} + \\
&4FF'F^{(3)2} - 4q(F')^2F^{(3)2} - 3FF'F''F^{(4)} + 3q(F')^2F''F^{(4)} \,]\, \mathrm{d}p\mathrm{d}x + \\
&[-30F(F'')^3 + 30qF'(F'')^3 - 20q^2(F'')^4 - \\
&10qF(F'')^2F^{(3)} + 10q^2F'(F'')^2F^{(3)} + 4F^2F^{(3)2} - \\
&8qFF'F^{(3)2} + 4q^2(F')^2F^{(3)2} - 3F^2F''F^{(4)} + \\
&6qFF'F''F^{(4)} - 3q^2(F')^2F''F^{(4)} \,]\, \mathrm{d}x^2.
\end{aligned} \tag{3.1}$$

As noted in Ref. [7] this metric is conformal to a Ricci flat metric $\hat{g}_F = \mathrm{e}^{2\Upsilon(q)} g_F$ with a conformal scale $\Upsilon = \Upsilon(q)$ satisfying second order ODE:

$$90F''^2(\Upsilon'' - \Upsilon'^2) - 60F''F^{(3)}\Upsilon' + 3F''F^{(4)} - 4F^{(3)2} = 0.$$

[2]The metric presented here differs from this of [7] by a convenient conformal factor equal to $-15(F'')^{10/3}$.

Thus, since for each $F = F(q)$ the conformal class $[g_F]$ contains a Ricci flat metric, its conformal holonomy must be a proper subgroup of the noncompact form of G_2. An interesting feature of this conformal class is that it is very special among all the conformal classes associated with equation (1.1). Not only has g_F very special conformal holonomy, making it very similar to the Lorentzian 4-dimensional Brinkman metrics; moreover, since its Weyl tensor has essentially only one nonvanishing component (see Ref. [7] for details) it is *not* weakly generic (see Ref. [3] for definition). This makes $[g_F]$ analogous to the Lorentzian type N metrics in 4-dimensions, such as for example, Fefferman metrics.

Having g_F of (3.1) we used the symbolic computer calculation program Mathematica to calculate its associated form γ of (2.2). We checked that this form *identically vanishes.* We further used Mathematica to calculate the corresponding approximate ambient metric $\bar{g}_F$. On doing that we obsereved that, surprisingly, the bilinear form β is also *identically vanishing.* The explicit formula for the approximate ambient metric is given below:

$$\bar{g}_F = t^2 g_F - 2\ \mathrm{d}t\mathrm{d}u\ -\ 2tuF''^{4/3}P\mathrm{d}q^2, \tag{3.2}$$

with

$$P = \frac{4F^{(3)2} - 3F''F^{(4)}}{90(F'')^{10/3}},$$

and g_F given by (3.1). The metric $\bar{g}_F$ is defined locally on $J \times \mathbb{R}_+ \times \mathbb{R}$ with coordiantes (x, y, p, q, z, t, u). It obviously has signature $(4, 3)$. We also checked, again using Mathematica, that $Ric(\bar{g}_F) \equiv 0$. Thus, we fulfiled the strategy outlined in Section 2. This enables us to conclude that $\bar{g}_F$ of (3.2) coincides with the ambient metric $\tilde{g}_F$ for g_F. To give expressions for the Cartan normal conformal connection for g_F and the Levi-Civita connection for $\tilde{g}_F = \bar{g}_F$ we first introduce a nonholonomic coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ on J given by

$$\begin{aligned}
&\theta^1 = \mathrm{d}y - p\mathrm{d}x\\
&\theta^2 = \mathrm{d}z - F\mathrm{d}x - F'(\mathrm{d}p - q\mathrm{d}x)\\
&\theta^3 = -\tfrac{2}{\sqrt{3}}(F'')^{1/3}(\mathrm{d}p - q\mathrm{d}x)\\
&30(F'')^{10/3}\theta^4 = \big(3F'F''F^{(4)} - 4F'F^{(3)2} - 10(F'')^2F^{(3)}\big)\big(\mathrm{d}p - q\mathrm{d}x\big) +\\
&\big(4F^{(3)2} - 3F''F^{(4)}\big)\big(\mathrm{d}z - F\mathrm{d}x\big) + 30(F'')^3\mathrm{d}x\\
&\theta^5 = -(F'')^{2/3}\mathrm{d}q.
\end{aligned}$$

