3-dimensional Cauchy-Riemann structures and $2^{\underline{\text{nd}}}$ order ordinary differential equations

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Abstract

The equivalence problem for second order ODEs given modulo point transformations is solved in full analogy with the equivalence problem of nondegenerate 3-dimensional CR structures. This approach enables an analog of the Feffereman metrics to be defined. The conformal class of these (split signature) metrics is well defined by each point equivalence class of second order ODEs. Its conformal curvature is interpreted in terms of the basic point invariants of the corresponding class of ODEs.

1 Introduction

This paper aims to explain the relations between two classical geometries: the geometry associated with 2nd order ordinary differential equations defined modulo point transformations of variables and the geometry of the 3-dimensional Cauchy-Riemann structures.

The geometry associated with 2nd order ordinary differential equations, considered modulo contact transformations, is trivial - all 2nd order ODEs are locally contact equivalent to the equation $y''=0$. If one considers (more natural) point transformations - all diffeomorphisms of the plane (x,y) - then their action on the space of all 2nd order ODEs has nontrivial orbits - there exist 2nd order ODEs that are not (even locally) point equivalent. An example of such point inequivalent equations is given by $y''=0$ and $y''=y^2$.

In general, the equation

$$y''=Q(x,y,y')$$

with the total differential

$$D=\partial_x+y'\partial_y+Q\partial_{y'},$$

can be characterized by a number of relative invariants whose vanishing or not is a point invariant property of the equation. The two of these invariants of lowest order are

$$w_1=D^2Q_{y'y'}-4DQ_{yy'}-DQ_{y'y'}Q_{y'}+4Q_{y'}Q_{yy'}-3Q_{y'y'}Q_y+6Q_{yy},$$

and

$$w_2=Q_{y'y'y'y'}.$$

These were known to S. Lie [18] and used by M. A. Tresse [43, 44] in his systematic study of an equivalence problem for second order ODEs given modulo point transformations. E. Cartan, in his celebrated paper [4] on projective connections, used the class of 2nd order ODEs for which the invariant w_2 vanished as an example of a geometry that naturally give rise to a Cartan normal projective connection[1].

The study of the geometry of Cauchy-Riemann (CR) structures was initiated by H. Poincare [30], who looked for a higher dimensional generalization of the well known fact that two real analytic arcs in $\mathbf{C}$ are locally biholomorphically equivalent. Using a heuristic argument he showed that generic two real 3-dimensional hypersurfaces N_1 and N_2 embedded in $\mathbf{C}^2$ are not, even locally, biholomorphically equivalent. This led B. Segre [34] to study the equivalence problem for real hypersurfaces of codimension 1 in $\mathbf{C}^2$, given modulo the biholomorphisms, a problem which was later solved in full generality by E. Cartan [5]. Generalization of the problem to $\mathbf{C}^n$ with $n>2$ led to the theory of CR-structures which is a part of several complex variables theory and lies on the borders between analysis, geometry and studies of PDEs. In this theory a particular role is played by the conformal Fefferman metrics [7] which are *Lorentzian* metrics naturally defined on a circle bundle over each CR manifold. These metrics were defined by Ch. Fefferman in 1976 and, surprisingly, were unnoticed by E. Cartan in his pioneering paper [5].

The appearence of Lorentzian metrics in the CR-structure theory is not an accident. In the lowest dimension ($n=2$) this is due to the well known fact [32, 33, 37, 39] that 3-dimensional Cauchy-Riemann structures are in one-to-one correspondence with congruences of null geodesics without shear in 4-dimensional space-times. Many physically interesting space-times, such as Minkowski,

[1] Cartan's observation has recently been understood from the twistorial point of view in [10] and generalized in [19].

Schwarzschild, Kerr-Newman, Taub-NUT, Hauser, plane gravitational waves and Robinson-Trautman, etc., admit congruences of such geodesics. The understanding of space-times admitting congruences of shear-free and null geodesics from the point of view of the corresponding CR-geometry has been quite fruitful in the process of solving Einstein vacuum equations [15, 16, 17, 20, 38]. In these papers the construction of the solutions of the Einstein equations in terms of the CR-functions of the corresponding CR geometry is very much in the spirit of the Penrose's twistor theory [27, 28]. More importantly, from quite another point of view, space-times admitting shear-free congruences of null geodesics are the Lorentzian analogs of Hermitian geometries in 4-dimensions. Since I. Robinson played the crucial role of introducing the shear-free property to General Relativity A. Trautman has called such manifolds Robinson manifolds [24, 41, 42].

Although it is not immediately self-evident, the geometries associated with 2nd order ODEs and the 3-dimensional CR structures are closely related. This fact was known to B. Segre who, in this context, was quoted in Cartan's paper [5]. Strangely enough, Cartan in [5] only mentioned that such relations existed but did not spend much time explaining what they were. In addition he does not appear to have used these relations to simplify his approach to the CR structures. In this paper we explain Segre's observation in detail and reconstruct and devolop the results of the two Cartan's papers [4, 5] from this point of view.

Section 2 consists of two parts. The first part contains a review of the concept of a 3-dimensional CR-structure. This is defined as a natural generalization of the notion of classes of real 3-dimensional hypersurfaces embedded, modulo biholomorphisms, in $\mathbf{C}^2$. Our definition is much more in the spirit of Cartan's treatment of such hypersurfaces than in the spirit of the modern theory of CR manifolds. This point of view will be adapted in the whole paper. The first part of Section 2 ends with the quotation of Cartan's theorem (Theorem 1) solving the equivalence problem for 3-dimensional CR structures. In the second part of Section 2 we give the modern description of this theorem in terms of Cartan's $\mathbf{SU}(2,1)$ connection. We also show how Cartan might have used his theorem to associate with each nondegenerate 3-dimensional CR structure the Fefferman class of metrics. We analyze the Fefferman metrics using Cartan's normal conformal connection associated with them, and give a new proof,based on the use of Baston-Mason conditions [1], of the fact [14] that these metrics are conformal to Einstein metrics only if the curvature of their CR structure's $\mathbf{SU}(2,1)$ connection vanishes. We close this section with a formula for the Bach tensor for the Fefferman metrics, expressed in terms of the curvature of the corresponding CR structure's connection.

Section 3, the main Section of the paper, explains the analogy between 3-dimensional CR structures and 2nd order ODEs defined modulo point transformations. The basic ingredients of this analogy are given just before Definition 3, which states what it means for two second order ODEs to be point equivalent to each other. Comparison between Definitions 2 and 3 makes the analogy self evident. Using this analogy we are able to formulate Theorem 2 which solves the equivalence problem for 2nd order ODEs given modulo point transformations. By the analogy this theorem is literally the same as Theorem 1. The only difference is that now the symbols appearing in the Theorem have different interpretations. This new interpretation implies that behind the equivalence problem for 2nd order ODEs modulo point transformations is a certain Cartan $\mathbf{SL}(3,\mathbf{R})$ connection. This fact was, of course, known to Cartan [4], but we are not sure if Cartan would present it in the spirit of our paper even if he had a time machine at his disposal (Cartan's ODE paper [4] dates from 1924, whereas his CR paper [5] is from 1932). After Theorem 2 we give a local representation of the point invariants of a 2nd order ODE obtaining, in particular, Lie's basic relative invariants w_1 and w_2. We proceed, exploiting the analogy, to define an ODE analog of Fefferman metrics, which now have *split* signature. The conformal class of split signature metrics which is naturally associated with each 2nd order ODE given modulo point transformations turn out to encode all the point invariant information about the underlying class of ODEs. In particular, all the Cartan invariants of the point equivalent class of ODEs are derived from the Weyl curvature of the corresponding Fefferman-like metric. These metrics are characterized by Proposition 1 and the Remark following it, and, some time ago, were considered by one of us (GAJS) within the general framework discused in [35]. Unlike the CR structures case there are point equivalent classes of 2nd order ODEs which have nonvanishing curvature of the Cartan $\mathbf{SL}(3,\mathbf{R})$ connection and for which the Fefferman-like metrics are conformal to Einstein metrics. Such metrics may only correspond to the ODEs for which the Lie relative invariants w_1 and w_2 satisfy $w_1 w_2 = 0$ and all of them are presented in the Appendix.

Like all the Cartan invariants, the Lie invariants w_1 and w_2 are interpreted in terms of the Fefferman-like metrics associated with the class of 2nd order ODEs that defines them. It turns out that the Fefferman metrics associated with a point equivalence class of 2nd order ODEs is always of the algebraic type $N \times N'$ in the Cartan-Petrov-Penrose [3, 26, 29] classification of real-valued 4-dimensional metrics. This means, in particular, that both the self-dual and the anti-self-dual parts of their Weyl tensor have only one independent component. It turns out that the self-dual part of this tensor is proportional to w_1 and the anti-self-dual part is proportional to w_2. Thus, the vanishing of one of Lie's relative invariants makes the associated Fefferman-like metric half-flat. This partially explains why, in such cases, these metrics may be conformal to Einstein metrics. The rest of Section 3 is devoted to understanding the fact that it is easy to find all 2nd order ODEs for which $w_2 = 0$, since all of them are of the form

$$y'' = A_0(x,y) + A_1(x,y)y' + A_2(x,y)(y')^2 + A_3(x,y)(y')^3,$$

and it is quite hard to find $Q = Q(x,y,y')$ for which $w_1 = 0$. From the Fefferman-like metrics point of view the switch bewteen w_2 and w_1 is the switch between the self-dual and the anti-self-dual part of their Weyl tensor. This suggest that invariants w_1 and w_2 should be on an equal footing of complexity. To see that this is indeed the case requires another notion of duality - the duality between the point equivalent classes of 2nd orderd ODEs. This duality was mentioned by Cartan in [4]. We explain it in detail at the end of Section 3. In particular, in Proposition 2, and in the Example preceeding it, we show how to construct solutions $Q = Q(x,y,y')$ of $w_1 = 0$ knowing Qs which satisfy $w_2 = 0$. The understanding of this duality in terms of the natural double fibration of the first jet bundle associated with the ODE is also given.

Finally, in Section 4 we give two applications of the theory presented in Sections 2 and 3. The first consists of an algorithm for associating a point equivalence class of 2nd order ODEs with a given 3-dimensional CR structure. This may be of some use in General Relativity theory and may provide a new understanding of well known congruences of shear-free and null geodesics. The second application is, as far as we know, the first example of a large class of split signature 4-metrics which satisfy the Bach equations, are genuinely of algebraic type $N \times N'$ and are not conformal to Einstein metrics.

