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CALCULUS AND INVARIANTS ON ALMOST COMPLEX MANIFOLDS, INCLUDING PROJECTIVE AND CONFORMAL GEOMETRY

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Abstract. We construct a family of canonical connections and surrounding basic theory for almost complex manifolds that are equipped with an affine connection. This framework provides a uniform approach to treating a range of geometries. In particular, we are able to construct an invariant and efficient calculus for conformal almost Hermitian geometries, and also for almost complex structures that are equipped with a projective structure. In the latter case, we find a projectively invariant tensor the vanishing of which is necessary and sufficient for the existence of an almost complex connection compatible with the path structure. In both the conformal and projective setting, we give torsion characterisations of the canonical connections and introduce certain interesting higher order invariants.

1. Introduction

Let M be a smooth manifold of even dimension $n = 2m$. An almost complex structure (ACS) J on M is an endomorphism of the tangent bundle TM such that $J^2 = -1$. The study of almost complex structures has a rich history, especially in connection with complex geometry such as the theory of Kähler manifolds and closely linked themes. There is by now rather sophisticated machinery available for the treatment of almost complex geometries [4], [14], [23], [24], [31], [33], [34], [35], [38]. However, the basic calculus has been typically developed starting from the assumption that there is an almost

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Hermitian metric given as part of the data. From there, it is often not clear what parts of the results may be applied to different geometric structures, or in more general settings.

Our aim in this article is to develop a uniform approach to the calculus for almost complex manifolds which are also equipped with some additional geometric structure such as a conformal structure, or a projective structure. We indicate how this may be applied to the construction of invariants of the structure; we treat in more detail some of the less obvious new invariants that are seen to arise naturally from this perspective. Because of the nature of our endeavour there are inevitable close links with many results in the literature, especially in the case where we specialise to almost Hermitian geometry. Within the scope of this article, it would be impossible to do justice to the very nice work that has been done in this direction by many authors. However the works of Libermann, Obata, and Lichnerowicz [29], [31], [36] are particularly relevant. Much of that work is put into a uniform context by Gauduchon in [18], where also some extensions and Dirac operators are discussed.

Briefly the treatment and strategy for the calculus development is as follows. In Section 2, we treat almost complex affine manifolds. This means the data of the structure is an almost complex manifold M, of any even dimension, equipped with an almost complex structure J and an affine connection ∇. For this setting, we develop a basic calculus that includes a family of connections determined by (M, J, ∇) that are canonical and almost complex, meaning that they preserve J. The point here is that this is developed in such a way that it then easily specialises to a range of other geometries where there is additional structure, and so provides a treatment of these that is uniform. The structures we treat are almost Hermitian geometry, conformal almost Hermitian geometry, and finally projective almost complex geometry. A conformal almost Hermitian geometry is the structure given by (M^{2m}, J, c) where $m \geq 2$, and c is a conformal equivalence class of almost Hermitian metrics. A projective almost complex geometry consists of (M^{2m}, J, p) where $m \geq 1$, and p is a projective equivalence class of torsion-free connections; two connections ∇ and $\widehat{\nabla}$ are said to be projectively equivalent if they have the same geodesics as unparameterised curves. A key point, for our development, is that each of these structures can be shown to have a canonical affine connection and so using this one may immediately employ the general machinery developed in Section 2. Now we outline in more detail the developments and some of the main results.

As mentioned, Section 2 develops the basic tangent bundle calculus for general affine almost complex manifolds. We prove that the affine connection ∇ and J determine a fundamental $(1,2)$-tensor G that plays a central role throughout the article. Using this we prove, for example, in Proposition 2.4 that the structure determines a 1-parameter family of canonical connections on TM that preserve the almost complex structure J. There is a distinguished connection $\overset{KN}{\nabla}$ in the class with anti-Hermitian torsion. This has the property that in the case that ∇ is torsion free then the torsion of $\overset{KN}{\nabla}$ is precisely the Nijenhuis tensor, see Proposition 2.7. Section 2 also contains many technical results for use later in the article. For example, we introduce there an important notion of compatibility between an affine connection and J, this amounts to G being completely trace-free in the sense of Lemma 2.9.

Next, Section 3 treats the case of almost Hermitian geometry. Of course on almost Hermitian manifolds the basic tensor calculus has been treated considerably in the literature. So the main points of this section are first: to indicate how the usual objects arise by simple specialisation of the tools from the G-calculus of Section 2; and second to lay out the almost Hermitian results for comparison with the results for conformal and projective structures which follow in the later sections. Building on work of Gilkey [21], and others, there is a classification by Gray–Hervella [24] of almost Hermitian manifolds according to a $U(m)$-decomposition of $\nabla\omega$. We describe in Proposition 3.2 how certain key cases from the Gray–Hervella list, such as nearly Kähler, Hermitian, and almost Kähler, are identified in terms of G. Beginning with the Levi-Civita connection ∇ then from the family of almost complex connections of Proposition 2.4 there is a unique connection that preserves the metric. We show in Theorem 3.4 how, in our framework, there is a torsion characterisation of this distinguished connection. This (or its equivalents in the literature) provides a universal solution to the problem of finding a type of characteristic connection for each of the structures of the Gray–Hervella classification and this is the subject of Corollary 3.5.

Section 4 begins the more involved application of the approach. On a conformal almost Hermitian manifold there is a canonical and unique Weyl connection $\overset{c}{\nabla}$ that is compatible with J (cf. [39], [2]). This provides the basic input to generate the conformal version of the tools from Section 2. Using this one concludes there is a conformally invariant connection $\overset{gc}{\nabla}$ determined by structure (M, J, c) which preserves J and the conformal structure, see Proposition 4.4. Theorem 4.9 shows that connections with torsion that both, preserve the conformal structure and are suitably compatible with the complex structure, are parametrised by their torsions. This is then used to show that the vanishing of a canonical conformal torsion invariant suffices to characterise the connection $\overset{gc}{\nabla}$ among all almost complex connections on the structure (M, J, c), see Proposition 4.11 and Theorem 4.12. This means that the in conformal setting the results are really as strong as in the almost Hermitian case, which we find surprising. In the Sections 4.5, 4.6, and 4.7, we show how the structures of the Gray–Hervella conformal almost Hermitian classification are described and treated via the G-calculus. Finally in Section 4.8, we show that the canonical Weyl structure of the manifold (M, J, c) leads to some interesting higher order conformal invariants, including global invariants and objects that are analogues of Q-curvature.

Section 5 is the last of the theoretical developments and treats almost complex manifolds that are also equipped with a projective structure p. In analogy with the conformal case, we prove in Proposition 5.1 that there is a unique connection $\overset{p}{\nabla}$ in p that is compatible with J. We observe in Corollary 5.3 that this implies a distinguished class of parametrised curves, namely those curves which are the geodesics of the connection $\overset{p}{\nabla}$. (For an affine connection ∇, its *geodesics* are those curves whose tangent field X satisfies $\nabla_X X = 0$.) Theorem 5.5 determines a distinguished almost complex connection $\overset{JP}{\nabla}$. This has the property that if a certain projective invariant $\overset{p}{G}{}_-^{\mathrm{symm}}$ vanishes then $\overset{JP}{\nabla}$ has as geodesics the mentioned distinguished class of curves, see Corollary 5.7. ($\overset{p}{G}{}_-^{\mathrm{symm}}$ it is the anti-Hermitian symmetric part of the fundamental G-invariant $\overset{p}{G}$ for the structure (M, J, p) see e.g. Section 5.2.) The main result of Section 5 is Theorem 5.10 which proves that if ∇' is any affine connection, that preserves J and agrees with the path structure p, then necessarily this invariant $\overset{p}{G}{}_-^{\mathrm{symm}}$ vanishes identically, ∇' is simply related to $\overset{JP}{\nabla}$, and the geodesics of ∇' agree with the distinguished curves. This is a fundamental result concerning the relation between almost complex and projective geometry. Among other things this shows that the connection $\overset{JP}{\nabla}$ is optimal and that the condition of vanishing of $\overset{p}{G}{}_-^{\mathrm{symm}}$ is an important and canonical condition of *compatibility* between the complex structure J and the projective structure p (hence the Definition 5.6). There is a torsion characterisation of $\overset{JP}{\nabla}$ given in Corollary 5.11, so for the compatible projective almost complex structures the results are again as strong as for the almost Hermitian case. In Section 5.2, we describe projective analogues of the objects in the Gray–Hervella conformal classification and discuss related issues. Finally, higher projective invariants are discussed briefly in Section 5.3.

Section 6 shows that examples are available for the various structures. In fact, we treat just a few cases here as for most of the structures it is rather obvious that there will be structures available satisfying the various conditions.

1.1. Conventions. For simplicity, all structures will be assumed smooth, meaning C^∞. Unless otherwise stated, X, Y denote arbitrary sections of the tangent bundle TM. We also from time to time, as convenient, employ Penrose's abstract index formalism [37]. For example, $\mathcal{E}^a$ is an alternative notation for TM (or its section space, we shall not distinguish) and X^a, Y^a denote sections thereof. The almost complex structure J is written via abstract indices as $J^a{}_b$, so that JX may be written $J^a{}_b X^b$, and $J^a{}_b J^b{}_c = -\delta^a_c$, where δ^a_b is the pointwise identity endomorphism on TM. Here the repeated indices indicate contractions.

2. Calculus on an almost complex affine manifold

Here we develop a canonical calculus for almost complex manifolds that are also equipped with an affine connection. This then forms the basis for our subsequent treatment of other geometries.

2.1. Almost complex affine connections. Given an almost complex manifold (M, J) an affine connection ∇ on M is called *almost complex* if it preserves J. We first observe that any affine connection can be modified to yield such a connection.

Let ∇ be any affine connection on an almost complex n-manifold M. Let H be a $(1,2)$ tensor field on M and consider the connection $\overset{H}{\nabla}$ defined by

$$\overset{H}{\nabla}_X Y := \nabla_X Y + H(Y, X) \quad \text{for } X, Y \in \Gamma(TM).$$

We seek H such that $\overset{H}{\nabla}_X (JY) = J \overset{H}{\nabla}_X Y$. This is equivalent to

$$\nabla_X (JY) - J\nabla_X Y = JH(Y, X) - H(JY, X). \tag{2.1}$$

Evidently if H is a solution then we obtain another solution by adding a $(1,2)$ tensor field K which is J linear in the first argument: $K(JY, X) = JK(Y, X)$. To remove this freedom, we replace H with a $(1,2)$ tensor G that is assumed to be J-antilinear in the first argument: $G(JY, X) = -JG(Y, X)$. Then (2.1) becomes

$$\nabla_X (JY) - J\nabla_X Y = 2JG(Y, X),$$

which may be solved for G to yield $G(Y, X) = -\frac{1}{2} J(\nabla_X J)Y$. Moreover for any H solving (2.1), G is the complex anti-linear part over the first argument, that is, $G(\cdot, \cdot) = \frac{1}{2}(H(\cdot, \cdot) + JH(J\cdot, \cdot))$. Thus the general solution to (2.1) is of the form $H = G + K$, for some K as above. We summarise as follows.

PROPOSITION 2.1. *Let (M, J) be an almost complex n-manifold and ∇ an arbitrary affine connection on M. Then $\overset{G}{\nabla}$ is an almost complex connection, that is $\overset{G}{\nabla} J = 0$, where*

$$\overset{G}{\nabla}_X Y := \nabla_X Y + G(Y, X),$$

and

$$G(X, Y) := \frac{1}{2}(\nabla_Y J)JX = -\frac{1}{2} J(\nabla_Y J)X. \tag{2.2}$$

Moreover, if $\overset{H}{\nabla}$ *is any connection preserving* J *then*

$$\overset{H}{\nabla}_X Y = \overset{G}{\nabla}_X Y + K(Y, X),$$

where K *is a* $(1,2)$ *tensor which is complex linear in the first argument.*

In places below, it is convenient to use abstract index notation for G and the related connections: we write $G^a{}_{bc} Y^b X^c$ for the vector field $G(Y, X)$ and

$$\overset{G}{\nabla}_a Y^b := \nabla_a Y^b + G^b{}_{ca} Y^c.$$

In this notation,

$$G^b{}_{ca} := \frac{1}{2}\big(\nabla_a J^b{}_d\big) J^d{}_c = -\frac{1}{2} J^b{}_d \nabla_a J^d{}_c.$$

We note some properties of G for later use.