In this coframe the metric g_F is simply:

$$g_F = 2\theta^1\theta^5 - 2\theta^2\theta^4 + (\theta^3)^2.$$

By means of the canonical projection

$$\pi(x, y, p, q, z, t, u) = (x, y, p, q, z)$$

the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ can be pulbacked to five linearly independent forms $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ on $J \times \mathbb{R}_+ \times \mathbb{R}$. They can be suplemented by

$$\theta^0 = \mathrm{d}t \qquad \text{and} \qquad \theta^6 = \mathrm{d}u$$

to form a coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6)$ on the ambient space $J \times \mathbb{R}_+ \times \mathbb{R}$.

The Cartan normal conformal connection, when written on J in the coframe $(\theta^1,\theta^2,\theta^3,\theta^4,\theta^5)$ reads:

$$\omega_{G_2}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -P\theta^5 & 0\\ \theta^1 & 0 & Q\theta^2+\frac{9}{2\sqrt{3}}P\theta^3 & \frac{1}{\sqrt{3}}\theta^4 & -\frac{1}{2\sqrt{3}}\theta^3 & 0 & -P\theta^5\\ \theta^2 & 0 & 0 & \frac{1}{\sqrt{3}}\theta^5 & 0 & -\frac{1}{2\sqrt{3}}\theta^3 & 0\\ \theta^3 & 0 & -2\sqrt{3}P\theta^5 & 0 & \frac{1}{\sqrt{3}}\theta^5 & -\frac{1}{\sqrt{3}}\theta^4 & 0\\ \theta^4 & 0 & 0 & -2\sqrt{3}P\theta^5 & 0 & Q\theta^2+\frac{9}{2\sqrt{3}}P\theta^3 & 0\\ \theta^5 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \theta^5 & -\theta^4 & \theta^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix}.$$

Here:

$$Q=\frac{40F^{(3)3}-45F''F^{(3)}F^{(4)}+9F''^2F^{(5)}}{90F''^5}.$$

Now we use coframe $(\theta^0,\theta^1,\theta^2,\theta^3,\theta^4,\theta^5,\theta^6)$ to write down the Levi-Civita connection for $\tilde{g}_F$. We have

$$\tilde{g}_F=g_{ij}\theta^i\theta^j,$$

with the indices range: $i,j=0,1,2,\ldots 6$, and the matrix g_{ij} given by

$$g_{ij}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & t^2 & 0\\ 0 & 0 & 0 & 0 & -t^2 & 0 & 0\\ 0 & 0 & 0 & t^2 & 0 & 0 & 0\\ 0 & 0 & -t^2 & 0 & 0 & 0 & 0\\ 0 & t^2 & 0 & 0 & 0 & -2tuP & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Levi-Civita connection for $\tilde{g}_F$ on $J\times\mathbb{R}_+\times\mathbb{R}$, when written in the coframe $(\theta^0,\theta^1,\theta^2,\theta^3,\theta^4,\theta^5,\theta^6)$ reads:

$$\omega_{LC}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -tP\theta^5 & 0\\ \frac{1}{t}\theta^1+\frac{u}{t^2}P\theta^5 & \frac{1}{t}\theta^0 & Q\theta^2+\frac{9}{2\sqrt{3}}P\theta^3 & \frac{1}{\sqrt{3}}\theta^4 & -\frac{1}{2\sqrt{3}}\theta^3 & \frac{u}{t^2}P\theta^0-\frac{u}{3t}Q\theta^5-\frac{1}{t}P\theta^6 & -\frac{1}{t}P\theta^5\\ \frac{1}{t}\theta^2 & 0 & \frac{1}{t}\theta^0 & \frac{1}{\sqrt{3}}\theta^5 & 0 & -\frac{1}{2\sqrt{3}}\theta^3 & 0\\ \frac{1}{t}\theta^3 & 0 & -2\sqrt{3}P\theta^5 & \frac{1}{t}\theta^0 & \frac{1}{\sqrt{3}}\theta^5 & -\frac{1}{\sqrt{3}}\theta^4 & 0\\ \frac{1}{t}\theta^4 & 0 & 0 & -2\sqrt{3}P\theta^5 & \frac{1}{t}\theta^0 & Q\theta^2+\frac{9}{2\sqrt{3}}P\theta^3 & 0\\ \frac{1}{t}\theta^5 & 0 & 0 & 0 & 0 & \frac{1}{t}\theta^0 & 0\\ 0 & t\theta^5 & -t\theta^4 & t\theta^3 & -t\theta^2 & t\theta^1-uP\theta^5 & 0 \end{pmatrix}.$$