Note on the conventions and the notation

We emphasize that in this paper all our considerations are purely local and concerned with nonsingular points of the introduced constructions. We also mention that, following the old tradition in PDEs, we denote the partial derivatives with respect to the variable associated with index i by the corresponding subscript, e.g. $\frac{\partial G}{\partial z_i} = G_i$.

2 3-dimensional CR structures

A 3-dimensional CR structure is a structure which a 3-dimensional hypersurface N embedded in $\mathbf{C}^2$ acquires from the ambient complex space. Following Elie Cartan [5] this structure can be described in the language of differential forms as follows.

Consider a 3-dimensional hypersurface N in $\mathbf{C}^2$ defined by means of a real function $G = G(z_1, z_2, \bar{z}_1, \bar{z}_2)$, such that $G_1 \neq 0$, via

$$N = \{\ (z_1, z_2) \in \mathbf{C}^2 \mid G(z_1, z_2, \bar{z}_1, \bar{z}_2) = 0\ \}.$$

All information about the structure acquired by N from $\mathbf{C}^2$ can be encoded in the two 1-forms

$$\lambda = i(G_1 \mathrm{d}z_1 + G_2 \mathrm{d}z_2) \qquad \text{and} \qquad \mu = \mathrm{d}z_2. \tag{1}$$

These forms have the following properties:

- λ is real, μ is complex
- $\lambda \wedge \mu \wedge \bar{\mu} \neq 0$ on N.

Moreover, if N underlies the biholomorphism

$$z_1 = z_1(z_1', z_2'), \quad z_2 = z_2(z_1', z_2')$$

the forms transform according to

$$\lambda \to \lambda' = a\lambda \quad \text{and} \quad \mu \to \mu' = b\mu + c\lambda,$$

where $a \neq 0$ (real) and $b \neq 0, c$ (complex) are appropriate functions on N. It is easy to see that the vanishing of the 3-form $\lambda \wedge \mathrm{d}\lambda$ is an invariant property under the biholomorphisms of $\mathbf{C}^2$. Thus, the two hypersurfaces

$$N_1 = \{\ (z_1, z_2) \in \mathbf{C}^2\ :\ z_1 - \bar{z}_1 = 0\ \}$$

and

$$N_2 = \{\ (z_1, z_2) \in \mathbf{C}^2\ :\ |z_1|^2 + |z_2|^2 - 1 = 0\ \}, \tag{2}$$

with the corresponding forms $\lambda_1 = i\mathrm{d}z_1$ and $\lambda_2 = i(\bar{z}_1 \mathrm{d}z_1 + \bar{z}_2 \mathrm{d}z_2)$ are not biholomorphically equivalent.

The above considerations motivate an introduction of the following structure on 3-manifolds.

Definition 1 *A CR-structure $[(\lambda, \mu)]$ on a 3-dimensional manifold N is an equivalence class of pairs of 1-forms (λ, μ) such that*

- *λ is real, μ is complex,*
- *$\lambda \wedge \mu \wedge \bar{\mu} \neq 0$ on N*
- *two pairs (λ, μ) and (λ', μ') are in the equivalence relation iff there exist functions $a \neq 0$ (real), $b \neq 0, c$ (complex) on N such that*

$$\lambda' = a\lambda, \qquad \mu' = b\mu + c\lambda, \qquad \bar{\mu}' = \bar{b}\bar{\mu} + \bar{c}\lambda.$$

A CR-structure is called nondegenerate iff

$$\mathrm{d}\lambda \wedge \lambda \neq 0;$$

otherwise a CR-structure is degenerate.

An obvious class of examples of CR-structures is given by biholomorphically equivalent classes of hypersurfaces in $\mathbf{C}^2$. The problem of classifying biholomorphically nonequivalent hypersurfaces in $\mathbf{C}^2$ is therefore a part of the equivalence problem of CR-structures.

Definition 2 *Let $(N,[(\lambda,\mu)])$ and $(N',[(\lambda',\mu')])$ be two CR-structures on two 3-dimensional manifolds N and N'. We say that $(N,[(\lambda,\mu)])$ and $(N',[(\lambda',\mu')])$ are (locally) equivalent iff, for any two representatives $(\lambda,\mu)\in[(\lambda,\mu)]$ and $(\lambda',\mu')\in[(\lambda',\mu')]$, there exists a (local) diffeomorphism $\phi: N\to N'$ and functions $a\neq 0$ (real), $b\neq 0, c$ (complex) on N such that*

$$\phi^*(\lambda')=a\lambda,\qquad \phi^*(\mu')=b\mu+c\lambda,\qquad \phi^*(\bar\mu')=\bar b\bar\mu+\bar c\lambda.$$

It is easy to see that all 3-dimensional *degenerate* CR-structures are locally equivalent to the structure associated with a biholomorphic class of hypersurfaces equivalent to the hypersurface $N_1=\mathbf{C}\times\mathbf{R}$. The equivalence problem for *nondegenerate* 3-dimensional CR-structures was solved by Elie Cartan [5]. Given a nondegenerate CR-structure $(N,[(\lambda,\mu)])$ he considered the forms

$$\theta^1=b\mu+c\lambda,\qquad \theta^2=\bar b\bar\mu+\bar c\lambda,\qquad \theta^3=a\lambda,\tag{3}$$

with some unspecified functions $a\neq 0$ (real), $b\neq 0$ and c (complex). He viewed the forms as being well defined on an 8-dimensional space P_0 parametrized by the points of N and by the coordinates $(a,b,\bar b,c,\bar c)$. Using his equivalence method (see e.g. [11, 25]) he then constructed another 8-dimensional manifold P on which the coframe consisting of the forms $(\theta^1,\theta^2,\theta^3)$ and the five additional well defined 1-forms $(\Omega_2,\bar\Omega_2,\Omega_3,\bar\Omega_3,\Omega_4)$ constitued the system of basic biholomorphic invariants of the CR-structure. More precisely, he proved the following theorem.

Theorem 1 *Every nondegenerate CR-structure $(N,[(\lambda,\mu)])$ uniquely defines an 8-dimensional manifold P, 1-forms $\theta^1,\theta^2,\theta^3,\Omega_2,\bar\Omega_2,\Omega_3,\bar\Omega_3,\Omega_4$ and functions $\mathcal{R},\bar{\mathcal{R}},\mathcal{S},\bar{\mathcal{S}}$ on P such that*

- *$\theta^1,\theta^2,\theta^3$ are as in (3), $\bar\Omega_2,\bar\Omega_3$ are respective complex conjugates of Ω_2,Ω_3 and Ω_4 is real,*
- *$\theta^1\wedge\theta^2\wedge\theta^3\wedge\Omega_2\wedge\bar\Omega_2\wedge\Omega_3\wedge\bar\Omega_3\wedge\Omega_4\neq 0$ at each point of P.*

The forms satisfy the following equations

$$\begin{aligned}
\mathrm{d}\theta^1&=\Omega_2\wedge\theta^1+\Omega_3\wedge\theta^3\\
\mathrm{d}\theta^2&=\bar\Omega_2\wedge\theta^2+\bar\Omega_3\wedge\theta^3\\
\mathrm{d}\theta^3&=i\theta^1\wedge\theta^2+(\Omega_2+\bar\Omega_2)\wedge\theta^3\\
\mathrm{d}\Omega_2&=2i\theta^1\wedge\bar\Omega_3+i\theta^2\wedge\Omega_3+\Omega_4\wedge\theta^3\\
\mathrm{d}\bar\Omega_2&=-2i\theta^2\wedge\Omega_3-i\theta^1\wedge\bar\Omega_3+\Omega_4\wedge\theta^3\\
\mathrm{d}\Omega_3&=\Omega_4\wedge\theta^1+\Omega_3\wedge\bar\Omega_2+\mathcal{R}\theta^2\wedge\theta^3\\
\mathrm{d}\bar\Omega_3&=\Omega_4\wedge\theta^2+\bar\Omega_3\wedge\Omega_2+\bar{\mathcal{R}}\theta^1\wedge\theta^3\\
\mathrm{d}\Omega_4&=i\Omega_3\wedge\bar\Omega_3+\Omega_4\wedge(\Omega_2+\bar\Omega_2)+\bar{\mathcal{S}}\theta^1\wedge\theta^3+\mathcal{S}\theta^2\wedge\theta^3.
\end{aligned}\tag{4}$$

The functions $\mathcal{R}$, $\mathcal{S}$, and their respective complex conjugates $\bar{\mathcal{R}}$, $\bar{\mathcal{S}}$, satisfy

$$\begin{aligned}
\mathrm{d}\mathcal{R}&=-\mathcal{R}(\Omega_2+3\bar\Omega_2)-\mathcal{S}\theta^1+\mathcal{R}_2\theta^2+\mathcal{R}_3\theta^3\\
\mathrm{d}\bar{\mathcal{R}}&=-\bar{\mathcal{R}}(\bar\Omega_2+3\Omega_2)-\bar{\mathcal{S}}\theta^2+\bar{\mathcal{R}}_2\theta^1+\bar{\mathcal{R}}_3\theta^3
\end{aligned}\tag{5}$$

and

$$\begin{aligned}
\mathrm{d}\mathcal{S}&=-\mathcal{S}(2\Omega_2+3\bar\Omega_2)-i\mathcal{R}\bar\Omega_3+\mathcal{S}_1\theta^1+\mathcal{S}_2\theta^2+\mathcal{S}_3\theta^3\\
\mathrm{d}\bar{\mathcal{S}}&=-\bar{\mathcal{S}}(2\bar\Omega_2+3\Omega_2)+i\bar{\mathcal{R}}\Omega_3+\bar{\mathcal{S}}_2\theta^1+\mathcal{S}_1\theta^2+\bar{\mathcal{S}}_3\theta^3,
\end{aligned}\tag{6}$$

with appropriate functions $\mathcal{R}_2, \mathcal{R}_3, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ and their conjugates.
The function $\mathcal{S}_1$ satisfies

$$\mathcal{S}_1 = \bar{\mathcal{S}}_1. \tag{7}$$

The above theorem, stated in the modern language, means the following. The manifold P is a Cartan bundle $H \to P \to N$, with H a 5-dimensional parabolic subgroup of $\mathbf{SU}(2,1)$. This latter group preserves the $(2,1)$-signature hermitian form

$$h(X,X) = \begin{pmatrix} X^1, & X^2, & X^3 \end{pmatrix} \hat{h} \begin{pmatrix} \bar{X}^1 \\ \bar{X}^2 \\ \bar{X}^3 \end{pmatrix}, \qquad \hat{h} = \begin{pmatrix} 0 & 0 & 2i \\ 0 & 1 & 0 \\ -2i & 0 & 0 \end{pmatrix}.$$