LEMMA 2.2. *For all tangent vectors* X, $G(\cdot, X)$ *is trace-free and* $JG(\cdot, X)$ *is trace-free. If* ∇ *preserves a volume form on* M, *then* $\overset{G}{\nabla}$ *preserves the same volume form.*

Proof. We have

$$2G^b{}_{ba} = \big(\nabla_a J^b{}_d\big) J^d{}_b = \frac{1}{2}\nabla_a \big(J^b{}_d J^d{}_b\big) = 0.$$

This proves first claim and the last statement follows immediately.

For the second claim, we re-express $2J^a{}_b G^b{}_{cd}$. This is

$$J^a{}_b \big(\nabla_d J^b{}_e\big) J^e{}_c = \nabla_d J^a{}_c. \tag{2.3}$$

Since the almost complex structure J is trace-free the result follows. □

LEMMA 2.3. $G(JX, Y) = -JG(X, Y)$ *and hence its Hermitian part*

$$G_+(X, Y) := \frac{1}{2}\big(G(X, Y) + G(JX, JY)\big)$$

and its anti-Hermitian part

$$G_-(X, Y) := \frac{1}{2}\big(G(X, Y) - G(JX, JY)\big)$$

have the following properties:

$$G_\pm(X, JY) = \pm JG_\pm(X, Y), \tag{2.4}$$

and

$$G_\pm(JX, Y) = -JG_\pm(X, Y). \tag{2.5}$$

Proof. The property $G(JX,Y) = -JG(X,Y)$ is a consequence of the proof of Proposition 2.1; alternatively it is easily verified from (2.2). Using this, we have

$$\begin{aligned}
2G_\pm(X,JY) &= G(X,JY) \pm JG(X,Y)\\
&= J\big(\pm G(X,Y) - JG(X,JY)\big)\\
&= J\big(\pm G(X,Y) + G(JX,JY)\big)\\
&= \pm 2JG_\pm(X,Y).
\end{aligned}$$

A similar calculation yields (2.5). □

In Proposition 2.1, we showed that on an almost complex manifold (M,J) the space of almost complex connections is affine modelled on the space of $(1,2)$ tensor fields which are complex linear in the first argument. Fix an affine connection ∇, as in that proposition. It follows immediately from the property (2.4) of G_+, in the Lemma 2.3, that we may, in particular, use multiples of G_+ to modify the connection $\overset{G}{\nabla}$, while retaining the property that the new connection preserves J.

Proposition 2.4. *Let ∇ be an affine connection on an almost complex manifold (M,J), and $\overset{G}{\nabla}$ the corresponding almost complex connection as above. For any $t\in\mathbb{R}$, $\overset{t}{\nabla}$ is an almost complex connection where*

$$\overset{t}{\nabla}_X Y := \overset{G}{\nabla}_X Y + tG_+(X,Y).$$

2.2. Torsion and integrability. In the above, we have not made any assumptions concerning the torsion of ∇. Beginning with any connection $\tilde\nabla$ with torsion $\tilde T$ the related connection ∇ defined by $\nabla_X Y := \tilde\nabla_X Y - \frac{1}{2}\tilde T(X,Y)$ is torsion free. In the case that ∇ is torsion free, then we obtain very useful formulae for the Nijenhuis tensor N_J. (The normalisation of N_J is for convenience.)

Proposition 2.5. *For any torsion free connection ∇ on an almost complex manifold (M,J), we have*

$$4N_J(X,Y) = (\nabla_X J)JY - (\nabla_Y J)JX + (\nabla_{JX}J)Y - (\nabla_{JY}J)X, \tag{2.6}$$

and hence

$$N_J(X,Y) = G_-(Y,X) - G_-(X,Y). \tag{2.7}$$

Proof. The formula (2.6) is well known, see, for example, [28]. Using (2.2) to rewrite this in terms of G yields (2.7). □

It is immediate from the definition of $\overset{G}{\nabla}$ that if ∇ is a torsion free connection then

$$T^G(X,Y) = G(Y,X) - G(X,Y). \tag{2.8}$$

Thus we obtain the following consequence of the Proposition 2.5.

Corollary 2.6. *Let ∇ be a torsion free connection on an almost complex manifold (M,J), and $\overset{G}{\nabla}$ the corresponding almost complex connection given by Proposition 2.1. Then the anti-Hermitian part of the torsion of $\overset{G}{\nabla}$ is the Nijenhuis tensor,*

$$T^G_-(X,Y) = N_J(X,Y). \tag{2.9}$$

Proof. Taking the anti-Hermitian part of (2.8) gives the result by (2.7). □

Evidently we may use these observations to select a distinguished connection $\overset{KN}{\nabla}$ from the class given in Proposition 2.4:

$$\overset{KN}{\nabla}_X Y = \overset{G}{\nabla}_X Y + G_+(X,Y). \tag{2.10}$$

For this connection, the Hermitian part of the torsion is zero, while the anti-Hermitian part of its torsion agrees with the anti-Hermitian part of the $\overset{G}{\nabla}$ torsion. From this observation and Proposition 2.5, we have the following result.

Proposition 2.7. *Beginning with any affine connection ∇, the associated connection $\overset{KN}{\nabla}$ has anti-Hermitian torsion* $\mathrm{Tor}(\overset{KN}{\nabla})$. *In the case that ∇ is torsion free, we have* $\mathrm{Tor}(\overset{KN}{\nabla}) = N_J$, *the Nijenhuis tensor.*

Remark 2.8. The connection $\overset{KN}{\nabla}$ is readily verified to be the classical connection of [27, Theorem 3.4, Section IX], where the property of its torsion is also noted. Note that an immediate corollary of Proposition 2.7 is that an almost complex manifold admits an almost complex torsion free connection if and only if J is integrable.

2.3. Compatible affine connections. Let (M,J) be an almost complex structure. An affine connection ∇ on M will be said to be *compatible* with J if

$$\nabla_a J^a{}_b = 0. \tag{2.11}$$

Lemma 2.9. *On an almost complex structure (M,J) an affine connection ∇ is compatible if and only if G^∇ is trace-free; equivalently, if and only if JG^∇ is trace-free; equivalently, if and only if $G^\nabla(\cdot,J\cdot)$ is trace-free; equivalently, if and only if $G^\nabla(J\cdot,J\cdot)$ is trace-free.*

Proof. Let ∇ be any affine connection. We have already in Lemma 2.2 that in any case the first trace of G and the first trace of JG both vanish (i.e., this feature of G is not related to compatibility).

Recall $G^b{}_{ca} := \frac{1}{2}(\nabla_a J^b{}_d)J^d{}_c$ so

$$2G^b{}_{cb} = \left(\nabla_b J^b{}_d\right)J^d{}_c,$$

which is clearly zero if and only if $\nabla_b J^b{}_d = 0$.

On the other hand recall from Lemma 2.2 that $2JG$ is

$$J^a{}_b\left(\nabla_d J^b{}_e\right)J^e{}_c = \nabla_d J^a{}_c \tag{2.12}$$

and so the only available trace yields $\nabla_a J^a{}_b$.

The trace of $2G(X, J\cdot)$ is $X^b\nabla_a J^a{}_b$, so this case is also clear. Then the final statement thus follows. □

3. Almost Hermitian geometry

Let (M^n, J) be an almost complex manifold of dimension $n \geq 4$, and g a Riemannian metric on M. The triple (M, J, g) is said to be *almost Hermitian* if J is orthogonal with respect to g, that is

$$g(JX, JY) = g(X, Y)$$

for all tangent vector fields X, Y.

Remark 3.1. Note that if g is any Riemannian metric on (M, J), then the Hermitian part of g, that is

$$g_+(X, Y) = \frac{1}{2}\big(g(X, Y) + g(JX, JY)\big),$$

is positive definite, and so (M, J, g_+) is an almost Hermitian structure.

Henceforth in this section, we shall assume (M, J, g) is an almost Hermitian (AH) structure. In this setting, we also have the skew symmetric Kähler form

$$\omega(X, Y) := g(X, JY). \tag{3.1}$$

Let us make two comments regarding this section. First, the results in this section are for the most part well known. Nevertheless, we want to understand some of the standard structures from almost Hermitian geometry in terms of the G-calculus developed above. This serves to put our discussion in context and gives us a basis from which we may compare the conformal and projective treatments in the next sections. The second point is that for the reason that the material is known we are brief here and some of the key results we shall use are drawn from the later Section 4; the point is that there the results are proved in a broader context.

Proceeding now, in this section we shall use G to denote the tensor of (2.2) where $\nabla = \overset{LC}{\nabla}$ is the Levi-Civita connection of g. With this specialisation, G

is what in the literature is an example of an intrinsic torsion and $\overset{G}{\nabla}$ is then what is usually called the *canonical Hermitian connection*, see, for example, [11], [34]. This classical object is the *first canonical connection* of [31] and to the best of our knowledge originated in the work [29] of Libermann. In fact the latter source gives a 1-parameter family of canonical almost Hermitian connections and this family is discussed in detail in [18] where it explained how the various connections of [4], [31], as well as a torsion minimising connection introduced in [18], arise from Libermann's family ∇^t; the connection $\overset{G}{\nabla}$ of this section is the operator ∇^0 from there. Such a family arises because the almost Hermitian metric enables a finer decomposition of G than is available in Section 2 above. Nevertheless, we shall not explore that here, since without geometry specific refinement the G-calculus from above is both simple and universally applicable, and these are the features that we apply in the later sections.

Let us write $G(X,Y,Z) := g(X,G(Y,Z))$, and $G_\pm(X,Y,Z) := g(X,G_\pm(Y,Z))$. First, we introduce some general facts that we shall use. From Proposition 4.5 (below), we have that

$$G(X,Y,Z) = -G(Y,X,Z) \tag{3.2}$$

and so $\overset{G}{\nabla}$ is a metric connection. The same proposition also proves that

$$G_-(X,Y,Z) = -G_-(Y,X,Z). \tag{3.3}$$

Also from there, or alternatively from the skew symmetry of ω, it follows that on an AH structure $g(\cdot, JG(\cdot,\cdot))$ is also skew over first and second arguments, that is

$$\begin{aligned} &g\big(X, JG(Y,\cdot)\big) + g\big(Y, JG(X,\cdot)\big) = 0 \\ \Leftrightarrow \quad &G(X,JY,Z) + G(Y,JX,Z) = 0, \end{aligned} \tag{3.4}$$

where the equivalence uses Lemma 2.3. Together (3.2) and (3.4) imply that $G(X,Y,Z)$ is anti-Hermitian in the first pair. That is

$$G(X,Y,Z) = -G(JX,JY,Z). \tag{3.5}$$

Now recall that an AH structure (M,J,g) is said to be a *Hermitian* structure if $N_J = 0$. Thus from Proposition 2.5 $G_-(X,Y,Z) = G_-(X,Z,Y)$. But this with (3.3) implies that $G_- = 0$. Conversely, again using Proposition 2.5, $G_- = 0$ implies $N_J = 0$. Thus, g is Hermitian if and only if $G_- = 0$.

An AH structure that satisfies

$$\delta\omega = 0 \tag{3.6}$$

is said to be *semi-Kähler* (or *co-symplectic*). Here δ is the formal adjoint of the exterior derivative; so note that this condition (3.6) is precisely that the Levi-Civita connection is compatible with J, as in the definition (2.11).