Note that on $\Sigma=\{(x,y,p,q,z,t,u):u=0,\ t=1\}$ we trivially have $\theta^0\equiv 0\equiv\theta^6$. Thus, restricting the formula for ω_{LC} to Σ, we see that $\omega_{G_2}\equiv\omega_{LC|\Sigma}$. Off this set the two connections: ω_{LC} and the pullbacked-by-π-connection ω_{G_2}, differ significantly. To see this it is enough to observe that contrary to ω_{LC}, the connection $\pi^*(\omega_{G_2})$ has *torsion*. Indeed writing the first Cartan structure equations for the $\pi^*(\omega_{G_2})$ in the coframe $(\theta^0,\theta^1,\theta^2,\theta^3,\theta^4,\theta^5,\theta^6)$ we find that the torsion is:

$$\mathrm{d}\theta^i+\pi^*(\omega_{G_2})^i{}_j\wedge\theta^j=\begin{pmatrix}0\\ -\theta^0\wedge\theta^1-P\theta^5\wedge\theta^6\\ -\theta^0\wedge\theta^2\\ -\theta^0\wedge\theta^3\\ -\theta^0\wedge\theta^4\\ -\theta^0\wedge\theta^5\\ 0\end{pmatrix}.$$

The vanishing of this torsion on the initial hypersurface Σ confirms our earlier statemant that the two connections ω_{G_2} and ω_{LC} coincide there.

It is interesting to note that the curvature $\mathrm{d}\omega_{LC}+\omega_{LC}\wedge\omega_{LC}$ does not depend on t, u and is anihilated by ∂_t and ∂_u. Thus it can be considered to be a 2-form on Σ. As such it is precisely equal to the curvature $\mathrm{d}\omega_{G_2}+\omega_{G_2}\wedge\omega_{G_2}$ of the connection ω_{G_2}:

$$\mathrm{d}\omega_{G_2}+\omega_{G_2}\wedge\omega_{G_2}=\mathrm{d}\omega_{LC}+\omega_{LC}\wedge\omega_{LC}=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&A_5&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&A_5&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{pmatrix}\theta^2\wedge\theta^5,$$

where[3]

$$A_5=\frac{-224F^{(3)4}+336F''F^{(3)2}F^{(4)}-51F''^2F^{(4)2}-80F''^2F^{(3)}F^{(5)}+10F''^3F^{(6)}}{100F''^{20/3}}.$$

4. Non-conformally Einstein example

To get quite different example of $[g_F]$ we consider equation (1.1) in the form:

$$z'=y''^2+a_6y'^6+a_5y'^5+a_4y'^4+a_3y'^3+a_2y'^2+a_1y'+a_0+bz,$$

where $a_i, i=0,1,...,6$, and b are real constants. This equation has the defining function

$$F=q^2+a_6p^6+a_5p^5+a_4p^4+a_3p^3+a_2p^2+a_1p+a_0+bz$$

and, via (1.3), leads to a conformal class $[g_F]$ represented by a metric

$$\begin{aligned}&15(2)^{-2/3}g_F=[9a_2+2b^2+27a_3p+54a_4p^2+90a_5p^3+135a_6p^4]\mathrm{d}y^2+\\&[15a_0+2(b^2-3a_2)p^2-3a_3p^3+9a_4p^4+30a_5p^5+60a_6p^6-\\&20bpq+5q^2+15bz]\mathrm{d}x^2+ \qquad (4.1)\\&[15a_1+4(3a_2-b^2)p-9a_3p^2-48a_4p^3-105a_5p^4-180a_6p^5+\end{aligned}$$

[3] We use the letter A_5 to denote the nonvanishing component of the curvature to be in accordance with [7] and Cartan's paper [1]. Note however that in order to avoid collision of notations between the present and the next sections we use capital A_5 instead of a_5 of paper [7].