The forms $\theta^1, \theta^2, \theta^3, \Omega_2, \bar{\Omega}_2, \Omega_3, \bar{\Omega}_3, \Omega_4$ of the theorem can be collected into a matrix of 1-forms

$$\omega = \begin{pmatrix} \frac{1}{3}(2\Omega_2 + \bar{\Omega}_2) & i\bar{\Omega}_3 & -\frac{1}{2}\Omega_4 \\ & & \\ \theta^1 & \frac{1}{3}(\bar{\Omega}_2 - \Omega_2) & -\frac{1}{2}\Omega_3 \\ & & \\ 2\theta^3 & 2i\theta^2 & -\frac{1}{3}(2\bar{\Omega}_2 + \Omega_2), \end{pmatrix}$$

satisfying

$$\omega \hat{h} + \hat{h}\omega^\dagger = 0,$$

which is an $\mathbf{su}(2,1)$-valued Cartan connection [12] on P. It follows from equations (4) that the curvature of this connections is

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega = \begin{pmatrix} 0 & i\bar{\mathcal{R}}\theta^1 \wedge \theta^3 & -\frac{1}{2}\bar{\mathcal{S}}\theta^1 \wedge \theta^3 - \frac{1}{2}\mathcal{S}\theta^2 \wedge \theta^3 \\ & & \\ 0 & 0 & -\frac{1}{2}\mathcal{R}\theta^2 \wedge \theta^3 \\ & & \\ 0 & 0 & 0 \end{pmatrix}.$$

It yields all the invariant information about the corresponding CR-structure, very much in the way as the Riemann curvature yields all the information about the Riemannian structure.

Remark Note that the assumption that $\mathcal{R}$ or $\mathcal{S}$ (and, therefore $\bar{\mathcal{R}}$ or $\bar{\mathcal{S}}$) are constant on P is compatible with (5) iff $\mathcal{R} = \mathcal{S} = 0$ (and, therefore $\bar{\mathcal{R}} = \bar{\mathcal{S}} = 0$). In such case the curvature Ω of the Cartan connection ω vanishes, and it follows that there is only one, modulo local equivalence, CR-structure with this property. It coincides with the CR-structure, which the hypersurface $N_2 = \mathbf{S}^3$ acquires from the ambient space $\mathbf{C}^2$ via equations (1), (2).

Using the matrix elements $\omega^i{}_j$ of the Cartan connection ω it is convenient to consider the bilinear form

$$G = -i\omega^3{}_j\omega^j{}_1.$$

This form, when writen explicitly in terms of $\theta^1, \theta^2, \theta^3, \Omega_2, \bar{\Omega}_2, \Omega_3, \bar{\Omega}_3, \Omega_4$, is given by

$$G = 2\theta^1\theta^2 + \frac{2}{3i}\theta^3(\Omega_2 - \bar{\Omega}_2).$$

Introducing the basis of vector fields $X_1, X_2, X_3, Y_2, \bar{Y}_2, Y_3, \bar{Y}_3, Y_4$, the respective dual of $\theta^1, \theta^2, \theta^3$, $\Omega_2, \bar{\Omega}_2, \Omega_3, \bar{\Omega}_3, \Omega_4$, one sees that G is a form of signature $(+++-0000)$ with four degenerate directions corresponding to four vector fields $Z_I = (Y_2 + \bar{Y}_2, Y_3, \bar{Y}_3, Y_4)$. These four directions span a 4-dimensional distribution which is integrable due to equations (4). Thus, the Cartan bundle P is foliated by 4-dimensional leaves tangent to the degenerate directions of G. Moreover, equations (4) guarantee that

$$\mathcal{L}_{Z_I}\ G = A_I\ G,$$

with certain functions A_I on P, so that the bilinear form G is preserved up to a scale when Lie transported along the leaves of the foliation. Therefore the 4-dimensional space $P/\!\sim$ of leaves of the foliation is naturally equipped with a conformal class of Lorentzian metrics $[g_F]$, the class to which the bilinear form G naturally descends. The Lorentzian metrics

$$g_F = 2\theta^1\theta^2 + \frac{2}{3i}\theta^3(\Omega_2 - \bar{\Omega}_2) \tag{8}$$

on $P/\!\sim$ coincide with the so called Fefferman metrics [7] (see also [9]) which Charles Fefferman associated with any nondegenerate CR-structure $(N, [(\lambda, \mu)]$.

Introducing the volume form

$$\eta = \frac{1}{3}\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge (\Omega_2 - \bar{\Omega}_2)$$

on $P/\!\sim$ we observe that the Hodge dualization $*$ of the forms $\theta^3 \wedge \theta^1$ and $\theta^3 \wedge \theta^2$ read

$$*(\theta^3 \wedge \theta^1) = -\frac{1}{i}(\theta^3 \wedge \theta^1) \quad \text{and} \quad *\,(\theta^3 \wedge \theta^2) = \frac{1}{i}(\theta^3 \wedge \theta^2).$$

Thus $\theta^3 \wedge \theta^1$ is self-dual and $\theta^3 \wedge \theta^2$ is anti-self-dual.

A convenient way of analyzing the Fefferman metrics is to look for the Cartan normal conformal connection associated with them. Given a nondegenerate CR-structure $(N, [(\lambda, \mu)]$ we define an $\mathbf{so}(4,2)$-valued matrix of 1-forms $\tilde{\omega}$ on P via

$$\tilde{\omega} = \begin{pmatrix} \frac{1}{2}(\Omega_2 + \bar{\Omega}_2) & \frac{i}{2}\bar{\Omega}_3 & -\frac{i}{2}\Omega_3 & -\Omega_4 & \frac{i}{12}(\Omega_2 - \bar{\Omega}_2) & 0 \\ \theta^1 & -\frac{1}{3}(\Omega_2 - \bar{\Omega}_2) & 0 & -\Omega_3 & \frac{i}{2}\theta^1 & -\frac{i}{2}\Omega_3 \\ \theta^2 & 0 & \frac{1}{3}(\Omega_2 - \bar{\Omega}_2) & -\bar{\Omega}_3 & -\frac{i}{2}\theta^2 & \frac{i}{2}\bar{\Omega}_3 \\ \theta^3 & \frac{i}{2}\theta^2 & -\frac{i}{2}\theta^1 & -\frac{1}{2}(\Omega_2 + \bar{\Omega}_2) & 0 & \frac{i}{12}(\Omega_2 - \bar{\Omega}_2) \\ \frac{1}{3i}(\Omega_2 - \bar{\Omega}_2) & \bar{\Omega}_3 & \Omega_3 & 0 & \frac{1}{2}(\Omega_2 + \bar{\Omega}_2) & -\Omega_4 \\ 0 & \theta^2 & \theta^1 & \frac{1}{3i}(\Omega_2 - \bar{\Omega}_2) & \theta^3 & -\frac{1}{2}(\Omega_2 + \bar{\Omega}_2) \end{pmatrix}. \tag{9}$$

This is a pullback of the Cartan normal conformal connection associated with the Fefferman metric from the Cartan $\mathbf{SO}(4,2)$ conformal bundle to P. With a slight abuse of the language we call $\tilde{\omega}$ the Cartan conformal coonnection. The pulback of the curvature of this connection

$$\tilde{\Omega} = \mathrm{d}\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega},$$

is given by

$$\tilde\Omega = \tilde\Omega^+ + \tilde\Omega^- = \tag{10}$$

$$= \begin{pmatrix} 0 & -\frac{i}{2}\bar{\mathcal{R}} & 0 & \bar{\mathcal{S}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\mathcal{R}} & 0 & -\frac{i}{2}\bar{\mathcal{R}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\bar{\mathcal{R}} & 0 & 0 & 0 & \bar{\mathcal{S}} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \theta^3\wedge\theta^1 + \begin{pmatrix} 0 & 0 & \frac{i}{2}\mathcal{R} & \mathcal{S} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{R} & 0 & \frac{i}{2}\mathcal{R} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{R} & 0 & 0 & \mathcal{S} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \theta^3\wedge\theta^2.$$

Here $\tilde\Omega^+$ and $\tilde\Omega^-$ denote the self-dual and the anti-selfdual parts of $\tilde\Omega$, respectively.

The theory of the conformal connections [8, 12, 13, 21] then implies that that the Weyl curvature 2-form C of g_F is given by[2]

$$C = C^+ + C^- = \tag{11}$$

$$= \bar{\mathcal{R}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \theta^3\wedge\theta^1 + \mathcal{R} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \theta^3\wedge\theta^2,$$

i.e. denoting the matrix elements of $\tilde\Omega$ by $\tilde\Omega^A{}_B$, $A, B = 0, 1, ...5$, it is given by $\tilde\Omega^A{}_B$ with $A, B = 1, 2, 3, 4$. The very simple form of the Weyl curvature C shows that the Fefferman metric g_F of any nondegenerate CR-structure $(N, [(\lambda, \mu)])$ is of the Petrov type N.

Remark Note that the curvature of the Cartan normal conformal connection $\tilde\omega$ of the Fefferman metric g_F yields essentially the same information as the curvature of the $\mathbf{su}(2,1)$-valued connection ω. This is due to the fact [2] that ω is simply an $\mathbf{su}(2,1)$ reduction of the Cartan normal conformal connection associated with the Fefferman metric g_F. In addition, this indicates the well known fact that the Fefferman conformal class of metrics $[g_F]$ associated with a given nondegenrate CR-structure $(N, [(\lambda, \mu)])$ yields all the invariant information about $(N, [(\lambda, \mu)])$. In particular, the $\mathbf{su}(2,1)$-curvature properties of the CR-structure are totally encoded in the Weyl tensor 2-forms C of the corresponding Fefferman metrics. Note that although C explicitely involves only $\mathcal{R}$ and $\bar{\mathcal{R}}$, the $\mathcal{S}$ and $\bar{\mathcal{S}}$ functions can be derived from them by means of equations (5).