A stronger condition on an AH manifold (M, J, g) is given by

$$(\overset{LC}{\nabla}_X J)X = 0, \quad \forall X \in \Gamma(TM)$$

and this defines structures that are called *nearly Kähler*. (In dimension $n = 4$, this is equivalent to Kähler, as below, but in higher dimensions it is a strictly weaker condition.) This condition is obviously that same as requiring that G be anti-symmetric: $G(X, Y) = -G(Y, X)$.

Next, an AH structure is said to be *almost Kähler* (or *symplectic*) if

$$d\omega = 0.$$

This is easily rewritten directly in terms of G:

$$d\omega = 0 \quad \Leftrightarrow \quad \mathrm{Alt}_{(X,Y,Z)}\, g\big(X, JG(Y, Z)\big) = 0, \tag{3.7}$$

where Alt is the projection to the completely skew part.

An AH structure is called *Kähler* if we have the two conditions

$$d\omega = 0 \quad \text{and} \quad N_J = 0. \tag{3.8}$$

The Kähler condition, when expressed in terms of the Levi-Civita connection $\overset{LC}{\nabla}$ of the metric g, may be expressed:

$$\overset{LC}{\nabla}_X J = 0, \quad \forall X \in TM.$$

Using the machinery here, this well known characterisation is easily recovered as follows. First, if J is parallel for the Levi-Civita connection then it follows at once that ω is parallel, and thus $d\omega = 0$ since the Levi-Civita connection is torsion free. On the other hand from Proposition 2.7 we also have that $N_J = 0$ (see Remark 2.8). For the other direction, suppose that the conditions (3.8) hold. Since $G_- = 0$ it follows that $G(X, JY, Z)$ is Hermitian on the argument pair Y, Z (i.e. $G(X, JY, Z) = G(X, J^2Y, JZ)$). But using (3.7) it follows that $G(X, JY, Z) = G(Z, JY, X) - G(Z, JX, Y)$, and so $G(X, JY, Z)$ is also Hermitian on the argument pair X, Y. But comparing with (3.5) it then follows that $G = 0$.

To summarise, we have the following.

Proposition 3.2. *If $\nabla = \overset{LC}{\nabla}$ is the Levi-Civita connection of an almost Hermitian manifold (M, J, g), then the structure is*:

(H) *Hermitian iff*

$$G_-(X, Y) = 0;$$

(NK) *nearly Kähler iff*

$$G(X, Y) + G(Y, X) = 0;$$

(AK) *almost Kähler iff*

$$\mathrm{Alt}_{(X,Y,Z)}\, g\big(X, JG(Y, Z)\big) = 0;$$

(K) *Kähler iff*

$$G = 0.$$

The Gray–Hervella classification of AH structures [24] is based around the $U(m)$ decomposition of $\nabla\omega$, where $\nabla = \overset{LC}{\nabla}$. But

$$\nabla_a \omega_{bc} = g_{be} \nabla_a J^e{}_c = 2 g_{be} J^e{}_b G^b{}_{ca}$$

or equivalently

$$\nabla_X \omega(Y, Z) = 2g\big(Y, JG(Z, X)\big). \tag{3.9}$$

Thus the Gray–Hervella classification could equivalently be formulated as a $U(m)$ decomposition of the G for the Levi-Civita connection. In the almost Hermitian setting G has a number of additional symmetries and properties (some mentioned in (3.2) to (3.5) above) that simplify the situation considerably. This leads to the next observation.

PROPOSITION 3.3. *An AH structure is nearly Kähler if and only if*

$$G(\cdot, \cdot, \cdot) := g\big(\cdot, G(\cdot, \cdot)\big)$$

is completely alternating. If this holds, then

$$G_+ = 0$$

while

$$\mathrm{Tor}(\overset{G}{\nabla}) = -2G_- = N_J,$$

and is completely alternating.

Proof. From Lemma 4.5 below (with ∇ the Levi-Civita connection), we have that in any case $G(\cdot, \cdot, \cdot)$ is skew on the first two arguments. If (M, J, g) is nearly Kähler, then this is also skew on the last pair. That used Proposition 3.2 and from that proposition the converse direction is immediate.

Since $G(\cdot, \cdot, \cdot)$ is completely alternating and anti-Hermitian over the first pair (by (3.5)) it follows that $G(\cdot, \cdot, \cdot)$ is anti-Hermitian over any pair of arguments. Thus, $G_+ = 0$. The final statements are then immediate from Proposition (2.5) and the expression (2.8) for the torsion of $\overset{G}{\nabla}$. □

Beginning with the Levi-Civita connection ∇ then from the family of almost complex connections of Proposition 2.4 it is easily verified that $\overset{G}{\nabla}$ is the unique connection that preserves the metric. This follows by a minor adaption of the proof of Theorem 4.8 below.

A powerful feature of the almost Hermitian setting is that the torsion carries the same information as G. This enables us to characterise the special connection $\overset{G}{\nabla}$ in terms of torsion, as follows (and cf. [18], [30], [31]).

THEOREM 3.4. *Let (M,J,g) an almost Hermitian structure. On this there is a unique almost complex metric connection with torsion T satisfying the algebraic condition*

$$G_g^T(X,Y) - JG_g^T(JX,Y) = 0 \quad \forall X,Y \in \Gamma(TM), \tag{3.10}$$

where

$$\left(G_g^T\right)^a{}_{bc} := \frac{1}{2}\left(T_c{}^a{}_b - T^a{}_{bc} - T_{bc}{}^a\right). \tag{3.11}$$

This connection is $\overset{G}{\nabla}$, based on the Levi-Civita connection.

Proof. First, observe that beginning with the Levi-Civita connection and forming from it $\overset{G}{\nabla}$ we have that $\overset{G}{\nabla}$ is an almost complex metric connection.

Now let $\overset{T}{\nabla}$ be any connection that is metric, that is

$$\overset{T}{\nabla} g = 0$$

and write T for its torsion. Then the difference tensor (as connections on TM) $\overset{T}{\nabla} - \overset{LC}{\nabla}$ is G_g^T (that is $G_g^T(X,Y) = \overset{T}{\nabla}_X Y - \overset{LC}{\nabla}_X Y$) as given in (3.11). The condition (3.10) is the statement that the complex linear part of $G_g^T(\cdot,X)$ is zero, for all $X \in \Gamma(TM)$. The G from Proposition 2.1 (with ∇ the Levi-Civita for g) has this property, so $\overset{G}{\nabla}$ provides an almost complex metric connection satisfying the conditions (3.10) and (3.11).

Let us now consider any metric connection $\overset{T}{\nabla}$ satisfying (3.10). Then $G_g^T(\cdot,X)$ is complex anti-linear and so by the second part of Proposition 2.1 (again applied using ∇ set to be the Levi-Civita connection for g) $G_g^T = G$. Thus $\overset{T}{\nabla} = \overset{G}{\nabla}$. □

For each structure in the Gray–Hervella classification one might hope that there is a corresponding *characteristic connection* [5]. Here this means an almost complex metric connection with torsion, but with torsion in some sense algebraically minimal so that with this torsion condition there exists a connection satisfying the given conditions, and it is unique. (This generalises the use of the term "characteristic connection" in the works of for example, Friedrich [16], see also [1] for a review.)

The Theorem 3.4 (or any of its equivalents in the literature) provides such a connection. One simply translates each of the structures in the Gray–Hervella into a condition on the G formed from the Levi-Civita connection (as for the examples in Proposition 3.2). Now one uses the formula (3.11) to recast the condition on G as a restriction on torsion. This combined with (3.10) give the total conditions to be imposed on the torsion. The existence and uniqueness then follow from Theorem 3.4. Although this result is well known, we summarise it here for comparison with the conformal and projective cases below.

COROLLARY 3.5. *There is a canonical characteristic connection for each of the structures in the Gray–Hervella classification of almost Hermitian manifolds.*

4. Conformal almost Hermitian manifolds

Throughout this section, we take (M^n, J) to be an almost complex manifold of dimension $n \geq 4$. A (Riemannian) conformal structure c on M is an equivalence class of Riemannian metrics such that if $g, \widehat{g} \in c$ then $\widehat{g} = e^{2\phi} g$ for some $\phi \in C^\infty(M)$. Here we observe that the structure (M^n, J, c) determines several canonical affine connections with different characterising properties. Much of the below will work for Hermitian metrics in signatures $(2p, 2q)$, but for simplicity we restrict to the Riemannian setting.

Note that if J is orthogonal for $g \in c$ then it is orthogonal for all metrics in c. In this case, we shall say that (M, J, c) is a *conformal almost Hermitian* structure. Note also that, by the observation of Remark 3.1, a Riemannian conformal structure on (M, J) determines an almost Hermitian Riemannian conformal structure c_+. We shall henceforth assume that any conformal structure c is almost Hermitian.

4.1. A canonical torsion free connection. An affine connection ∇ on a Riemannian manifold (M, g) will be said to be *conformal* if it preserves the conformal class of the metric, that is

$$\nabla_a g_{bc} = 2B_a g_{bc}, \tag{4.1}$$

for some 1-form field B that we shall term the *Weyl potential.* A *Weyl connection* $\overset{W}{\nabla}$ is an affine connection which is conformal and torsion free [40].

On a conformal structure (M, c), we shall say an affine connection ∇ is *conformal* (or *Weyl* if torsion free) if (4.1) holds for all $g \in c$. This means that on a conformal structure (M, c) there is not a Weyl potential B_a, but rather an equivalence class of such over the conformal equivalence relation: given $g \in c$, B^g_a is a 1-form field and if $\widehat{g} = e^{2\phi} g$, for some smooth function ϕ, then

$$B^{\widehat{g}}_a = B^g_a + \Upsilon_a, \tag{4.2}$$

where $\Upsilon := d\phi$. (In fact B is a connection coefficient, as we explain in Section 4.8 below.) That such structures arise naturally is illustrated by the following result of [39] (and see also [2]).

PROPOSITION 4.1. *Let (M, J, c) be a conformal almost Hermitian structure of dimension $n \geq 4$. There is a canonical and unique Weyl connection $\overset{c}{\nabla}$ that* *is compatible with J; that is satisfying*

$$\overset{c}{\nabla}_a J^a{}_b = 0.$$

Given a choice of $g \in c$, $\overset{c}{\nabla}$ is given explicitly in terms of the Levi-Civita connection ∇ and J by

$$\overset{c}{\nabla}_a Y^b = \nabla_a Y^b - B_a Y^b + B^b Y_a - B_c Y^c \delta^b_a, \tag{4.3}$$

where

$$B^g_a := \frac{1}{n-2} J^c{}_b \nabla_c J^b{}_a. \tag{4.4}$$

Proof. Let us fix $g \in c$. The formula (4.3) for $\overset{c}{\nabla}$ is equivalent to the formula for its dual:

$$\overset{c}{\nabla}_a U_b = \nabla_a U_b + B_a U_b + B_b U_a - U^c B_c g_{ab}, \tag{4.5}$$

where U is any 1-form field. From this (4.1) follows, and conversely it is easily verified that (4.1), with the torsion free condition, implies (4.5).

Next note that using (4.3) and (4.5), we have

$$\overset{c}{\nabla}_i J^k{}_\ell = \nabla_i J^k{}_\ell + \big(B^k J_{i\ell} + B_\ell J^k{}_i - B_a J^a{}_\ell \delta^k_i - J^k{}_a B^a g_{\ell i}\big). \tag{4.6}$$

Contracting this yields that

$$0 = \overset{c}{\nabla}_i J^i{}_\ell \quad \Leftrightarrow \quad 0 = \nabla_i J^i{}_\ell + \big(B_i J^i{}_\ell - n B_i J^i{}_\ell + J^i{}_\ell B_i\big),$$

and thus

$$0 = \overset{c}{\nabla}_i J^i{}_\ell \quad \Leftrightarrow \quad \nabla_i J^i{}_\ell = (n-2) B_i J^i{}_\ell \quad \Leftrightarrow \quad B_a = \frac{1}{n-2} J^c{}_b \nabla_c J^b{}_a.$$

Since B_a is uniquely determined, and $g \in c$ is arbitrary, this proves the proposition and that in particular $\overset{c}{\nabla}$ depends only on J and c (but not the further information of $g \in c$). ☐

Observe that in the proof above the factor $(n-2)$ arising shows that the property of compatibility is stable under conformal rescaling. In fact, in dimension 2 every Weyl connection is complex.