$$20bq]\mathrm{d}x\mathrm{d}y + 20\mathrm{d}p^2 -$$
$$10(bp+q)\mathrm{d}p\mathrm{d}x + 10b\mathrm{d}p\mathrm{d}y - 30\mathrm{d}q\mathrm{d}y - 15\mathrm{d}x\mathrm{d}z + 30p\mathrm{d}q\mathrm{d}x.$$

This metric is *not* conformal to an Einstein metric. The quickest way to check this is the calculation of the Cotton, C_{ijk}, and the Weyl, W_{ijkl}, tensors for g_F. Once these tensors are calculated, it is easy to observe that they do not admit a vector field K^i such that $C_{ijk} + K^l W_{lijk} = 0$. As a consequence the metric is *not* a *conformal C-space* metric. This proves our statement since every conformally Einstein metric is neccessarily a conformal C-space metric (see e.g. Ref. [3]).

Recall that g_F of (4.1), as a member of the family of metrics (1.3), defines a conformal class $[g_F]$ with *conformal* holonomy H *reduced* to the noncompact group G_2 or to one of its subgroups. But since the metric (4.1) is not conformal to an Einstein metric, we do not have an immediate reason to conclude that $H \neq G_2$. We *conjecture* that $H = G_2$ here and try to prove it in a subsequent paper [6].

It is remarkable that the ambient metric $\tilde{g}_F$ for g_F of (4.1) assumes a very compact form:

$$\tilde{g}_F = t^2 g_F \; - 2\;\mathrm{d}t\mathrm{d}u \; -$$
$$2\;tu\;[\;\tfrac{1}{20}(-2a_2 + 4b^2 + 3a_3p + 6a_4p^2 - 20a_5p^3 - 120a_6p^4)\mathrm{d}x^2 -$$
$$\tfrac{9}{20}(a_3 - 10a_5p^2 - 40a_6p^3)\mathrm{d}x\mathrm{d}y - \tfrac{9}{10}(a_4 + 5a_5p + 15a_6p^2)\mathrm{d}y^2\;]\; +$$
$$u^2\;[\;\tfrac{3}{20(2)^{2/3}}(a_4 - 10a_5p + 60a_6p^2)\mathrm{d}x^2 + \tfrac{9}{4(2)^{2/3}}(a_5 - 12a_6p)\mathrm{d}x\mathrm{d}y + \tfrac{81}{4(2)^{2/3}}a_6\mathrm{d}y^2\;].$$

This is checked by applying our strategy described in Section 2 to the metric (4.1). As in the previous example, using Mathematica, we calculated the bilinear form γ for (4.1). It turned out to be equal to *zero*, $\gamma \equiv 0$. Then we calculated $\bar{g}_F$, and checked that it is *Ricci flat*. Thus we concluded that $\bar{g}_F$ coincides with the ambient metric for $\tilde{g}_F$. The above given formula for $\tilde{g}_F$ is therefore just $\bar{g}_F$, which we calculated using (2.2).

We find this example as a sort of miracle. Apriori there is no reason for g_F to have the ambient metric *truncated* at the *second* order in terms of the ambient parameters t and u. We are intrigued by this fact.

Now, following the general procedure outlined in [7], we introduce a special coframe for g_F given by:

$$\theta^1 = \mathrm{d}y - p\mathrm{d}x$$
$$\theta^2 = \mathrm{d}z - F\mathrm{d}x - 2q(\mathrm{d}p - q\mathrm{d}x)$$
$$\theta^3 = -\tfrac{2^{4/3}}{\sqrt{3}}(\mathrm{d}p - q\mathrm{d}x)$$
$$\theta^4 = 2^{-1/3}\mathrm{d}x$$
$$15(2)^{1/3}\theta^5 = (9a_2 + 2b^2 + 27a_3p + 54a_4p^2 + 90a_5p^3 + 135a_6p^4)(\mathrm{d}y - p\mathrm{d}x) +$$
$$10b(\mathrm{d}p - q\mathrm{d}x) - 30\mathrm{d}q +$$
$$15(a_1 + 2a_2p + 3a_3p^2 + 4a_4p^3 + 5a_5p^4 + 6a_6p^5 + 2bq)\mathrm{d}x.$$