It is known [14] that the Fefferman metrics are conformal to the Einstein metrics only in the case when the corresponding CR-structure is flat ($\Omega = 0$). To see this we recall the Baston-Mason result

[2] Here C has tensor indices $C^\mu{}_\nu$, $\mu, \nu = 1, 2, 3, 4$ which are associated with the null tetrad $\theta^1, \theta^2, \theta^3, \theta^4 = \frac{1}{3i}(\Omega_2 - \bar\Omega_2)$ of g_F. In this tetrad $g_F = 2\theta^1\theta^2 + 2\theta^3\theta^4$.

[1] stating that there are two neccessary conditions for a 4-dimensional metric $g = g_{\mu\nu}\theta^\mu\theta^\nu$ to be conformal to an Einstein metric. These, when expressed in terms of the Cartan normal conformal connection $\tilde{\omega}$, are[3]

$$(i)\ \ \mathrm{d}*\tilde{\Omega}+\tilde{\omega}\wedge *\tilde{\Omega}-*\tilde{\Omega}\wedge\tilde{\omega}=0, \qquad \text{and} \qquad (ii)\ \ [\tilde{\Omega}^{+}{}_{\mu\nu},\tilde{\Omega}^{-}{}_{\rho\sigma}]=0, \tag{12}$$

where $\tilde{\Omega}^{\pm}=\frac{1}{2}\tilde{\Omega}^{\pm}{}_{\mu\nu}\theta^\mu\wedge\theta^\nu$. Note that condition (i) is equivalent to the vanishing of the Bach tensor of g.

Calculating $[\tilde{\Omega}^{-}{}_{32},\tilde{\Omega}^{+}{}_{31}]$ for the Fefferman metrics (8) yields

$$[\tilde{\Omega}^{-}{}_{32},\tilde{\Omega}^{+}{}_{31}]=i\mathcal{R}\bar{\mathcal{R}}\begin{pmatrix}0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{pmatrix},$$

so that the above condition (ii) is satisfied iff $\mathcal{R}=0$. This means that the corresponding CR-structure is flat. It follows that if $\mathcal{R}=0$ the corresponding Fefferman metrics are conformal to the Minkowski metric. In the non-flat ($\mathcal{R}\neq 0$) case the Fefferman metrics are always not conformal to Einstein metrics. Note also that despite of this fact the principal null direction of the Fefferman metrics (which in the notation of (8) is tangent to the vector filed dual to the form θ^3) is geodesic and shear-free. It has nonvanishing twist and generates a 1-parameter conformal symmetry of g_F.

We close this section with the formula for $\tilde{D}*\tilde{\Omega}=\mathrm{d}*\tilde{\Omega}+\tilde{\omega}\wedge *\tilde{\Omega}-*\tilde{\Omega}\wedge\tilde{\omega}$ which, for the Fefferman metrics (8), reads

$$\tilde{D}*\tilde{\Omega}=-\frac{2}{i}\mathcal{S}_1\begin{pmatrix}0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{pmatrix}\theta^1\wedge\theta^2\wedge\theta^3, \tag{13}$$

where $\mathcal{S}_1$ is defined by (6). This formula implies that the Fefferman metrics (8) satisfy the Bach equations iff

$$\mathcal{S}_1=0, \tag{14}$$

or, what is the same,

$$\mathrm{d}\mathcal{S}\wedge\theta^2\wedge\theta^3\wedge(2\Omega_2+3\bar{\Omega}_2)\wedge\bar{\Omega}_3=0.$$

The only known example of a CR-structure with a Fefferman metric satisfying this condition is presented in [22].

3 Second order ODEs modulo point transformations

A second order ODE

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = Q(\ x,\ y,\ \frac{\mathrm{d}y}{\mathrm{d}x}\ ) \tag{15}$$

[3] It is worthwile to note that for algebraically general metrics the Baston-Mason conditions $(i)-(ii)$ are also sufficient for the conformal Einstein property [1].

for a function $\mathbf{R} \ni x \to y = y(x) \in \mathbf{R}$, can be alternatively written as a system of the two first order ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p, \qquad\qquad \frac{\mathrm{d}p}{\mathrm{d}x} = Q(x,y,p)$$

for two functions $\mathbf{R} \ni x \to y = y(x) \in \mathbf{R}$ and $\mathbf{R} \ni x \to p = p(x) \in \mathbf{R}$. This system defines two (contact) 1-forms

$$\omega^1 = \mathrm{d}y - p\mathrm{d}x, \qquad\qquad \omega^2 = \mathrm{d}p - Q\mathrm{d}x, \tag{16}$$

on a 3-dimensional manifold J^1, the *first jet space*, parametrized by coordinates (x, y, p). All the information about the ODE (15) is encoded in these two forms. For example, any solution to (15) is a curve $\gamma(x) = (\ x, y(x), p(x)\ ) \subset J^1$ on which the forms (16) vanish.

The two contact 1-forms (ω^1, ω^2) can be supplemented by

$$\omega^3 = \mathrm{d}x, \tag{17}$$

so that the three 1-forms $(\omega^1, \omega^2, \omega^3)$ constitute a basis of 1-forms on J^1. This basis will be the basic object of study in the following.

Under the point transformation of variables

$$y \to \tilde{y} = \tilde{y}(x,y), \qquad\qquad x \to \tilde{x} = \tilde{x}(x,y),$$

the function $Q = Q(x, y, y')$ defining the differential equation (15) changes in a rather complicated way. The corresponding change of the basis $(\omega^1, \omega^2, \omega^3)$ is

$$\omega^1 \to \tilde{\omega}^1 = a_1\omega^1, \qquad \omega^2 \to \tilde{\omega}^2 = a_2\omega^2 + a_3\omega^1, \qquad \omega^3 \to \tilde{\omega}^3 = a_4\omega^3 + a_5\omega^1, \tag{18}$$

where a_1, a_2, a_3, a_4, a_5 are real functions on J^1 such that $a_1a_2a_4 \neq 0$ on J^1.

It is now convenient, to introduce the following (a bit unusual) notation. The reason for this will eventually become apparent.

Let $i \neq 0$ denote a *real* number. In addition, let the *real* 1-forms $(\lambda, \mu, \bar{\mu})$ be defined by

$$\lambda = -i\omega^1, \qquad \mu = \omega^2, \qquad \bar{\mu} = \omega^3. \tag{19}$$

It follows from the definition of $(\omega^1, \omega^2, \omega^3)$ that

$$\lambda \wedge \mu \wedge \bar{\mu} \neq 0, \tag{20}$$

$$\mathrm{d}\lambda \wedge \lambda \neq 0, \tag{21}$$

and that the forms $(\lambda, \mu, \bar{\mu})$ are given up to transformations

$$\lambda \to a\lambda, \qquad \mu \to b\mu + c\lambda, \qquad \bar{\mu} \to \bar{b}\bar{\mu} + \bar{c}\lambda, \tag{22}$$

with real functions $a, b, \bar{b}, c, \bar{c}$ such that $ab\bar{b} \neq 0$.
Conversely, given a 3-dimensional manifold N equipped with three real 1-forms $(\lambda, \mu, \bar{\mu})$ satisfying (20)-(21) and defined up to transformations (22), we can associate with them a point equivalent class of a 2nd order ODE as follows. Since $\dim J^1 = 3$ we have

$$\mathrm{d}\lambda \wedge \lambda \wedge \bar{\mu} = 0 \quad \text{and} \quad \mathrm{d}\bar{\mu} \wedge \lambda \wedge \bar{\mu} = 0.$$

Hence the Fröbenius theorem [25] applied to the forms $\lambda, \bar{\mu}$ implies that there exist coordinates (x, y, z) on N such that $\lambda = A\mathrm{d}x + B\mathrm{d}y$ and $\bar{\mu} = C\mathrm{d}x + H\mathrm{d}y$, where A, B, C, H are appropriate functions on N. Thus, modulo the freedom (22), the forms λ, $\bar{\mu}$ can be transformed to $\lambda = \mathrm{d}y - p\mathrm{d}x$, $\bar{\mu} = \mathrm{d}x$, where p is a certain function of coordinates (x, y, z) on N. But, $0 \neq \mathrm{d}\lambda \wedge \lambda = \mathrm{d}p \wedge \mathrm{d}y \wedge \mathrm{d}x$ so (x, y, p) can be considered a new coordinate system on N. In this coordinates the form μ can be written as $\mu = U\mathrm{d}x + V\mathrm{d}y + Z\mathrm{d}p$ so, by means of transformations (22), can be reduced to $\mu = \mathrm{d}p - Q\mathrm{d}x$ with $Q = Q(x, y, p)$ a certain real function on N. Thus, the original forms $(\lambda, \mu, \bar{\mu})$ define a point equivalent class of a second order ODE $y'' = Q(x, y, y')$. The above considerations prove the one-to-one correspondence between second order ODEs given modulo point transformations and equivalence classes of the triples of real 1-forms $(\lambda, \mu, \bar{\mu})$ on 3-manifolds satisfying (20),(21) and given up to (22). This enables us to reformulate an equivalence problem for second order ODEs modulo point transformations in much the same way as an equivalence problem for *nondegenerate* 3-dimensional CR-structures.

Definition 3 *Two second order ODEs, represented, by the respective real 1-forms $(\lambda, \mu, \bar{\mu})$ and $(\lambda', \mu', \bar{\mu}')$, on the respective 3-manifolds N and N', are locally* point equivalent, *iff there exist a local diffeomorphism*

$$\phi : N \to N'$$

and real *functions $a \neq 0, b \neq 0, \bar{b} \neq 0, c, \bar{c}$ on N such that*

$$\phi^*(\lambda') = a\lambda, \qquad \phi^*(\mu') = b\mu + c\lambda, \qquad \phi^*(\bar{\mu}) = \bar{b}\mu + \bar{c}\lambda.$$

This definition, when compared with Definition 2, indicates that we can treat the forms $(\lambda, \mu, \bar{\mu})$ representing second order ODEs as the respective analogs of the forms $(\lambda, \mu, \bar{\mu})$ representing nondegenerate 3-dimensional CR-structures. It also indicates that the solution for the equivalence problem for 2nd order ODEs modulo point transformations should be given by a theorem analogous to Theorem 1. Actually, with the above introduced notation, in which all the three 1-forms $(\lambda, \mu, \bar{\mu})$ are *real*, $i \neq 0$ is a *real* number and the 'bar' symbol merely denotes that a given variable (a function, or a form) is totally *independent* of its non-bared counterpart, we obtain the solution of the equivalence problem for ODEs by the following reinterpretation of Theorem 1. First, given a point equivalence class of 2nd order ODEs, represented by forms $(\lambda, \mu, \bar{\mu})$, we associate with it the forms

$$\theta^1 = b\mu + c\lambda, \qquad \theta^2 = \bar{b}\bar{\mu} + \bar{c}\lambda, \qquad \theta^3 = a\lambda. \tag{23}$$

Then the analog of Theorem 1 is as follows.