Remark 4.2. Note that it is immediate from the Proposition (4.1) that B_a, as defined in (4.4), must satisfy the conformal transformation formula (4.2); this can also be verified using the conformal transformation properties of the Levi-Civita connection (see, e.g., [3]). In the literature (e.g., [24]) $2B^g$ is usually called the *Lee form*, cf. [26].

Since $\overset{c}{\nabla}$ is conformally invariant it follows that (4.6) defines an invariant of (M, J, c); this is precisely the conformal invariant μ of [24, Section 4].

With a canonical conformally invariant connection, as we have with $\overset{c}{\nabla}$, many geometric consequences are immediate. For example, we have an immediate consequence of the Proposition 4.1.

Corollary 4.3. *Let (M, J, c) be a conformal almost Hermitian manifold of any dimension. Then M has a preferred class of parametrised curves, viz. the geodesics of $\overset{c}{\nabla}$.*

Furthermore, since $\overset{c}{\nabla}$, from Proposition 4.1, is canonically determined by a conformal almost Hermitian structure, it may be used to proliferate (conformally invariant) invariants of the structure. For example, the curvature $\overset{c}{R}$ of $\overset{c}{\nabla}$ is an invariant of the structure (M, J, c). Its $\overset{c}{\nabla}$ covariant derivatives $\overset{c}{\nabla}\cdots\overset{c}{\nabla}\overset{c}{R}$, the $\overset{c}{\nabla}$ derivatives of J and contractions thereof also yield invariants and so forth. However for many purposes it is obviously more natural to work rather with a connection that is both conformally invariant and preserves the almost complex structure J.

4.2. Canonical conformal almost complex connections. Since the conformal almost Hermitian structure (M, J, c) determines $\overset{c}{\nabla}$ it follows that also canonically associated to the structure is the invariant

$$\overset{c}{G}{}^b{}_{ca} := \frac{1}{2}\big(\overset{c}{\nabla}_a J^b{}_d\big) J^d{}_c; \tag{4.7}$$

from the earlier developments it is clear this should play a fundamental role.

More generally, by using the canonical torsion-free affine connection $\overset{c}{\nabla}$ as the initial connection, and using the results of Section 2, we can form a range of geometric objects determined canonically by the conformal structure and the compatible almost complex structure J. We begin this with the following.

Proposition 4.4. *Let (M, J, c) be a conformal almost Hermitian structure of dimension $n \geq 4$ and $g \in c$. This structure determines a canonical affine connection $\overset{gc}{\nabla}$ defined by*

$$\overset{gc}{\nabla}_X Y = \overset{c}{\nabla}_X Y + \overset{c}{G}(Y, X), \quad \textit{where } \overset{c}{G}{}^b{}_{ca} := \frac{1}{2}\big(\overset{c}{\nabla}_a J^b{}_d\big) J^d{}_c.$$

This has the properties:

- $\overset{gc}{\nabla}_a g_{bc} = 2B_a g_{bc}$;
- $\overset{gc}{\nabla} J = 0$;
- *The anti-Hermitian part of its torsion gives the Nijenhuis tensor $N_J(X, Y) = T^{gc}_{-}(X, Y)$.*

Proof. All results are simply specialisations of statements in Proposition 2.1 and Corollary 2.6 except for the fact $\overset{gc}{\nabla}_a g_{bc} = 2B_a g_{bc}$. This is a consequence of the corresponding property of $\overset{c}{\nabla}$ in Proposition 4.1, and the first part of the lemma below. $\square$

LEMMA 4.5. *On an almost Hermitian manifold* (M,J,g) *let* ∇ *be any affine connection with the property that* $\nabla_a g_{bc} = 2B_a g_{bc}$ *for some* 1*-form* B (*i.e.* ∇ *is conformal*). *Then with* ${G^a}_{bc}$ *defined as in Proposition* 2.1, $G(X,Y,Z) := g(X,G(Y,Z))$, *and* $G_\pm(X,Y,Z) := g(X,G_\pm(Y,Z))$, *we have*

- $G(X,Y,Z) = -G(Y,X,Z)$;
- $G(X,JY,Z) = -G(Y,JX,Z)$;
- $G_+(X,Y,Z) = -G_+(Y,X,Z)$, *and* $G_-(X,Y,Z) = -G_-(Y,X,Z)$;
- $G(JX,JY,Z) = -G(X,Y,Z)$, *and* $G_\pm(JX,JY,Z) = -G_\pm(X,Y,Z)$.

Proof. The last claim is immediate from the complex anti-linearity of G, as in Lemma 2.3, and the fact that J is orthogonal for g.

It remains to establish the first two claims, since these imply the third. (The parts G_+ and G_- are as defined in Section 2.) For the first, we have

$$\begin{aligned}
&2\big(G(Y,Z,X) + G(Z,Y,X)\big)\\
&\quad = g\big(J(\nabla_X J)Y, Z\big) + g\big(Y, J(\nabla_X J)Z\big)\\
&\quad = -g\big((\nabla_X J)Y, JZ\big) - g\big(JY, (\nabla_X J)Z\big)\\
&\quad = -g\big(\nabla_X(JY), JZ\big) - g\big(JY, \nabla_X(JZ)\big) + g(J\nabla_X Y, JZ) + g(JY, J\nabla_X Z)\\
&\quad = -X\cdot g(JY,JZ) + (\nabla_X g)(JY,JZ) + X\cdot g(Y,Z) - (\nabla_X g)(Y,Z)\\
&\quad = 0,
\end{aligned}$$

where the last equality follows using again that g is almost Hermitian and that $\nabla_X g = 2B(X)g$.

Now for the second identity, we calculate

$$G(X,JY,Z) + G(Y,JX,Z).$$

Setting $X' = -JX$ this is

$$\begin{aligned}
g\big(JX', G(JY,Z)\big) - g\big(Y, G\big(X',Z\big)\big) &= g\big(JX', G(JY,Z)\big) - g\big(JY, JG\big(X',Z\big)\big)\\
&= g\big(JX', G(JY,Z)\big) + g\big(JY, G\big(JX',Z\big)\big)\\
&= 0,
\end{aligned}$$

where in the last line we have used the previous result. $\square$

In particular, the identities of Lemma 4.5 hold for $\overset{c}{G}$. From these, we obtain the following conformal analogue of Proposition 3.2, part (H).

PROPOSITION 4.6. *If* (M,J,c) *is a conformal Hermitian structure, then* $\overset{c}{G}_- = 0$ *and the connection* $\overset{gc}{\nabla}$ *has Hermitian torsion.*

Proof. From Proposition 4.4, we have $N_J(X,Y) = \overset{c}{G}_-(Y,X) - \overset{c}{G}_-(X,Y)$. So if $N_J = 0$ then $\overset{c}{G}_-(Y,X) = \overset{c}{G}_-(X,Y)$. This with the third bullet point of Lemma 4.5 (applied in the case $\nabla = \overset{c}{\nabla}$, so G there is $\overset{c}{G}$) implies $\overset{c}{G}_- = 0$. Thus, the torsion is Hermitian: $\overset{gc}{T}(X,Y) = \overset{c}{G}_+(Y,X) - \overset{c}{G}_+(X,Y)$. □

4.3. Compatibility in the conformal setting. We note here that in this setting there is a refinement of Lemma 2.9 (related to the forming of metric traces).

LEMMA 4.7. *On a conformal almost Hermitian structure (M,J,c) a Weyl connection $\overset{W}{\nabla}$ is compatible, and then agrees with $\overset{c}{\nabla}$, if and only if $\overset{W}{G}$ is totally trace-free; equivalently, if and only if $J\overset{W}{G}$ is totally trace-free; equivalently if and only if $\overset{W}{G}(\cdot, J\cdot)$ is completely trace-free; equivalently if and only if $\overset{W}{G}(J\cdot, J\cdot)$ is completely trace-free.*

Proof. Let $\overset{W}{\nabla}$ be a Weyl connection as in (4.1) (and $\overset{W}{\nabla}$ is torsion free). Recall from Lemma 2.2 that $\overset{W}{G}{}^i{}_{ik} = 0$, $(J\overset{W}{G})^i{}_{ik} = 0$, and these properties are not linked to compatibility.

From Lemma 2.9 (and using Lemma 2.2), we have that compatibility is equivalent to the vanishing of the trace, for all tangent fields X, of any one of $\overset{W}{G}(X,\cdot)$, $J\overset{W}{G}(X,\cdot)$, $\overset{W}{G}(X, J\cdot)$.

Since a conformal structure is available, we may also use a metric $g \in c$ to form a trace:

$$\begin{aligned} 2g^{jk}\overset{W}{G}{}^i{}_{jk} &= g^{jk}\big(\overset{W}{\nabla}_k J^i{}_\ell\big)J^\ell{}_j = -g^{jk}J^i{}_\ell \overset{W}{\nabla}_k J^\ell{}_j \\ &= -J^i{}_\ell \overset{W}{\nabla}_k\big(g^{jk}J^\ell{}_j\big) + J^i{}_\ell J^\ell{}_j \overset{W}{\nabla}_k g^{jk} \\ &= J^i{}_\ell \overset{W}{\nabla}_k\big(g^{\ell m}J^k{}_m\big) - \overset{W}{\nabla}_k g^{ik} \\ &= J^i{}_\ell g^{\ell m}\overset{W}{\nabla}_k J^k{}_m + J^i{}_\ell J^k{}_m \overset{W}{\nabla}_k g^{\ell m} - \overset{W}{\nabla}_k g^{ik} \\ &= J^{im}\overset{W}{\nabla}_k J^k{}_m - 2B^i + 2B^i = J^{im}\overset{W}{\nabla}_k J^k{}_m. \end{aligned}$$

This vanishes if and only if $\overset{W}{\nabla}$ is compatible. If it is compatible, then by Proposition 4.1 $\overset{W}{\nabla} = \overset{c}{\nabla}$.

On the other hand

$$g^{jk}(J\overset{W}{G})^i{}_{jk} = J^i{}_\ell g^{jk}\overset{W}{G}{}^\ell{}_{jk},$$

and so this trace vanishes if and only if $g^{jk}\overset{W}{G}{}^\ell{}_{jk} = 0$, whence the claim for JG follows from the previous.

For the last two parts, note that the metric trace of $\overset{W}{G}(J\cdot, J\cdot)$ is obviously the same as the metric trace of G (since J is orthogonal) while the metric trace of $2\overset{W}{G}(\cdot, J\cdot)$ is $-g^{ca}\nabla_b J^b{}_a$. □

4.4. Characterising the distinguished connection. We may use $\overset{gc}{\nabla}$ as the $\overset{G}{\nabla}$ in Proposition 2.4 to give a 1-parameter family $\overset{c,t}{\nabla}$ of connections determined by the conformal almost Hermitian structure. Acting with this on $g \in c$, we have

$$\overset{c,t}{\nabla}_a g_{bc} = 2B_a g_{bc} - t\big(g_{dc}\overset{c}{G}{}^d_{+ab} + g_{bd}\overset{c}{G}{}^d_{+ac}\big)$$

so this satisfies a conformal condition $\overset{c,t}{\nabla}_a g_{bc} = B'_a g_{bc}$ if and only if $t(\overset{c}{G}_{+cab} + \overset{c}{G}_{+bac}) = 2(B_a - B'_a)g_{bc}$. But, from Lemma 4.7, $\overset{c}{G}_+$ is trace-free and so either $t=0$ or

$$\overset{c}{G}_+(X,Y,Z) = -\overset{c}{G}_+(Z,Y,X).$$

But from Lemma 4.5 $\overset{c}{G}_+$ is also alternating on the first two arguments and hence if the display holds then altogether we have $\overset{c}{G}_+ \in \Lambda^3$. But this implies $\overset{c}{G}_+ = 0$ as by definition $\overset{c}{G}_+$ is Hermitian in the last pair of arguments, whereas from Lemma 4.5 it is also anti-Hermitian in the first pair. In summary, we have the following.