In this coframe the metric g_F is:

$$g_F = 2\theta^1\theta^5 - 2\theta^2\theta^4 + (\theta^3)^2.$$

As in the previous section, we use the canonical projection

$$\pi(x, y, p, q, z, t, u) = (x, y, p, q, z)$$

to pullback the coframe $(\theta^1,\theta^2,\theta^3,\theta^4,\theta^5)$ to five linearly independent forms $(\theta^1,\theta^2,\theta^3,\theta^4,\theta^5)$ on $J\times\mathbb{R}_+\times\mathbb{R}$, which are further suplemented by

$$\theta^0 = {\rm d}t \qquad \text{and} \qquad \theta^6 = {\rm d}u$$

to form a coframe $(\theta^0,\theta^1,\theta^2,\theta^3,\theta^4,\theta^5,\theta^6)$ on the ambient space $J\times\mathbb{R}_+\times\mathbb{R}$.

It turns out that if $b=0$ the coframes on J and $J\times\mathbb{R}_+\times\mathbb{R}$ defined in this way are suitable to analyze the relations between the Cartan normal conformal connection ω_{G_2} for $[g_F]$ and the Levi-Civita connection ω_{LC} for $\tilde{g}_F$. If $b\neq 0$ the conection ω_{G_2} in the coframe $(\theta^1,\theta^2,\theta^3,\theta^4,\theta^5)$ and the connection ω_{LC} in the coframe $(\theta^0,\theta^1,\theta^2,\theta^3,\theta^4,\theta^5,\theta^6)$ do not coincide on $t=1$, $u=0$. We will not analyze this case here.

Restricting to the

$$b=0$$

case we find the following:

- the connections ω_{G_2} in the coframe $(\theta^1,\theta^2,\theta^3,\theta^4,\theta^5)$ and the connection ω_{LC} in the coframe $(\theta^0,\theta^1,\theta^2,\theta^3,\theta^4,\theta^5,\theta^6)$ coincide on $t=1$, $u=0$.
- the torsion of $\pi^*(\omega_{G_2})$ in the coframe $(\theta^0,\theta^1,\theta^2,\theta^3,\theta^4,\theta^5,\theta^6)$ is nonvanishing off the set $t=1$, $u=0$
- unlike the example of the previous section the curvature ${\rm d}\omega_{LC}+\omega_{LC}\wedge\omega_{LC}$ siginificantly depends on t and u.
- even on $t=1$, $u=0$, the curvature ${\rm d}\omega_{G_2}+\omega_{G_2}\wedge\omega_{G_2}$ and the restriction of ${\rm d}\omega_{LC}+\omega_{LC}\wedge\omega_{LC}$ do not coincide.

5. Acknowledgements

I am very grateful to T P Branson, M Eastwood and W Miller Jr, the organizers of the 2006 IMA Summer Program "Symmetries and Overdetermined Systems of Partial Differential Equations", for invitating me to Minneapolis to particpate in this very fruitful event. The topic covered by this note is inspired by the talk of C R Graham which I heard in Minneapolis during the program. In particular, I am very obliged to C R Graham for sending me the formulas (2.2), which I used to prepare the examples included in this note.

References

[1] Cartan E, "Les systemes de Pfaff a cinq variables et les equations aux derivees partielles du seconde ordre" *Ann. Sc. Norm. Sup.* **27** 109-192 (1910)

[2] Fefferman C, Graham C R, "Conformal invariants", in *Elie Cartan et mathematiques d'aujourd'hui*, Asterisque, hors serie (Societe Mathematique de France, Paris) 95-116 (1985)

[3] Gover A R, Nurowski P, "Obstructions to conformally Einstein metrics in n dimensions" *Journ. Geom. Phys.* **56** 450-484 (2006)

[4] Graham C R, private communications, unpublished

[5] de Haro S, Skenderis K, Solodukhin S N, "Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence", *Comm. Math. Phys.* **217** (2001), 594–622, hep-th/0002230

[6] Leistner Th, Nurowski P, in preparation

[7] Nurowski P, "Differential equations and conformal structures" *Journ. Geom. Phys.* **55** 19-49 (2005)

Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, ul. Hoża 69, Warszawa, Poland

E-mail address: nurowski@fuw.edu.pl