Theorem 2 *Every second order ODE given modulo point transformations uniquely defines an 8-dimensional manifold P,* real *1-forms $\theta^1, \theta^2, \theta^3, \Omega_2, \bar{\Omega}_2, \Omega_3, \bar{\Omega}_3, \Omega_4$ and* real *functions $\mathcal{R}, \bar{\mathcal{R}}, \mathcal{S}, \bar{\mathcal{S}}$ on P such that*

- *$\theta^1, \theta^2, \theta^3$ are as in (23),*
- *$\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \Omega_2 \wedge \bar{\Omega}_2 \wedge \Omega_3 \wedge \bar{\Omega}_3 \wedge \Omega_4 \neq 0$ at each point of P.*

The forms satisfy the following equations

$$\begin{aligned}
\mathrm{d}\theta^1 &= \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^3 \\
\mathrm{d}\theta^2 &= \bar{\Omega}_2 \wedge \theta^2 + \bar{\Omega}_3 \wedge \theta^3 \\
\mathrm{d}\theta^3 &= i\theta^1 \wedge \theta^2 + (\Omega_2 + \bar{\Omega}_2) \wedge \theta^3 \\
\mathrm{d}\Omega_2 &= 2i\theta^1 \wedge \bar{\Omega}_3 + i\theta^2 \wedge \Omega_3 + \Omega_4 \wedge \theta^3 \\
\mathrm{d}\bar{\Omega}_2 &= -2i\theta^2 \wedge \Omega_3 - i\theta^1 \wedge \bar{\Omega}_3 + \Omega_4 \wedge \theta^3 \\
\mathrm{d}\Omega_3 &= \Omega_4 \wedge \theta^1 + \Omega_3 \wedge \bar{\Omega}_2 + \mathcal{R}\theta^2 \wedge \theta^3 \\
\mathrm{d}\bar{\Omega}_3 &= \Omega_4 \wedge \theta^2 + \bar{\Omega}_3 \wedge \Omega_2 + \bar{\mathcal{R}}\theta^1 \wedge \theta^3 \\
\mathrm{d}\Omega_4 &= i\Omega_3 \wedge \bar{\Omega}_3 + \Omega_4 \wedge (\Omega_2 + \bar{\Omega}_2) + \bar{\mathcal{S}}\theta^1 \wedge \theta^3 + \mathcal{S}\theta^2 \wedge \theta^3.
\end{aligned} \tag{24}$$

The functions $\mathcal{R}$, $\bar{\mathcal{R}}$, $\mathcal{S}$, $\bar{\mathcal{S}}$, satisfy in addition

$$\begin{aligned} \mathrm{d}\mathcal{R} &= -\mathcal{R}(\Omega_2 + 3\bar{\Omega}_2) - \mathcal{S}\theta^1 + \mathcal{R}_2\theta^2 + \mathcal{R}_3\theta^3 \\ \mathrm{d}\bar{\mathcal{R}} &= -\bar{\mathcal{R}}(\bar{\Omega}_2 + 3\Omega_2) - \bar{\mathcal{S}}\theta^2 + \bar{\mathcal{R}}_2\theta^1 + \bar{\mathcal{R}}_3\theta^3 \end{aligned} \tag{25}$$

and

$$\begin{aligned} \mathrm{d}\mathcal{S} &= -\mathcal{S}(2\Omega_2 + 3\bar{\Omega}_2) - i\mathcal{R}\bar{\Omega}_3 + \mathcal{S}_1\theta^1 + \mathcal{S}_2\theta^2 + \mathcal{S}_3\theta^3 \\ \mathrm{d}\bar{\mathcal{S}} &= -\bar{\mathcal{S}}(2\bar{\Omega}_2 + 3\Omega_2) + i\bar{\mathcal{R}}\Omega_3 + \bar{\mathcal{S}}_2\theta^1 + \mathcal{S}_1\theta^2 + \bar{\mathcal{S}}_3\theta^3, \end{aligned}$$

with appropriate functions $\mathcal{R}_2, \bar{\mathcal{R}}_2, \mathcal{R}_3, \bar{\mathcal{R}}_3, \mathcal{S}_1, \bar{\mathcal{S}}_1, \mathcal{S}_2, \bar{\mathcal{S}}_2, \mathcal{S}_3, \bar{\mathcal{S}}_3$.
The function $\mathcal{S}_1$ satisfies

$$\mathcal{S}_1 = \bar{\mathcal{S}}_1. \tag{26}$$

Given an equation $y'' = Q(x, y, y')$ and the standard coordinate system (x, y, p) on $N = J^1$ we introduce the vector field

$$D = \partial_x + p\partial_y + Q\partial_p, \qquad Q = Q(x, y, p).$$

The coordinates (x, y, p) can be extended to a coordinate system $(x, y, p, \rho, \phi, \gamma, \bar{\gamma}, r)$ on P in which the forms and functions of the above theorem can be written as follows:

$$\lambda = -i(\mathrm{d}y - p\mathrm{d}x), \qquad \mu = \mathrm{d}p - Q\mathrm{d}x, \qquad \bar{\mu} = \mathrm{d}x$$

$$\theta^1 = \rho\mathrm{e}^{i\phi}(\mu + \gamma\lambda), \qquad \theta^2 = \rho\mathrm{e}^{-i\phi}(\bar{\mu} + \bar{\gamma}\lambda), \qquad \theta^3 = \rho^2\lambda \tag{27}$$

$$\Omega_2 = i\mathrm{d}\phi + \tfrac{\mathrm{d}\rho}{\rho} + \tfrac{1}{4i\rho^2}\ [\ 6\gamma\bar{\gamma}i^2 - 6\bar{\gamma}iQ_p - Q_{pp} - 4ir\rho\ ]\ \theta^3 - \tfrac{2i\bar{\gamma}}{\rho}\mathrm{e}^{-i\phi}\theta^1 - \tfrac{\mathrm{e}^{i\phi}}{\rho}(i\gamma - Q_p)\theta^2$$

$$\bar{\Omega}_2 = -i\mathrm{d}\phi + \tfrac{\mathrm{d}\rho}{\rho} - \tfrac{1}{4i\rho^2}\ [\ 6\gamma\bar{\gamma}i^2 - 2\bar{\gamma}iQ_p - Q_{pp} + 4ir\rho\ ]\ \theta^3 + \tfrac{i\bar{\gamma}}{\rho}\mathrm{e}^{-i\phi}\theta^1 + \tfrac{\mathrm{e}^{i\phi}}{\rho}(2i\gamma - Q_p)\theta^2$$

$$\begin{aligned} \Omega_3 = \tfrac{\mathrm{e}^{i\phi}}{\rho}\ [\ \mathrm{d}\gamma &- \tfrac{1}{6i^2\rho^2}(DQ_{pp} + 6\gamma^2\bar{\gamma}i^3 - 6\gamma\bar{\gamma}i^2Q_p - 3\gamma iQ_{pp} - 4Q_{py} - 6\bar{\gamma}iQ_y)\theta^3 + \\ &\tfrac{\mathrm{e}^{-i\phi}}{4i\rho}(2\gamma\bar{\gamma}i^2 - 2\bar{\gamma}iQ_p - Q_{pp} - 4ir\rho)\theta^1 + \tfrac{\mathrm{e}^{i\phi}}{i\rho}(\gamma^2i^2 - \gamma iQ_p - Q_y)\theta^2\ ] \end{aligned}$$

$$\begin{aligned} \bar{\Omega}_3 = \tfrac{\mathrm{e}^{-i\phi}}{\rho}\ [\ \mathrm{d}\bar{\gamma} &+ \tfrac{1}{6i^2\rho^2}(6\gamma\bar{\gamma}^2i^3 - 6\bar{\gamma}^2i^2Q_p - 3\bar{\gamma}iQ_{pp} - Q_{ppp})\theta^3 - \tfrac{\mathrm{e}^{-i\phi}}{\rho}\bar{\gamma}^2i\theta^1 - \\ &\tfrac{\mathrm{e}^{i\phi}}{4i\rho}(2\gamma\bar{\gamma}i^2 - 2\bar{\gamma}iQ_p - Q_{pp} + 4ir\rho)\theta^2\ ] \end{aligned}$$

$$\begin{aligned} \Omega_4 = &-\tfrac{i}{2\rho^2}\bar{\gamma}\mathrm{d}\gamma + \tfrac{1}{2\rho^2}(i\gamma - Q_p)\mathrm{d}\bar{\gamma} - \tfrac{\mathrm{d}r}{\rho} - \tfrac{r\mathrm{d}\rho}{\rho^2} + \\ &\tfrac{1}{48i^2\rho^4}\ [\ 8DQ_{ppp} - 3Q_{pp}^2 + 8Q_pQ_{ppp} - 12Q_{ppy} - 12\gamma iQ_{ppp} + \bar{\gamma}(12iDQ_{pp} - 24iQ_{py}) - 12\gamma\bar{\gamma}i^2Q_{pp} + \\ &\bar{\gamma}^2(24i^2DQ_p + 12i^2Q_p^2 - 48i^2Q_y) - 48i^3\gamma\bar{\gamma}^2Q_p + 36\gamma^2\bar{\gamma}^2i^4 + 48\bar{\gamma}r\rho i^2Q_p + 48i^2\rho^2r^2\ ]\ \theta^3 - \\ &\tfrac{\mathrm{e}^{i\phi}}{12i\rho^3}\ [\ 6\gamma\bar{\gamma}^2i^3 - 6\bar{\gamma}^2i^2Q_p + 3\bar{\gamma}iQ_{pp} - Q_{ppp} - 12\bar{\gamma}i^2r\rho\ ]\ \theta^1 - \\ &\tfrac{\mathrm{e}^{i\phi}}{12i\rho^3}\ [\ DQ_{pp} - 4Q_{py} - 3i\gamma Q_{pp} + 6i\bar{\gamma}(DQ_p - 2Q_y) - 6i^2\gamma\bar{\gamma}Q_p + 6\gamma^2\bar{\gamma}i^3 + 12\gamma i^2r\rho\ ]\ \theta^2, \end{aligned}$$

$$\mathcal{R} = -\tfrac{\mathrm{e}^{2i\phi}}{6i^2\rho^4}w_1, \qquad \mathcal{S} = -\tfrac{\mathrm{e}^{i\phi}}{3i^2\rho^5}\ [\partial_pw_1 + i\bar{\gamma}w_1\ ], \tag{28}$$

$$\bar{\mathcal{R}} = -\tfrac{\mathrm{e}^{-2i\phi}}{6i^2\rho^4}w_2, \qquad \bar{\mathcal{S}} = -\tfrac{\mathrm{e}^{-i\phi}}{3i^2\rho^5}\ [\ Dw_2 + (2Q_p - i\gamma)w_2\ ], \tag{29}$$

where we have introduced functions

$$w_1 = D^2Q_{pp} - 4DQ_{py} - DQ_{pp}Q_p + 4Q_pQ_{py} - 3Q_{pp}Q_y + 6Q_{yy},$$

and

$$w_2 = Q_{pppp}$$

which are the relative point invariants of the ODE.