Theorem 4.8. *The connection* $\overset{c,t}{\nabla}$ *defined by*

$$\overset{c,t}{\nabla}_X Y := \overset{gc}{\nabla}_X Y + t\overset{c}{G}_+(X,Y) \tag{4.8}$$

is almost complex for any $t \in \mathbb{R}$. *It is conformal if and only if* $t=0$.

The theorem shows that, on any conformal almost Hermitian structure, $\overset{gc}{\nabla}$ is certainly distinguished among the natural class $\overset{c,t}{\nabla}$. We seek a characterisation which is in the spirit of the characterisation of the Levi-Civita connection on Riemannian manifolds as the unique torsion free connection preserving the metric. First, a preliminary result that in fact is stronger than we need.

Here we weaken the property of preserving J to just compatibility and conformality. We show that compatible conformal connections are parametrised by the algebraic torsion tensors, that is sections of $TM \otimes (\Lambda^2 T^*M)$.

Theorem 4.9. *Let* (M,J,g) *be an almost Hermitian manifold. Let* T *be a* $(1,2)$ *tensor satisfying* $T^a{}_{bc} = -T^a{}_{cb}$ *but which is otherwise arbitrary. Then there exists a unique* 1*-form* B' *and a unique connection* $\overset{T}{\nabla}$ *such that*:

- $\overset{T}{\nabla} g = 2B'g$;
- T *is the torsion of* $\overset{T}{\nabla}$;

- $\overset{T}{\nabla}_a J^a{}_b = 0$.

Proof. It is a straightforward calculation to verify that the unique solution is given by the following:

$$B'_d = A_d + B_d,$$

where

$$\overset{T}{A}_d := \frac{1}{n-2}\Big(T^a{}_{ad} + \frac{1}{2} J^a{}_c J^b{}_d \big(T_a{}^c{}_b - T_{ba}{}^c - T^c{}_{ba}\big)\Big); \tag{4.9}$$

and the difference tensor $\overset{T}{G}$, defined by $\overset{T}{G}(Y,X) := \overset{T}{\nabla}_X Y - \overset{c}{\nabla}_X Y$, is given by

$$\overset{T}{G}_{abc} = \overset{T}{A}_a g_{bc} - \overset{T}{A}_b g_{ac} - \overset{T}{A}_c g_{ab} + \frac{1}{2}(T_{cab} - T_{abc} - T_{bca}). \tag{4.10}$$ □

Note that because g was arbitrary (apart from the condition of being almost Hermitian) in the calculations of the proof, and $B'_d = A_d + B_d$, it follows at once that A_b and hence $\overset{T}{G}$ (with the latter as a $(1,2)$ tensor) are conformally invariant. This is also clear by inspection of the formulae (4.9) and (4.10). These objects are purely dependent on the torsion of $\overset{T}{\nabla}$ and the structure (M, J, c).

The torsion $\overset{gc}{T}$ of the connection $\overset{gc}{\nabla}$ is a conformal invariant of a conformal almost Hermitian structure,

$$\overset{gc}{T}(X,Y) := \overset{c}{G}(Y,X) - \overset{c}{G}(X,Y).$$

On the other hand, on a fixed conformal almost Hermitian manifold it follows from Theorem 4.9 that the conformal almost complex connections are, in particular, parametrised by their torsions. Thus, we have the following characterisation of $\overset{gc}{\nabla}$.

Proposition 4.10. *Let (M, J, c) be a conformal Hermitian structure, then $\overset{gc}{\nabla}$ is the unique conformal almost complex connection with torsion $\overset{gc}{T}$.*

Theorem 4.9 and the proof of Proposition 2.1 enable us to invert the last observation and obtain a torsion characterisation of the canonical connection $\overset{gc}{\nabla}$ of Proposition 4.4. First, we summarise the presence of conformal invariants that we need.

Proposition 4.11. *On a conformal almost Hermitian manifold (M, J, c), there is a conformally invariant map from algebraic torsion tensors to 1-forms given by*

$$T^a{}_{bc} \mapsto \overset{T}{A}_d,$$

where $\overset{T}{A}$ *is given by* (4.9). *Thus for each algebraic torsion tensor field, we may form the conformally invariant* $(1,2)$ *tensor*

$$V(T) := J\overset{T}{G}(\cdot,\cdot) + \overset{T}{G}(J\cdot,\cdot), \tag{4.11}$$

where $\overset{T}{G}$ *is given by* (4.10).

THEOREM 4.12. *Let* ∇ *be an almost complex conformal connection on a conformal almost Hermitian structure* (M,J,c). *Then* $\nabla = \overset{gc}{\nabla}$ *if and only if* $V(\mathrm{Tor}\,\nabla) = 0$.

That is, on a conformal almost Hermitian structure, $\overset{gc}{\nabla}$ is the unique conformal almost complex connection with torsion satisfying the algebraic condition $V(\mathrm{Tor}\,\nabla) = 0$. The theorem provides a simple torsion condition which characterises this among all connections preserving the conformal almost Hermitian structure. Thus, it provides the sought conformal almost Hermitian analogue of the characterisation of the Levi-Civita connection. As in the Riemannian case (i.e., as in Proposition 3.5), we could impose further conditions on the torsion for each structure in the Gray–Hervella classification of conformal almost Hermitian structures and so obtain the following conclusion.

COROLLARY 4.13. *There is a canonical characteristic connection for each of the structures in the Gray–Hervella classification of conformal almost Hermitian manifolds.*

We next look at some of the standard conformal variants of the well known special almost Hermitian structures. In particular, we see how these fit into the current picture.

4.5. The nearly Kähler Weyl condition. It is natural to consider the conformal condition

$$(\overset{c}{\nabla}_X J)X = 0 \quad \text{for all } X \in \Gamma(TM);$$

equivalently

$$2J\overset{c}{G}(X,X) = 0 \quad \Leftrightarrow \quad \overset{c}{G}(X,X) = 0 \quad \forall X \in \Gamma(TM). \tag{4.12}$$

It is easily verified that this agrees with the manifold being in the class $\mathcal{W}_1 \oplus \mathcal{W}_4$ of [24], it is the natural conformal analogue of the nearly Kähler class, and is sometimes called *nearly Kähler Weyl* in the literature.

PROPOSITION 4.14. *A conformal almost Hermitian structure* (M,J,c) *is nearly Kähler Weyl if and only if* $\overset{c}{\nabla}$ *and* $\overset{gc}{\nabla}$ *have the same geodesics. In particular, they are in the same projective class.*

Proof. The geodesic equations $\overset{c}{\nabla}_X X = 0$ and $\overset{gc}{\nabla}_X X = 0$, for the two connections, differ by $\overset{c}{G}(X,X)$. □

Since those properties of the Levi-Civita connection and its associated G, that were used to prove Proposition 3.3, are also satisfied by $\overset{c}{\nabla}$ and $\overset{c}{G}$, we immediately have the analogous result, as follows.

PROPOSITION 4.15. *NKW structures satisfy the following conditions*

$$\overset{c}{G}_+ = 0$$

while

$$\mathrm{Tor}(\overset{NKW}{\nabla}) = -2\overset{c}{G}_- = N_J,$$

and is completely alternating.

4.6. The locally conformally almost Kähler condition. Recall that an almost Hermitian manifold (M, J, g) is said to be almost Kähler if the Kähler form $\omega = g(\cdot, J\cdot)$ is closed.

On a dimension $n = 4$ almost Hermitian manifold the map $\omega \wedge \cdot : \Lambda^1 \to \Lambda^3$ is an isomorphism. Thus, one has

$$d\omega = \theta \wedge \omega \tag{4.13}$$

for some 1-form θ. Note that it follows from this that $d\theta \wedge \omega = 0$. If $d\theta = 0$ then we say that the manifold is *locally conformally almost Kähler* (LCAK), since locally a conformally related metric has a closed Kähler form.

For an almost Hermitian manifold of dimension $n \geq 6$, the equation (4.13) does not hold generally, but if it does then $d\theta$ is zero, since then $\omega \wedge \cdot : \Lambda^2 \to \Lambda^4$ is injective. Thus for $n \geq 6$ the structure is LCAK if and only if (4.13) holds.

Using the metric g and its Levi-Civita connection ∇, (4.13) becomes

$$\nabla_a \omega_{bc} + \nabla_c \omega_{ab} + \nabla_b \omega_{ca} = \theta_a \omega_{bc} + \theta_c \omega_{ab} + \theta_b \omega_{ca}.$$

Contracting now with ω^{bc} (indices raised using g), and using $\omega^{bc}\omega_{bc} = n$, gives $(n-2)\theta = 2(n-2)B$, and so

$$\theta = 2B, \quad \text{if } n \geq 4,$$

where the 1-form B is as in (4.4) above. So θ is the Lee form.

Thus putting together the results for dimensions $n \geq 6$ with also the definition of LC(A)K in dimension 4, we see that in all dimensions $n \geq 4$ the invariant

$$F := dB$$

is an obstruction to an (almost) Hermitian manifold being LC(A)K; as we discuss below F is a conformal invariant. If F vanishes, and (4.13) holds, then the de Rham cohomology class of $[B] \in H^1(M)$ is the obstruction to (M, J, g) being conformally (almost) Kähler.

In dimensions $n \geq 6$, we can capture the full obstruction to the LCAK condition in terms of $\overset{cq}{G}$. Expanding the left- and right-hand side of $d\omega = \theta \wedge \omega$

in terms of $\overset{gc}{\nabla}$ we obtain

$$\overset{gc}{\nabla}_{g[c}\omega_{ab]} + \omega_{d[c}\overset{gc}{T}{}^{d}{}_{ab]} = \theta_{[c}\omega_{ab]},$$

where $[\cdots]$ indicates taking the totally skew part and $\overset{gc}{T}$ is the torsion of $\overset{gc}{\nabla}$. But $\overset{c}{\nabla}$ preserves J, and acts on g as in Proposition 4.4. So $\overset{gc}{\nabla}_{g[c}\omega_{ab]} = 2B_{[c}\omega_{ab]} = \theta_{[c}\omega_{ab]}$. Thus, $\omega_{d[c}\overset{gc}{T}{}^{d}{}_{ab]} = 0$ is equivalent to (4.13). In summary, and using the expression (2.8) for the torsion, we have the following. Here $\overset{gc}{T}(\cdot,\cdot,\cdot)$ means $g(\cdot,\overset{gc}{T}(\cdot,\cdot))$.

Proposition 4.16. *Let (M,J,g) be an (almost) Hermitian manifold of dimension $n \geq 6$. Then it is LC(A)K if and only if the conformal invariant*

$$\mathrm{Alt}\,\overset{c}{G}(J\cdot,\cdot,\cdot) \tag{4.14}$$

vanishes.

In the Hermitian case, there is also an alternative as follows. A Hermitian manifold is LCK if and only if the conformal invariant

$$\mathrm{Alt}\,\overset{c}{G}(\cdot,\cdot,\cdot) \quad \textit{or equivalently} \quad \mathrm{Alt}\,\overset{gc}{T}(\cdot,\cdot,\cdot) \tag{4.15}$$

vanishes.

Proof. The agreement (up to a constant) of (4.14) and $d\omega - \theta\wedge\omega$ was established above (and cf. e.g., [24]).