Similarly as in the CR case, Theorem 2 can be reinterpreted in terms of the language of Cartan connections. It follows that the manifold P of Theorem 2 is a Cartan bundle $H \to P \to J^1$, with H a 5-dimensional parabolic subgroup of $\mathbf{SL}(3, \mathbf{R})$ group. The forms $\theta^1, \theta^2, \theta^3, \Omega_2, \bar{\Omega}_2, \Omega_3, \bar{\Omega}_3, \Omega_4$ of the theorem can be collected into a matrix of 1-forms

$$\omega = \begin{pmatrix} \frac{1}{3}(2\Omega_2 + \bar{\Omega}_2) & i\bar{\Omega}_3 & -\frac{1}{2}\Omega_4 \\ \theta^1 & \frac{1}{3}(\bar{\Omega}_2 - \Omega_2) & -\frac{1}{2}\Omega_3 \\ 2\theta^3 & 2i\theta^2 & -\frac{1}{3}(2\bar{\Omega}_2 + \Omega_2), \end{pmatrix}$$

which is now an $\mathbf{sl}(3, \mathbf{R})$-valued Cartan connection on P (all the variables are real!). It follows from equations (24) that the curvature of this connections is

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega = \begin{pmatrix} 0 & i\bar{\mathcal{R}}\theta^1 \wedge \theta^3 & -\frac{1}{2}\bar{\mathcal{S}}\theta^1 \wedge \theta^3 - \frac{1}{2}\mathcal{S}\theta^2 \wedge \theta^3 \\ 0 & 0 & -\frac{1}{2}\mathcal{R}\theta^2 \wedge \theta^3 \\ 0 & 0 & 0 \end{pmatrix}.$$

It yields all the invariant information about the corresponding point equivalent class of second order ODEs. In particular, the ODEs corresponding to flat ($\mathcal{R} = 0$, $\bar{\mathcal{R}} = 0$, $\mathcal{S} = 0$, $\bar{\mathcal{S}} = 0$) connections are given by the conditions

$$w_1 = 0, \qquad w_2 = 0.$$

They are all point equivalent to the flat equation $y'' = 0$. We remark that the vanishing of $\mathcal{R}$ implies vanishing of $\mathcal{S}$. Each of these two conditions is a point invariant property of the corresponding ODE. However, the other pair of point invariant conditions $\bar{\mathcal{R}} = 0$, $\bar{\mathcal{S}} = 0$ is totally independent. This is the significant difference between the behaviour of CR structures and 2nd order ODEs. Indeed, in the classification of nondegenerate 3-dimensional CR-structures there are only two major branches: either $\mathcal{R} = 0$ (in which case the CR-structure is locally equivalent to $\mathbf{S}^3 \subset \mathbf{C}^2$) or $\mathcal{R} \neq 0$. In the ODE case $\mathcal{R}$ and $\bar{\mathcal{R}}$ are unrelated and we have four main branches, corresponding to (i) $\mathcal{R} = 0$, $\bar{\mathcal{R}} = 0$, (ii) $\mathcal{R} = 0$, $\bar{\mathcal{R}} \neq 0$, (ii') $\mathcal{R} \neq 0$, $\bar{\mathcal{R}} = 0$, and (iii) $\mathcal{R} \neq 0$, $\bar{\mathcal{R}} \neq 0$. It follows that branches (ii) and (ii') are, in a sense, dual to each other. To explain this duality we need to introduce the Fefferman metric associated with an ODE.

The system (24) defining the invariant forms of Theorem 2 has all the qualitative properties of system (4) of Theorem 1. Thus, introducing the basis $X_1, X_2, X_3, Y_2, \bar{Y}_2, Y_3, \bar{Y}_3, Y_4$ of vector fields, the respective dual of forms $\theta^1, \theta^2, \theta^3$, $\Omega_2, \bar{\Omega}_2, \Omega_3, \bar{\Omega}_3, \Omega_4$, we see that the distribution spanned by the four vector fields $Z_I = (Y_2 + \bar{Y}_2, Y_3, \bar{Y}_3, Y_4)$ is integrable. Moreover, the bilinear form

$$G = 2\theta^1\theta^2 + \frac{2}{3i}\theta^3(\Omega_2 - \bar{\Omega}_2),$$

which now has signature $(++--0000)$, has all Z_Is as degenerate directions. This, when compared with the fact that G is preserved up to a scale during the Lie transport along Z_Is, shows that the 4-dimensional space $P/\sim$ of leaves of the distribution spanned by Z_Is is naturally equipped with the conformal class of split signature metrics $[g_F]$, the class to which the bilinear form G naturally descends. We call the metrics

$$g_F = 2\theta^1\theta^2 + \frac{2}{3i}\theta^3(\Omega_2 - \bar{\Omega}_2) \tag{30}$$

on $P/\sim$ the Fefferman metrics associated with a point equivalence class of a second order ODE $y'' = Q(x, y, y)$.

The metrics g_F, when written in coordinates $(x, y, p, \rho, \phi, \gamma, \bar{\gamma}, r)$ on P, read

$$g_F = 2\rho^2 \ [\ (\mathrm{d}p - Q\mathrm{d}x)\mathrm{d}x - (\mathrm{d}y - p\mathrm{d}x)(\frac{2}{3}\mathrm{i}\mathrm{d}\phi + \frac{2}{3}Q_p\mathrm{d}x + \frac{1}{6}Q_{pp}(\mathrm{d}y - p\mathrm{d}x))\ ]. \tag{31}$$

This enables us to (locally) identify the space parametrized by (x, y, p, ϕ) with $P/\sim$ and the space parametrized by (x, y, p, ϕ, ρ) with the space of all Fefferman metrics associated with a given $y'' = Q(x, y, y')$.

Proposition 1 *The Fefferman conformal class of metrics $[g_F]$ associated with a point equivalent class of ODEs has the following properties.*

- *Each g_F has signature $(++--)$*
- *The Weyl tensor of each g_F has both, the self-dual and the anti-self-dual parts of Petrov type N. The self-dual part C^+ is proportional to $\mathcal{R}$ and the anti-self-dual part C^- is proportional to $\bar{\mathcal{R}}$.*
- *g_F satisfies the Baston-Mason conditions (12) if and only if the corresponding point equivalent class of equations satisfies either $\mathcal{R} = 0$ or $\bar{\mathcal{R}} = 0$.*

The first two statements of the above proposoition are abvious in view of formulae (9)and (11). To prove the last statement we calculate the Baston-Mason conditions (12) in coordinates (x, y, p, ϕ). A short calculation and the identity

$$w_{1pp} = (D^2 + 3Q_pD + 2DQ_p + 2Q_p^2 - Q_y)w_2 \tag{32}$$

show that these conditions are equivalent to

$$(i') \quad w_{1pp} = 0 \qquad\qquad \text{and} \qquad\qquad (ii') \quad w_1w_2 = 0, \tag{33}$$

where (i') corresponds to the vanishing of the Yang-Mills current of the Cartan normal conformal connection $\tilde{\omega}$ associated with g_F via (9) and (ii') corresponds to the Baston-Mason condition (ii) of (12). Comparing (32) and (33) proves that the necessary and sufficient conditions for (12) are $w_1 = 0$ or $w_2 = 0$. This in particular means that such metrics must be either self-dual or anti-selfdual.

Remark
Note that one of the principal null directions of g_F, generated by the vector field dual to the form θ^3 of (30), is a conformal Killing vector field for g_F. It generates a congruence of null shear-free and twisting geodesics on $P/\sim$. This statement, together with the above Proposition 1 totally characterizes the Fefferman metrics g_F [35].

Since metrics g_F are algebraically special (of type $N \times N'$ or its specializations) the Baston-Mason conditions (12) are not sufficient to guarantee the conformal Einstein property for them. All the Fefferman metrics which are conformal to Einstein metrics are given in the Appendix.

It is very easy to determine all classes of second order ODEs correponding to the self-dual Fefferman metrics. These are all equations for which $w_2 = 0$, i.e. all the equations of the form

$$y'' = A_0(x,y) + A_1(x,y)y' + A_2(x,y)(y')^2 + A_3(x,y)(y')^3,$$

for which the function Q is an arbitrary polynomial of the third order in the variable p. Finding classes of equations corresponding to the anti-self-dual metrics is more difficult but, surprisingly, possible, due to another notion of duality: the duality of second order ODEs.

Given a second order ODE in the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = Q(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}) \tag{34}$$

consider its general solution $y = y(x, X, Y)$, where X, Y are constants of integration. In the space $\mathbf{R}^2 \times \mathbf{R}^2$ parametrized by (x, y, X, Y) this solution can be considered a 3-dimensional hypersurface

$$N = \{(x, y, X, Y) \in \mathbf{R}^2 \times \mathbf{R}^2 \mid G(x, y, X, Y) = y - y(x, X, Y) = 0\ \}.$$

Assuming that $G(x, y, X, Y) = 0$ can be solved with respect to Y one gets a function $Y = Y(X, x, y)$. Treating x and y as constant parameters, we can eliminate them by double differentiation of Y with respect to X. This means that $Y = Y(X, x, y)$ can be considered a solution to a second order ODE

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}X^2} = q(X, Y, \frac{\mathrm{d}Y}{\mathrm{d}X}). \tag{35}$$

In a passage from (34) to (35) we have chosen X to be an independent variable and Y to be the dependent one. But another choices are possible. In general, we could have chosen two independent functions $\xi = \xi(X, Y)$ and $\zeta = \zeta(X, Y)$ and have treated ξ and ζ as an independent and dependent variables, respectively. Then, after double differentiation with respect to ξ, which eliminates the parameters (x, y), we would see that the function ζ also satisfied a second order ODE, which would be quite different then (35). It is, however, obvious that this other second order ODE would be in the same point equivalence class as (35). Thus, given a point equivalence class of ODEs generated by (34) there is a uniquely defined equivalence class of ODEs (35) associated with it. The class (35) is called a class of the *dual* equations to the equations from the class (34). The following example shows the usefulness of this concept.