From the first part, it follows that $\mathrm{Alt}\,g(\cdot,\overset{c}{G}(J\cdot,J\cdot))$ is an equivalent obstruction. But, if (M,J,c) is conformal Hermitian then $\overset{c}{G}$ is Hermitian, and so this is $\mathrm{Alt}\,g(\cdot,\overset{c}{G}(\cdot,\cdot))$. ☐

Remark 4.17. Our condition $\mathrm{Alt}\,\overset{c}{G}(J\cdot,\cdot,\cdot) = 0$ is the Gray–Hervella conformally invariant condition saying that the structure (M,J,c) is in the class $W_2 \oplus W_4$. The introduction of connection $\overset{gc}{\nabla}$ enables us to say that in the Hermitian case the obstruction for the LCK is the non-trivial presence of a totally skew symmetric part of the torsion of $\overset{gc}{\nabla}$.

4.7. The Gray–Hervella types. We summarise here how the conformal Gray–Hervella classification appears in terms of the conformal intrinsic torsion $\overset{c}{G}$. Since the situation is rather degenerate in dimension 4 we assume here that $n \geq 6$. See Table 1.

The W_i are $U(m)$ irreducible parts of the representation W inducing $\Lambda^1 \otimes \Lambda^{1,1}$, where $\Lambda^{1,1}$ indicates the Hermitian part of Λ^2, so $W \cong W_1 \oplus W_2 \oplus W_3 \oplus W_4$; for details see [24]. The point is that (on an almost Hermitian manifold) $\overset{LC}{\nabla}\omega$ is a section of $\Lambda^1 \otimes \Lambda^{1,1}$ and the Gray–Hervella type indicates irreducible

TABLE 1

Gray–Hervella type	Condition in terms of $\overset{c}{G}$	Name
W_4	$\overset{c}{G}=0$	LCK
$W_1 \oplus W_4$	$\overset{c}{G}(X,X)=0$	NKW
$W_2 \oplus W_4$	$\mathrm{Alt}(g(\cdot, J\overset{c}{G}(\cdot,\cdot)))=0$	LCAK
$W_3 \oplus W_4$	$\overset{c}{G}=\overset{c}{G}_+$	Hermitian
$W_1 \oplus W_2 \oplus W_4$	$\overset{c}{G}=\overset{c}{G}_-$	
$W_1 \oplus W_3 \oplus W_4$	$\overset{c}{G}_-(X,X)=0$	
$W_2 \oplus W_3 \oplus W_4$	$\mathrm{Alt}(g(\cdot, J\overset{c}{G}_-(\cdot,\cdot)))=0$	

Recall the abbreviations: LCAK: Locally conformally almost Kähler; LCK: Locally conformally Kähler; NKW: Nearly Kähler Weyl.

parts of $\overset{LC}{\nabla}\omega$ that may be not zero on the structure; more precisely projection to the complementary W_i is certainly 0.

We could equally use the representation corresponding to $\overset{LC}{G}$, and in that language W_4 corresponds to a trace-type part of $\overset{LC}{G}$ that we have denoted B_a (cf. (4.6)). With this understood we can recover conditions in terms of $\overset{c}{G}$ and B for any of the Gray–Hervella types by combining the various conditions with also the possibility of setting B_a to zero. The latter eliminates W_4, and breaks conformal invariance. So for example the "pure" types arise from either setting $\overset{c}{G}=0$, to obtain the W_4-type, or any of the next three conditions in the table with, in addition, $B_a=0$. At the other extreme imposing only $B_a=0$ gives the $W_1\oplus W_2\oplus W_3$ type known as semi-Kähler. This enables the possibility of treating any of the almost Hermitian Gray–Hervella types using the conformal machinery from this section.

4.8. Higher conformal invariants. The primary Gray–Hervella classification is based around the vanishing of invariants involving just one covariant derivative of J. There are obvious ways to extend this to obtain special structures linked to higher jets of J and the metric. For example, one may look at the $U(m)$-type decomposition of the Riemann curvature, or alternatively for conformal questions the $U(m)$-type decomposition of the Weyl tensor, or the curvature $\overset{gc}{R}$, of $\overset{gc}{\nabla}$, or even of the covariant derivatives of $\overset{gc}{R}$.

Here we wish to describe special conformal conditions which arise from equations on B. These are more subtle because B itself is not itself conformally invariant (recall (4.2)) and some could be of interest since they are related to key objects in conformal geometry, such as the GJMS operators of [22]. There is also a link with the following question:

> Consider an even dimensional manifold M that admits an almost complex structure J. Suppose that M is equipped with a Riemannian signature conformal structure c. Does the structure (M, J, c) have a distinguished metric $g \in c$?

Because (M, J, c) certainly has a distinguished Weyl connection $\overset{c}{\nabla}$, this question is closely related to the properties of B_a. In the case that M is closed (i.e., compact without boundary), there is a celebrated answer due to Gauduchon [17] (see also [9]): There is a unique (up to homothety) metric $g \in c$ such that $\delta B = -\nabla^a B_a = 0$ (i.e., δ is the formal adjoint of d, in the metric scale g). By construction then this *Gauduchon metric* is an invariant of the structure (M, J, c). We shall show below that, at least in suitably generic settings, there is another distinguished metric. To study this, and other conformal invariants, effectively we need a small amount of additional background which yields another interpretation of B.

On any smooth n-manifold M the highest exterior power of the tangent bundle $(\Lambda^n TM)$ is a line bundle. Its square $(\Lambda^n TM)^2$ is orientable; let us assume an orientation. We may forget the tensorial structure of $(\Lambda^n TM)^2$ and view this purely as a line bundle, we shall write L or $E[1]$ for the $2n$th positive root. For $w \in \mathbb{R}$, we denote L^w by $E[w]$.

If (M, g) is a Riemannian manifold, then $(\Lambda^n TM)^2$ is canonically trivialised (and oriented) since $\Lambda^n g^{-1}$ is a section; for a section μ of $(\Lambda^n TM)^2$ the component function in the trivialisation is obtained (up to a non-zero constant) by a complete contraction with $\wedge^n g$ (where this notation means the projection of $\otimes^n g$ onto its part in $(\Lambda^n T^*M)^2$).

Now consider a conformal structure (M, c). Given any $g \in c$, we write σ_g for the positive section of $E[1]$ with $(\sigma_g)^{2n} = \Lambda^n g^{-1}$, under the identification of $E[2n]$ with $(\Lambda^n TM)^2$. Evidently on (M, c) there is a canonical section $\mathbf{g}$ of $S^2T^*M[2] := S^2T^*M \otimes E[2]$ (where S^2 indicates the symmetric second tensor power) with the property that given any $g \in c$ we have

$$\mathbf{g} = \sigma_g^2 g.$$

This is called the *conformal metric*. Note that any positive section σ of $E[1]$ determines a metric by $g := \sigma^{-2}\mathbf{g}$, so we call such a σ a *scale*. That $L = E[1]$ is a $2n$th root (rather than some other choice) is a convenient for conformal geometry, thus we often term sections of $E[w]$ *conformal densities* of weight w.

In this section, we note that there are a number of conformal invariants associated to almost Hermitian manifolds. We have not tried to be exhaustive in the treatment here, but rather we have attempted to indicate some interesting directions. Suppose then that we have a conformal almost Hermitian manifold (M, J, c). First note that, from the definition of σ_g, it follows that for any $g \in c$, $\overset{c}{\nabla}_a \sigma_g = -B_a \sigma_g$, and similar for $\overset{gc}{\nabla}$, and so we have the following.

PROPOSITION 4.18. *The canonical connections* $\overset{c}{\nabla}$, *and* $\overset{gc}{\nabla}$ *preserve the conformal metric*

$$\overset{c}{\nabla}\mathbf{g} = 0 \quad \textit{and} \quad \overset{gc}{\nabla}\mathbf{g} = 0.$$

Recall that an AH structure that satisfies (3.6), i.e., $\delta\omega = 0$, is said to be semi-Kähler. We are interested here in related conformal conditions. From above, we have that each metric $g \in c$ determines, through the corresponding Levi-Civita connection ∇, a Weyl potential

$$B^g_a = \frac{1}{n-2} J^c{}_b \nabla_c J^b{}_a = -\frac{1}{n-2} J^b{}_a \nabla_c J^c{}_b.$$

Thus

$$J^a{}_b B^g_a = \frac{1}{n-2} \nabla_c J^c{}_b = -\frac{1}{n-2} \nabla^c \omega_{cb}.$$

That is

$$(n-2) JB = -\delta\omega,$$

and so an AH structure is semi-Kähler if and only if $B = 0$.

Now recall that B transforms conformally according to (4.2), that is,

$$B^{\widehat{g}}_a = B^g_a + \Upsilon_a,$$

where $\widehat{g} = e^{2\phi} g$, and $\Upsilon = d\phi$, and $\phi \in C^\infty(M)$. It follows that if B^g_a is exact then so is $B^{\widehat{g}}_a$, in which case we shall say that (M, J, c) is *conformally semi-Kähler* (CSK); on a conformally semi-Kähler manifold there is a distinguished metric $g \in c$, namely the metric satisfying

$$B^g_a = 0 \quad \text{equivalently} \quad \delta\omega = 0 \quad \text{equivalently} \quad \overset{c}{\nabla} = \overset{LC}{\nabla}{}^g.$$

It also follows from (4.2) that the *Faraday tensor*

$$F = dB$$

is a conformal invariant of (M, J, c). In fact this is the curvature of $\overset{c}{\nabla}$ as a connection on $L^{-1} = E[-1]$, from which its invariance is immediate. If $F = 0$ for a conformal almost Hermitian structure (M, J, c) then we are locally in the situation above, so the structure is *locally conformally semi-Kähler* (LCSK). (So, from the discussion of Section 4.6, in dimension 4 this agrees with LCAK while in higher dimensions LCAK implies LCSK.) In an obvious way, there are weakenings of this condition by decomposing F into its Hermitian and anti-Hermitian parts $F_\pm$ (which are obviously conformal invariants). We may specify that $F_+ = 0$ or $F_- = 0$.

There are also routes to more subtle conformal invariants, and we wish indicate these. In the discussion below, Λ^k denotes the bundle of k-forms or, by way of notational abuse, the sections of this bundle; and Λ_k is the tensor product of Λ^k with the conformal $(2k-n)$-densities.

The manifold M is oriented by the almost complex structure. On oriented dimension 4 manifolds the bundle map known as the Hodge-star operator is conformally invariant on 2-forms

$$\star : \Lambda^2 \to \Lambda^2$$

and this squares to 1. The 2-forms decompose orthogonally into the eigenspaces of this and applying the respective projections to F we obtain, respectively, the self-dual/anti-self-dual parts $F^{*\pm}$ of F as conformal invariants of (M, J, c). Since almost complex manifolds are naturally oriented we have these invariants on any 4-dimensional almost Hermitian manifold, and the vanishing of just one of these gives an obvious weakening of the locally conformally semi-Kähler condition. We may say that (M, J, c) is *Faraday (anti-)self-dual* if $F^{*-} = 0$ (respectively $F^{*+} = 0$); if one or the other condition holds we may say the structure is *Faraday half-flat.* In an obvious way, we may seek a finer grading by further decomposing the $F^{*\pm}$ into Hermitian and anti-Hermitian parts $F^{*\pm}_{\pm}$. This is partially successful. A straightforward calculation shows that:

$$\begin{aligned} &\operatorname{rank}\left(\Lambda^2\right)^{*+}_{+} = 1; \qquad &&\operatorname{rank}\left(\Lambda^2\right)^{*+}_{-} = 2;\\ &\operatorname{rank}\left(\Lambda^2\right)^{*-}_{+} = 3; \qquad &&\operatorname{rank}\left(\Lambda^2\right)^{*-}_{-} = 0. \end{aligned}$$

More precisely $(\Lambda^2)^{*+}_{+} = \mathbb{R}\omega$, $(\Lambda^2)^{*+}_{-} = \omega^{\perp} \cap (\Lambda^2)^{*+}$, $(\Lambda^2)^{*-}_{+} = (\Lambda^2)^{*-}$, $(\Lambda^2)^{*-}_{-} = 0$, meaning the zero bundle. Then the condition $F \wedge \omega = 0$ from Section 4.6 imples F has no component in $(\Lambda^2)^{*+}_{+}$.