Example
The relative invariants w_1 and w_2 calculated for the second order ODE

$$y'' = \frac{a}{y^3}, \tag{36}$$

where a is a real constant, read

$$w_1 = \frac{72a}{y^5} \qquad \text{and} \qquad w_2 = 0.$$

Therefore the Fefferman metrics

$$g_F = 2\rho^2\ [\ (\mathrm{d}p - \frac{a}{y^3}\mathrm{d}x)\mathrm{d}x - \frac{2}{3}i(\mathrm{d}y - p\mathrm{d}x)\mathrm{d}\phi\ ] \tag{37}$$

associated with the point equivalent class of ODEs generated by (36) are self-dual but not conformally flat [4].
The general solution $y = y(x, X, Y)$ of (36) depends on two arbitrary constants of integration (X, Y) and satisfies
$$y^2 = Y(x - X)^2 + \frac{a}{Y}. \tag{38}$$
This generates a hypersurface
$$N = \{(x, y, X, Y) \in \mathbf{R}^2 \times \mathbf{R}^2 \mid y^2 = \frac{a}{Y} + Y(x - X)^2 \}$$
in $\mathbf{R}^2 \times \mathbf{R}^2$. Now, we will treat equation (38) as an equation for a function $Y = Y(X, x, y)$ of an independent variable X and parametrized by x and y. Differentiating (38) with respect to X and keeping x and y constants we get
$$0 = Y'(x - X)^2 - 2Y(x - X) - a\frac{Y'}{Y^2}.$$
Solving for x and differentiating once more with respect to X we find that $Y = Y(X)$ satisfies the second order ODE
$$Y'' = -\frac{-Y^4Y'^2 + aY'^4 - 2Y^2Y'^2\sqrt{Y^4 + aY'^2}}{Y^5 + Y^3\sqrt{Y^4 + aY'^2}}.$$
This is an equation which generates the point equivalence class of equations dual to (36). This equation has $w_2 \neq 0$ for each $a \neq 0$. A direct (but lenghty!) calculation shows that
$$q = q(X, Y, P) = -\frac{-Y^4P^2 + aP^4 - 2Y^2P^2\sqrt{Y^4 + aP^2}}{Y^5 + Y^3\sqrt{Y^4 + aP^2}}$$
has $w_1 = 0$. This is a general fact known already to Elie Cartan [4]. More formally, we have the following Proposition.

Proposition 2 *The point equivalence class of dual ODEs to a point equivalence class of 2nd order ODEs for which $w_2 = 0$ and $w_1 \neq 0$ has $w_2 \neq 0$ and $w_1 = 0$.*

We have already noted (in the Example above) that this proposition enables one to find nontrivial solutions to a quite complicated differential equation $w_1 = 0$. Note also that applying the Proposition one can obtain quite nontrivial anti-self-dual metrics from a rather dull ones (Calculate the Fefferman metrics for q of the Example, and compare it with (37)). Finally, note that it follows from the Proposition that in the classification scheme of the 2nd order ODEs modulo point transformations, the classification of the $w_1 = 0$ and $w_2 \neq 0$ case can be obtained from the classification of the simpler $w_2 = 0$ and $w_1 \neq 0$ case.

In Ref. [4] Proposition 2 is only briefly mentioned[5]. It could be proven by the following line of argument.

[4] These are actually the Sparling-Tod metrics well known in the twistor theory [36]

[5] For an unexperienced reader it is rather hard to find a trace of the Proposition in the text of Ref. [4]. We are very grateful to Mike Crampin [6] for clarifying this point for us. We also take this opportunity to present his understanding of the last paragraph of Cartan's paper [4]. Due to equations (27), (28)-(29), which represent the transformation properties of the forms $(\omega^1, \omega^2, \omega^3)$ and the relative invariants w_1 and w_2, if both w_1 and w_2 are nonvanishing, the following forms are (modulo sign) well defined on J^1
$$I_1 = (w_1w_2)^{\frac{1}{4}}\omega^1, \qquad I_2 = (w_1w_2)^{\frac{1}{2}}\omega^1 \wedge \omega^2 \wedge \omega^3$$
$$I_3 = w_1^{\frac{1}{8}}w_2^{\frac{5}{8}}\omega^1 \wedge \omega^2, \qquad I_4 = w_1^{\frac{5}{8}}w_2^{\frac{1}{8}}\omega^1 \wedge \omega^3.$$

The switch between the dual equations $y''=Q(x,y,y')$ and $Y''=q(X,Y,Y')$ essentially means the switch between the contact forms $\omega^2={\rm d}p-Q{\rm d}x$ and $\omega^3={\rm d}x$. To see this consider the general solution $y=y(x,X,Y)$ of the original equation. Now, pass from the canonical coordinates (x,y,p) on the first jet bundle J^1 to the new coordinates (s,X,Y) defined via

$$x=s,\qquad y=y(s,X,Y)\qquad\text{and}\qquad p=y_s(s,X,Y).$$

In these new coordinates the contact forms

$$\omega^1={\rm d}y-p{\rm d}x,\qquad \omega^2={\rm d}p-Q{\rm d}x,\qquad \omega^3={\rm d}x$$

associated with the original equation are given by

$$\omega^1=y_X{\rm d}X+y_Y{\rm d}Y,\qquad \omega^2=y_{sX}{\rm d}X+y_{sY}{\rm d}Y,\qquad \omega^3={\rm d}s.$$

Thus in the point equivalence class of the forms ω^1 and ω^2 there are forms

$$\omega^1={\rm d}Y+\frac{y_X}{y_Y}{\rm d}X\qquad\text{and}\qquad \omega^2={\rm d}X.$$

Moreover the condition $0\neq{\rm d}\omega^1\wedge\omega^1={\rm d}(\frac{y_X}{y_Y})\wedge{\rm d}X\wedge{\rm d}Y$ implies that the three functions

$$X,\qquad Y\qquad\text{and}\qquad P=-\frac{y_X}{y_Y}$$

constitute a coordinate system on J^1. Therefore s must be a function of these three variables: $s=s(X,Y,P)$. This means that in the equivalence class of the form

$$\omega^3={\rm d}s=s_X{\rm d}X+s_Y{\rm d}Y+s_P{\rm d}P$$

there is ω^3 which can be writen as

$$\omega^3={\rm d}P-q(X,Y,P){\rm d}X,\qquad\qquad\text{where}\qquad\qquad q=-\frac{s_X+s_YP}{s_P}.$$

Summarizing, we are able to introduce a coordinate system (X,Y,P) on the first jet bundle J^1 in which the point equivalence class of the contact forms associated with the original equation $y''=Q(x,y,y')$ can be written as

$$\omega^1={\rm d}Y-P{\rm d}X,\qquad \omega^2={\rm d}X,\qquad \omega^3={\rm d}P-q(X,Y,P){\rm d}X.$$

But this enables us to interpret the (X,Y,P) coordinates as cannonical coordinates for the first jet bundle associated with the contact forms

$$\omega^1={\rm d}Y-P{\rm d}X,\qquad \omega^3={\rm d}P-q(X,Y,P){\rm d}X,\qquad \omega^2={\rm d}X$$

of the differential equation $Y''=q(X,Y,Y')$. Because of the original definitions of X and Y this is clearly the dual equation to $y''=Q(x,y,y')$. Note that this interpretation requires the switch between the forms ω^2 and ω^3. Note also that this switch is compatible with transformations (18) which treat ω^2 and ω^3 on the equal footing mixing each of them with ω^1 only.

Once the switch between $y''=Q(x,y,y')$ and $Y''=q(X,Y,Y')$ is understood from the point of view of the switch between ω^2 and ω^3 it is easy to see that the passage from a differential equation to its dual changes the role of the invariants w_1 and w_2. Indeed, looking at the curvature Ω of the Cartan connection ω associated with the equation $y''=Q(x,y,y')$ we see that the invariant w_1 is associated with the $\omega^3 \wedge \omega^1$ term and the invariant w_2 is associated with the $\omega^2 \wedge \omega^1$ term. Thus, the switch between ω^2 and ω^3 caused by the switch between the equation and its dual, amounts in the switch of w_1 and w_2.

From the geometric point of view the switch between the mutually dual 2nd order ODEs can be understood as a transformation that interchanges two naturally defined congruences of lines on the jet bundle J^1. This bundle is naturally fibred over J^0 - the plane parametrized by (x, y). The fibres of $J^1 \to J^0$ are 1-dimensional and, in the natural coordinates (x, y, p) on J^1, can be specified by fixing x and y. They generate the first congruence of lines on J^1. The other congruence is defined by the point equivalence class of the equation $y'' = Q(x, y, y')$ in the following way. The equation $y'' = Q(x, y, y')$ equippes J^1 with the total differential vector field $X_{\bar{\mu}} = D$. Any other equation from the point equivalence class of $y'' = Q(x, y, y')$ defines the total differential that differs from D by a scaling functional factor. Thus the lines tangent to all of these total diferentials are well defined on J^1 and generate the second natural congruence of lines. Each of the above congruences on J^1 defines a natural direction of vector fields tangent to them but, in the canonical coordinates (x, y, p) on J^1, only the lines of the first congruence can be defined as lines tangent to a particularly simple vector field $X_\mu = \partial_p$. The passage from the equation to its dual changes the picture: it switches between X_μ and $X_{\bar{\mu}}$, so that the jet bundle J^1 is now interpretd as a bundle with 1-dimensional fibres tangent to $X_{\bar{\mu}} = \partial_P$. The space of such fibres may then be identified with the plane parametrized by (X, Y) and the congruence tangent to X_μ by the congruence tangent to the total differential of the dual equation.