The formal adjoint of $d : \Lambda^1 \to \Lambda^2$, which we denote $\delta : \Lambda_2 \to \Lambda_1$, is also conformally invariant. Thus in dimension 4 the *Maxwell current* $\delta F = \delta dB$ is also conformally invariant, and if this is zero then we may say the conformal almost Hermitian structure is simply *Maxwell.*

In fact, there is analogue for all dimensions $n \geq 4$ of the last result. In order to state this, and for the subsequent developments, we shall need the following result from [6].

THEOREM 4.19. *On Riemannian manifolds of even dimension $n \geq 4$ there are natural formally self-adjoint differential operators*

$$Q^g_k : \Lambda^k \to \Lambda_k, \quad k = 0, 1, \ldots, n/2 + 1,$$

with $Q^g_{n/2} = 1$, $Q^g_{n/2+1} = 0$ and otherwise with properties as follows. Up to a non-zero constant scale, Q^g_k has the form

$$(d\delta)^{n/2-k} + \mathrm{LOT}$$

and $Q_0 1$ is the (Branson) Q-curvature. Then

- *Upon restriction to the closed k-forms $\mathcal{C}^k$, Q^g_k has the conformal transformation law*

$$Q^{\widehat{g}}_k u = Q^g_k u + \delta Q^g_{k+1} d(\phi u), \tag{4.16}$$

  where $\widehat{g} = e^{2\phi} g$ with ϕ a smooth function.
- *It follows that the differential operator $\delta Q^g_{k+1} d =: L_k : \Lambda^k \to \Lambda_k$ is conformally invariant.*
- *The operators $G_k : \Lambda^k \to \Lambda_{k-1}$ defined by the composition $G_k := \delta Q_k$ are conformally invariant on the null space of L_k. In particular, further restricting, the map $G_k : \mathcal{C}^k \to \Lambda_{k-1}$ is conformally invariant.*

Thus we have the following generalisation of the Maxwell current.

THEOREM 4.20. *The* (*conformally weighted*) 1*-form field $G_2 F = L_1 B$ in Λ_1 is a local* (*conformal*) *invariant of conformal almost Hermitian manifolds.*

REMARK 4.21. Thus, $G_2 F = 0$ gives a weakening of the LCSK condition.

Note that $G_2 F$ may be viewed as an analogue (for $\overset{c}{\nabla}$ viewed as a connection on L^{-1}) of the Fefferman–Graham obstruction tensor of [15], since the latter can be seen to arise from a non-linear analogue of G_2 applied to a certain curvature (the curvature of the conformal tractor connection) [20].

Next if the invariant $G_2 F = 0$ (which holds trivially if (M, J, c) is LCSK), then we are able to use the third bullet point of Theorem 4.19. However some care is required as B is not invariant. For $g \in c$, in this case we obtain that

$$Q^g_J := G_1 B^g$$

is a $-n$-density that transforms conformally like the Q-curvature: if $\widehat{g} = e^{2\phi} g$ as above then

$$Q^{\widehat{g}}_J = Q^g_J + L_0 \phi, \tag{4.17}$$

where we also note that L_0, from Theorem 4.19, is the dimension order conformally invariant Laplacian power operator of [22] (the so-called critical GJMS operator). Clearly locally constant functions are in the kernel of L_0, if these give the entire null space $\mathcal{N}(L_0)$ then L_0 is said to have *trivial kernel.* Since L_0 is formally self-adjoint [25], from standard Fredholm theory we have the following result.

THEOREM 4.22. *Suppose (M, J, c) is a closed conformal almost Hermitian manifold satisfying the conformal condition $G_2 F = 0$, and such that the GJMS operator L_0 has trivial kernel. There is a unique preferred unit volume metric $g \in c$ satisfying*

$$Q^g_J = 0.$$

In the special case that (M, J, c) is CSK, g is the unique (*unit volume*) *semi-Kähler metric in the conformal class.*

On a generic (Riemannian signature) conformal manifold the critical GJMS operator has trivial kernel, but there are examples where the kernel is non-trivial (see, e.g., [13], [10] and similar examples are easily constructed). Thus, we note the following partial generalisation of the previous theorem.

THEOREM 4.23. *Suppose* (M, J, c) *is a closed conformal almost Hermitian manifold satisfying the conformal condition* $G_2F = 0$. *Then*

$$I_\phi = \int_M \phi Q_J$$

is a well-defined (*i.e.*, *conformally invariant*) *invariant for any function* ϕ *in the kernel of* L_0.

These conformal invariants obstruct the conformal prescription of zero Q_J. *That is there is metric* $g \in c$ *satisfying*

$$Q_J^g = 0$$

if and only if $I_\phi = 0$ *for all* $\phi \in \mathcal{N}(L_0)$. *Such a metric*, *if it exists*, *is unique up to* $g \mapsto e^{2\phi} g$ *where* $\phi \in \mathcal{N}(L_0)$.

Proof. By construction, the operator L_0 is formally self-adjoint, since the Q_k^g are, so the first result follows from (4.17) and the conformal weight of Q_J.

The second part is again immediate from standard spectral theory. □

REMARK 4.24. As mentioned any (locally) constant function is in the kernel of L_0, but Q_J is a divergence so the I_ϕ defined in Theorem 4.23 may only possibly be non-trivial on structures where the kernel of L_0 includes locally non-constant functions. The above theorems are analogues of results for Branson's Q-curvature and its prescription in the cases where the conformal invariant $\int_M Q$ is zero. See, [32] and Proposition 3.5 of [19], which use [7], [8].

Note that in the language of physics, the role of G_1 in Theorem 4.22 and Theorem 4.23 is as a "gauge fixing" operator. Indeed the (L_k, G_k) form graded injectively elliptic systems and in dimension four G_1 is the Eastwood–Singer conformal gauge fixing operator of [12].

There are further global conformal invariants available as follows. From the third bullet point of Theorem 4.19 we have that G_1 is conformally invariant on closed 1-forms. The conformally invariant subspace $\mathcal{H}^1$ of $\mathcal{C}^1$ consisting of those closed 1-forms that are also annihilated by G_1 is termed the space of *conformal harmonics* (of degree 1) and has dimension at least as large as the first Betti number, see [6]. The following is an easy consequence of the properties of Q_1 and the conformal transformation formula for B.

THEOREM 4.25. *On a closed conformal almost Hermitian manifold* (M,J,c), *satisfying the conformal condition* $G_2F=0$,

$$I_u := \int_M (B, Q_1 u)$$

is conformally invariant, for any 1*-form* $u \in \mathcal{H}^1$.

REMARK 4.26. The invariants I_u here generalise the I_ϕ of Theorem 4.23. If $u = d\phi$ for some function ϕ, then the condition $u \in \mathcal{H}^1$ is equivalent to $L_0\phi = 0$, and integrating by parts we obtain $I_u = I_\phi$.

Finally, note that in view of the transformation law (4.17) we can obviously add to the Branson Q-curvature Q a multiple of Q_J so as to obtain a local conformally invariant $(-n)$-density.

4.9. Summary. Any conformal almost Hermitian structure (M,J,c) determines a canonical Weyl structure $\overset{c}{\nabla}$ with conformally invariant curvature F (viewing $\overset{c}{\nabla}$ as a connection on $E[-1]$). There is a natural hierarchy of curvature conditions that one can consider:

- $\overset{c}{\nabla}$, and therefore also F, general;
- (1) $G_2F=0$; or
  (2) $F_+=0$ or $F_-=0$; and in dimension 4 $F^{*+}_+=0$, or $F^{*-}_+=0$, and/or, $F^{*+}_-=0$ (and note that in dimension 4 if either $F^{*+}=0$ or $F^{*-}=0$ then $G_2F=\delta F=0$);
- $F=0$, this is LCSK; then
  (1) $0 \neq [B^g] \in H^1(M)$, where $g \in c$; or
  (2) B^g exact, so $\overset{c}{\nabla}$ is a Levi-Civita connection, for some metric in c.

The last of these is the conformally semi-Kähler (co-symplectic) condition; in case that is satisfied then $\overset{c}{\nabla}$ is the unique Levi-Civita connection in the conformal class, that is compatible with J.

REMARK 4.27. None of the results developed in Section 4.8 above depend on J being orthogonal for the conformal structure. The discussion has assumed this only to link with conditions that are familiar in the literature (such as LCSK).

Moreover, only the results using/claiming ellipticity rely on Riemannian signature.

5. Projective geometry

A projective (differential) geometry consists of a manifold M equipped with an equivalence class p of torsion free affine connections (we write (M,p)); the class is characterised by the fact that two connections ∇ and $\widehat{\nabla}$ in p have the same *path structure*, that is the same geodesics up to parametrisation. In the following to avoid difficult language it will be useful to understand p in a slightly different way. We shall also use p to mean *that set of parametrised curves* such that each curve in p is a geodesic for some $\nabla \in p$. We shall say that a connection (in general with torsion) is *compatible with the path structure* if its geodesics agree with the geodesics of some $\nabla \in p$.

Explicitly, the connections ∇ and $\widehat{\nabla}$ are related by the equation

$$\widehat{\nabla}_a Y^b = \nabla_a Y^b + \Upsilon_a Y^b + \Upsilon_c Y^c \delta_a^b, \tag{5.1}$$

where Υ is some smooth section of T^*M. Equivalently two connections (on T^*M) in the same projective class are related by

$$\widehat{\nabla}_a u_b = \nabla_a u_b - \Upsilon_a u_b - \Upsilon_b u_a, \tag{5.2}$$

on any 1-form field u.

Here, we consider almost complex manifolds (M,J) of dimension $n = 2m$ ($m \in \{1,2,\dots\}$) equipped with a projective structure p. Associated to any $\nabla \in p$ we may form

$$A_c^{\nabla} := -\frac{1}{n}\big(\nabla_a J^a{}_b\big) J^b{}_c = \frac{1}{n} J^a{}_b \big(\nabla_a J^b{}_c\big). \tag{5.3}$$

It is then easily verified that if we change ∇ to $\widehat{\nabla}$, as in (5.1), then

$$A_c^{\widehat{\nabla}} = A_c^{\nabla} + \Upsilon_c. \tag{5.4}$$

We have the following result, which is an analogue of Proposition 4.1.

Proposition 5.1. *Let (M,J) be an almost complex manifold of any (even) dimension. If this is also equipped with a projective structure p, then there is a unique connection $\overset{p}{\nabla} \in p$ which is compatible with J.*

Proof. Let ∇ and $\overset{p}{\nabla}$ be connections in p. Then it is a straightforward exercise to show that $\overset{p}{\nabla}$ is compatible with J (as defined in Section 2.3) if and only if (as a connection on T^*M) we have

$$\overset{p}{\nabla}_a u_b = \nabla_a u_b + A_a^{\nabla} u_b + A_b^{\nabla} u_a, \tag{5.5}$$

where A_a^{∇} is given in terms of ∇ by the formula (5.3). Showing this uses that J is trace-free (since $J^2 = -\mathrm{id}$). □

Note that from the uniqueness it follows that $\overset{p}{\nabla}$ is independent of $\nabla \in p$, as used in the proof. Alternatively one can see this by putting together (5.1), (5.2), and (5.4).

Remark 5.2. Putting things in another order, we may state things (somewhat informally) as follows. Given an even dimensional projective manifold (M,p), any almost complex structure J on M yields a canonical breaking of the projective symmetry, via $\overset{p}{\nabla}$.

There is an obvious consequence of the proposition. The set of curves p has a distinguished subset, as follows.

Corollary 5.3. *Let (M, J, p) be a projective almost complex manifold of any dimension. Then M has a preferred class of parametrised curves, namely the geodesics of $\overset{p}{\nabla}$.*

5.1. Canonical projective almost complex connections. The curvature of $\overset{p}{\nabla}$, and its $\overset{p}{\nabla}$ covariant derivatives, are all invariants of the structure (M, J, p). Another invariant is

$$\overset{p}{G}{}^b{}_{ca} := \frac{1}{2}\big(\overset{p}{\nabla}_a J^b{}_d\big) J^d{}_c; \tag{5.6}$$

in analogy with the conformal case, this, and its tensor parts, play a fundamental role.