4 Realification of a 3-dimensional CR-structure

The analogy between 3-dimensional CR-structures and 2nd order ODEs described in the previous two sections can be used to associate a point equivalent class of 2nd order ODEs with a 3-dimensional CR-structure. This may introduce a new insight in General Relativity, where the 3-dimensional CR structures correspond [37, 39, 40] to congruences of shear-free and null geodesics in space-times[6]. In particular, the shear-free congruence of null geodesics associated with the celebrated Kerr space-time, can be interpreted in terms of a certain class of point equivalent second order ODEs. In this section we provide a framework for this kind of considerations concentrating on 3-dimensional CR-structures that admit 2-dimensional group of local symmetries[7].

A 3-dimensional CR-structure with two symmetries can be locally described by real coordinates (u, x, y) in which a representative of the forms from the class $[(\lambda, \mu)])$ is given by

$$\lambda = \mathrm{d}u + f(y)\mathrm{d}x, \qquad \mu = \mathrm{d}x + i\mathrm{d}y, \qquad \bar{\mu} = \mathrm{d}x - i\mathrm{d}y. \tag{39}$$

Here the function $f = f(y)$ is real and $i^2 = -1$.

To pass from the above CR-structures to the corresponding point equivalence class of 2nd order ODEs we require that the symbol i is a *nonvanishing real constant* so that the forms (39) become *real* and we can interpret them as the forms that via (22) define a point equivalence class of 2nd order ODEs. To find a particular representative of on ODE in this class we introduce new coordinates $(\bar{u}, \bar{x}, \bar{y})$, which are related to (u, x, y) by

$$u = \bar{u}, \qquad y = \bar{y}, \qquad x = \bar{x} + i\bar{y}.$$

[6] Such space-times, the Lorentzian 4-dimensional manifolds admiting a null congruence of shear-free geodesics, are called *Robinson manifolds* [24, 41, 42] and are known to be the analogs of Hermitian manifolds of four dimensional Riemannian geometry.

[7] Note that the Kerr congruence is in this class of examples [20]

Since i is now *real* and nonvanishing this is a *real* transformation of the coordinates. It brings $(\lambda,\mu,\bar\mu)$ to the form

$$\lambda=\mathrm{d}(\ \bar u+i\int f(\bar y\ )\mathrm{d}\bar y\ )\ +\ f(\bar y)\mathrm{d}\bar x,\qquad\qquad \mu=\mathrm{d}\bar x+2i\mathrm{d}\bar y,\qquad\qquad \bar\mu=\mathrm{d}\bar x.$$

After another coordinate transformation

$$X=\bar x,\qquad Y=\bar u+i\int f(\bar y)\mathrm{d}\bar y,\qquad P=-f(\bar y)$$

and an application of the chain rule one sees that in the class (22) of 1-forms $(\lambda,\mu,\bar\mu)$ there are forms

$$\lambda=\mathrm{d}Y-P\mathrm{d}X,\qquad \mu=\mathrm{d}P\ -\ \frac{1}{2i}\ f'(y)_{|_{y=f^{-1}(-P)}}\mathrm{d}X,\qquad \bar\mu=\mathrm{d}X.$$

This means thet the point equivalence classes of 2nd order ODEs associated with the CR-structures generated by $f=f(y)$ are represented by the equations of the form

$$Y''=\frac{1}{2i}f'(y)_{|_{y=f^{-1}(-Y')}}.$$

Consider, in particular, the family of CR-structures which have 3-dimensional symmetries of Bianchi type VI_k [23]. They are parametrized by the Bianchi type parameter $k<0$ and are represented by the function $f=y^n$, where $n=n(k)$ is an apropriate [23] function of k. Then the point equivalent class of ODEs associated with each n is given by

$$Y''=\frac{n}{2i}(-Y')^{1-\frac{1}{n}}. \qquad (40)$$

The application of the above results to the new understanding of the congruences of shear-free and null geodesics in space-time and, in particular, to the Kerr congruence[8] may be of some use in General Relativity Theory. Here we focus on properties of Fefferman metrics associated with equations (40). It is known [22] that the Fefferman metrics associated with the 3-dimensional CR-structures admitting three symmetries of Bianchi type corresponding to $n=-3$ satisfy the Bach equations. They are the only known *Lorentzian* metrics of twisting type N which satisfy the Bach equations and which are not conformal to Einstein metrics. Similarly the Fefferman metrics for the equations point equivalent to the equation (40) with $n=-3$ provide an example of *split* signature metrics of type $N\times N'$ which satisfy the Bach equations and which are never conformal to Einstein metrics. Explicitely, these metrics are conformal to

$$g_F=2\ (\ iP^{\frac{4}{3}}\mathrm{d}P+\frac{3}{2}P^2\mathrm{d}X\ )\ \mathrm{d}X-2\ (\ \mathrm{d}Y-P\mathrm{d}X\ )\ (\ \frac{2}{3}i^2P^{\frac{2}{3}}\mathrm{d}\Phi-\frac{1}{9}\mathrm{d}Y-\frac{11}{9}P\mathrm{d}X\ ).$$

These metrics can be generalized by considering 2nd order ODEs point equivalent to

$$y''=h(y'),$$

where $h=h(p)$ is a sufficiently smooth real function. Then the Fefferman metrics associated with such equations are given by

$$g_F=2\rho^2\ [\ (\mathrm{d}p-h\mathrm{d}x)\mathrm{d}x-(\mathrm{d}y-p\mathrm{d}x)(\frac{2}{3}i\mathrm{d}\phi+\frac{2}{3}h'\mathrm{d}x+\frac{1}{6}h''(\mathrm{d}y-p\mathrm{d}x))\ ]. \qquad (41)$$

[8] This congruence may be represented by $f=\frac{1}{\cosh^2(y)}$ [20].

The invariants w_1 and w_2 for these metrics are

$$w_1 = h^2 h'''', \qquad\qquad w_2 = h''''$$

so the metrics are not conformal to Einstein metrics iff

$$h'''' \neq 0.$$

This, together with (33), shows that every solution to the equation

$$h^2 h'''' = ap + b,$$

where a and b are real constants such that at least one of them does not vanish defines, via (41), a *split* signature conformal class of metrics that satisfy the Bach equations, are of type $N \times N'$ and are not conformal to Einstein metrics. By an appropriate complexfication of these metrics one may get the generalization of the Lorentzian Bach non-Einstein metrics (45) of Ref. [22].

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6 Appendix

In this Appendix we find all Fefferman metrics (31) which are conformally equivalent to the Einstein metrics. First, we write them in the form

$$g_F = 2\theta^1\theta^2 + \theta^3\theta^4,$$

with

$$\begin{aligned} \theta^1 &= \rho(\mathrm{d}p - Q\mathrm{d}x) \\ \theta^2 &= \rho\mathrm{d}x \\ \theta^3 &= -\rho^2(\mathrm{d}y - p\mathrm{d}x) \\ \theta^4 &= \tfrac{2}{3}i\mathrm{d}\phi + \tfrac{2}{3}Q_p\mathrm{d}x + \tfrac{1}{6}Q_{pp}(\mathrm{d}y - p\mathrm{d}x), \end{aligned} \tag{42}$$

where $\rho = \rho(x, y, p, \phi)$ is a function on J^1. We search for all ρ and $Q = Q(x, y, p)$ for which the Einstein equations

$$Ric(g_F) = \Lambda g_F \tag{43}$$

are satisfied. Assuming that g_F obey equations (43) we use the implied Weyl tensor identity $\nabla_\mu C^\mu{}_{\nu\rho\sigma} = 0$. Its $\{113\}$ and $\{223\}$ components imply the equations

$$(3\rho_\phi - i\rho)w_2 = 0 \qquad \text{and} \qquad (3\rho_\phi + i\rho)w_1 = 0. \tag{44}$$

Thus, the metrics g_F may be conformal to Einstein metrics only in the following three cases:

($\bullet$) $w_1 = w_2 = 0$

($\bullet\bullet$) $w_2 = 0$, $w_1 \neq 0$ and $\rho = A(x,y,p)\exp(-\frac{i\phi}{3})$

($\bullet\bullet\bullet$) $w_2 \neq 0$, $w_1 = 0$ and $\rho = A(x,y,p)\exp(\frac{i\phi}{3})$

Case ($\bullet$) corresponds to ODEs with flat Cartan connection. These have conformally flat Fefferman metrics.

In the ($\bullet\bullet$) case the general solution for the Einstein equations (43) is given by (42), where either

$$\rho = \frac{\exp(-\frac{2}{3}a)}{p+2b}\exp(-\frac{i\phi}{3}), \tag{45}$$

and

$$Q = p^3 c + p^2(6bc - 2a_y) + p(12b^2c - 3b_y - 6ba_y - a_x) + 8b^3c - 2bb_y - 4b^2a_y - 2b_x - 2ba_x, \tag{46}$$

or

$$\rho = \exp(-\frac{2}{3}a)\exp(-\frac{i\phi}{3}), \tag{47}$$

and

$$Q = p^2 a_y + 2pa_x + b, \tag{48}$$

with $a = a(x,y)$, $b = b(x,y)$, $c = c(x,y)$ arbitrary functions of variables x, y. All these solutions are Ricci flat. They exhaust the list of Fefferman metrics which have $w_2 = 0$ and are conformal to non-flat Einstein metrics. Since Q in these solutions depend only on at most *three* arbitrary functions of two variables and generic Q for which $w_2 = 0$ depends on *four* functions of two variables, then not all ODEs with $w_2 = 0$ have Fefferman metrics which are conformal to Einstein metrics.

The Einstein equations in the ($\bullet\bullet\bullet$) case reduce to

$$\rho = a\exp(\frac{i\phi}{3}), \tag{49}$$

where the function $a = a(x,y)$ satsifies a single differential equation

$$36(Da)^2 + 6a(-3D^2a + Q_pDa) + a^2(6DQ_p - 18Q_y - 4Q_p^2) = 0. \tag{50}$$

It follows, as it should be, that one of the integrability conditions for this equation is $w_1 = 0$. It is not clear what other conditions should be imposed on Q to guarantee the solvability of (50). If, for example, Q satisfies

$$6DQ_p - 18Q_y - 4Q_p^2 = 0,$$

a simple solution is given by $a = const.$

Finally, we remark that all the metrics g_F which, via (42), correspond to the solutions (49)-(50) are Ricci flat. This, when compared with the results of case ($\bullet\bullet$), proves that conformally non flat Fefferman-Einstein metrics have vanishing cosmological constant.

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