Specialising the development and results of Section 2 to the case that $\overset{p}{\nabla}$ is the initial affine connection, we can form a range of connections and geometric objects that are determined canonically by the almost complex structure J and the projective structure. In particular, we obtain the following result.

Proposition 5.4. *Let (M, J, p) be a projective almost complex structure. This determines a canonical affine connection $\overset{gp}{\nabla}$ defined by*

$$\overset{gp}{\nabla}_X Y = \overset{p}{\nabla}_X Y + \overset{p}{G}(Y, X), \quad \textit{where } \overset{p}{G}{}^b{}_{ca} := \frac{1}{2}\big(\overset{p}{\nabla}_a J^b{}_d\big) J^d{}_c.$$

This has the properties:

- $\overset{gp}{\nabla} J = 0$;
- *The anti-Hermitian part of its torsion gives the Nijenhuis tensor $N_J(X, Y) = T_-^{gp}(X, Y)$.*

Proof. All results are simply specialisations of statements in Proposition 2.1 and Corollary 2.6. □

Now we may use $\overset{gp}{\nabla}$ as the $\overset{G}{\nabla}$ in Proposition 2.4 to give a family of connections $\overset{p,t}{\nabla}$ determined by the projective almost complex structure, and parametrised by $t \in \mathbb{R}$. Recall that in the conformal setting the canonical connection $\overset{gc}{\nabla}$ had the congenial property that it preserved both J and the conformal structure. The projective analogue would be a connection that preserves both J and the path structure of the projective geometry. Indeed since, even more, there are preferred geodesics (by Corollary 5.3) it is reasonable to seek a connection that shares this preference. We investigate using the family $\overset{p,t}{\nabla}$.

For each choice of t, the connection $\overset{p,t}{\nabla}$ has the same geodesics as the connection $\overset{p}{\nabla}$ (and so the same paths as any connection in p) if and only if

$$\overset{p}{G}(X,X) + t\overset{p}{G}_+(X,X) = 0 \quad \text{for all } X \in \Gamma(TM).$$

This must also hold if X is replaced by JX. Together we obtain an equivalent system

$$(1+t)\overset{p}{G}_+(X,X) + \overset{p}{G}_-(X,X) = 0,$$
$$(1+t)\overset{p}{G}_+(X,X) - \overset{p}{G}_-(X,X) = 0.$$

In solving this, there are two cases: if $t \neq -1$ then $\overset{p}{G}_+(X,X) = \overset{p}{G}_-(X,X) = 0$ whence $\overset{p}{G}(X,X) = 0$; if $t = -1$ then we have the weaker requirement $\overset{p}{G}_-(X,X) = 0$. Since the condition $2J\overset{p}{G}(X,X) = (\overset{p}{\nabla}_X J)X = 0$, for all $X \in \Gamma(TM)$, is an analogue of the nearly Kähler condition (on almost Hermitian structures), if this holds we shall say the structure (M,J,p) is *projective nearly Kähler*. We summarise as follows.

THEOREM 5.5. *The connection* $\overset{p,t}{\nabla}$ *defined by*

$$\overset{p,t}{\nabla}_X Y := \overset{gp}{\nabla}_X Y + t\overset{p}{G}_+(X,Y)$$

is almost complex. It has the same path structure as (M,J,p) *if and only if*:

- $t \neq -1$ *and the manifold is projective nearly Kähler, that is,*

$$(\overset{p}{\nabla}_X J)X = 0 \quad \textit{for all } X \in \Gamma(TM);$$

- $t = -1$ *and the manifold satisfies*

$$(\overset{p}{\nabla}_X J)X = (\overset{p}{\nabla}_{JX} J)(JX) \quad \textit{for all } X \in \Gamma(TM). \tag{5.7}$$

It is evident from the theorem that the connection

$$\overset{JP}{\nabla} := \overset{p,-1}{\nabla}$$

is distinguished, since it imposes the weakest condition on the structure (M,J,p) in order to have the geodesics from the distinguished set of Corollary 5.3. Since the value of any of these connections is limited without path structure preservation we make the following definition.

DEFINITION 5.6. On an even manifold M let J be an almost complex structure and p a projective structure. We shall say that p and J are *compatible* if

$$\overset{p}{G}_-(X,X) = 0, \quad \forall X,$$

or equivalently if (5.7) holds. When this holds we shall also say that (M,J,p) is a compatible projective almost complex structure.

Then for emphasis we may state the following.

COROLLARY 5.7. *On any compatible projective almost complex structure* (M, J, p) *the canonical connection* $\overset{JP}{\nabla}$ *preserves* J *and has as geodesics the distinguished curves determined by* p *and* J (*as in Corollary* 5.3).

REMARK 5.8. The term "compatible" as defined here is not easily confused with its use to describe properties of a particular affine connection, as defined in Section 2.3.

Together equation (5.7) and the compatibility equation $\overset{p}{\nabla}_a J^a{}_b = 0$ are *formal* analogues of the equations used by Gray and Hervella [24] to characterise the class $W_1 \oplus W_3$ in almost Hermitian geometry. It is more closely an analogue of the conformal class $W_1 \oplus W_3 \oplus W_4$.

The $t = 1$ case of Theorem 5.5 recovers the projectively canonical version of the $\overset{KN}{\nabla}$ connection which has torsion precisely agreeing with the Nijenhuis tensor.

LEMMA 5.9. *Two affine connections* ∇ *and* ∇' *have the same geodesics if and only if their difference tensor is an algebraic torsion tensor.*

Proof. Let T be a section of $TM \otimes (\Lambda^2 T^* M)$. Then it is clear that ∇' defined by $\nabla'_X Y = \nabla_X Y + \frac{1}{2} T(X, Y)$ has the same geodesics as ∇.

Conversely suppose that $\frac{1}{2} T$ is the difference tensor given by $\nabla' - \nabla$ and T is not a section of $TM \otimes (\Lambda^2 T^* M)$. In this case there exists a point $q \in M$ and $X_q \in T_q M$ such that at q we have $T(X_q, X_q) \neq 0$. Then the ∇-geodesic through q that has tangent there X_q is not a geodesic for ∇'. □

In the following we shall use the following notation. Given a $(1, 2)$ tensor H we will denote its symmetric and skew parts by H^{symm} and H^{skew}, respectively. That is

$$
\begin{aligned}
H^{\mathrm{symm}}(X, Y) &= \frac{1}{2}\big(H(X, Y) + H(Y, X)\big), \\
H^{\mathrm{skew}}(X, Y) &= \frac{1}{2}\big(H(X, Y) - H(Y, X)\big).
\end{aligned}
$$

The interpretation of the solution leading to Theorem 5.5 is clear via the Lemma 5.9. The difference $(\overset{p,t}{\nabla} - \overset{p}{\nabla})(X, Y)$ is $\overset{p}{G}(Y, X) + t\overset{p}{G}_+(X, Y)$. So for example at $t = -1$ this is $\overset{p}{G}{}^{\mathrm{skew}}_+(Y, X) + \overset{p}{G}_-(Y, X)$. Thus the J, p compatibility condition $\overset{p}{G}{}^{\mathrm{symm}}_- = 0$ is exactly what is required to achieve a skew difference tensor, as required since $\overset{p}{\nabla}$ torsion free.

At this point it is reasonable to ask if there is an almost complex connection with the same geodesics as $\overset{p}{\nabla}$, without imposing the J, p compatibility condition $\overset{p}{G}{}^{\mathrm{symm}}_- = 0$. In other words whether might be some connection that is "better" than $\overset{JP}{\nabla}$ in this sense. We shall now show that there is no such connection.

THEOREM 5.10. *Let (M, J, p) be a projective almost complex structure. Suppose that there exists an affine connection ∇' such that ∇' is almost complex and ∇' has geodesics in the projective class p. Then J and p are compatible, that is*

$$\overset{p}{G}{}^{\mathrm{symm}}_{-} = 0,$$

and

$$\nabla' = \overset{JP}{\nabla} + \frac{1}{2}T,$$

where T is an anti-Hermitian algebraic torsion tensor (that is $T(X, Y) = -T(Y, X)$ and $T(JX, JY) = -T(X, Y)$) and ∇' has the same geodesics as $\overset{p}{\nabla}$.

Proof. Let us assume that ∇' preserves J and actually has the same geodesics as $\overset{p}{\nabla}$. Then since ∇' preserves J, and using Proposition 2.1, we have that

$$\nabla'_X Y = \overset{gp}{\nabla}_X Y + K(Y, X),$$

where K is a $(1, 2)$ tensor which is complex linear in the first argument. Now using that ∇' has the same geodesics as $\overset{p}{\nabla}$ then, using the Lemma 5.9, this implies that

$$\overset{p}{G}(Y, X) + K(Y, X) = \frac{1}{2}T(Y, X)$$

for some $(1, 2)$ tensor T that is skew, i.e., $T(X, Y) = -T(Y, X)$. Taking the symmetric part of both sides gives

$$\overset{p}{G}{}^{\mathrm{symm}} + K^{\mathrm{symm}} = 0. \tag{5.8}$$

Further taking the anti-Hermitian part we have

$$\overset{p}{G}{}^{\mathrm{symm}}_{-} + K^{\mathrm{symm}}_{-} = 0. \tag{5.9}$$

Now from (5.8), we have

$$\overset{p}{G}(JX, X) + \overset{p}{G}(X, JX) + K(JX, X) + K(X, JX) = 0.$$

Multiplying through with J gives

$$\overset{p}{G}(X, X) - \overset{p}{G}(JX, JX) - K(X, X) + K(JX, JX) = 0,$$

where we have used $\overset{p}{G}(\cdot, X)$ is complex anti-linear, while $K(\cdot, X)$ is complex linear. Thus

$$\overset{p}{G}{}^{\mathrm{symm}}_{-} - K^{\mathrm{symm}}_{-} = 0,$$

which with (5.9) implies that $\overset{p}{G}{}^{\mathrm{symm}}_{-} = 0$, as claimed. Since (M, J, p) is thus seen to be a compatible projective almost complex structure, $\overset{JP}{\nabla}$ is a connection with the same geodesics as $\overset{p}{\nabla}$. Thus $\nabla' = \overset{JP}{\nabla} + \frac{1}{2}T$ for some algebraic torsion tensor field T. Using now that both ∇' and $\overset{JP}{\nabla}$ preserve J it follows immediately that T is anti-Hermitian.

In the above, we assumed that ∇' has the same geodesics as $\overset{p}{\nabla}$. Let us now relax that and assume only that ∇' (is an almost complex connection that) has geodesics in the set p given by the projective structure. Then $\widehat{\nabla} := \nabla' - \frac{1}{2}\operatorname{Tor}\nabla'$ is a torsion free connection with the same geodesics as ∇' and so

$$\widehat{\nabla}_a Y^b = \overset{p}{\nabla}_a Y^b + \Upsilon_a Y^b + \Upsilon_c Y^c \delta^b_a,$$

for some 1-form field Υ. Adding ($\frac{1}{2}$ of) the torsion of $\overset{JP}{\nabla}$, it now follows that

$$\overset{JP}{\nabla}_a Y^b + \Upsilon_a Y^b + \Upsilon_c Y^c \delta^b_a$$

has the same geodesics as ∇'. The difference between this connection and ∇' is some algebraic torsion tensor field, while both $\overset{JP}{\nabla}$ and ∇' preserve J. Thus, it is the case that there is an algebraic torsion tensor field T so that for any fixed Y

$$X \mapsto X \cdot \Upsilon(Y) + Y \cdot \Upsilon(X) + T(X, Y)$$

is J-linear. In particular, then taking $Y = JX$ and applying this map to JX, the J-linearity implies that

$$2(JX) \cdot \Upsilon(JX) = J\big(X \cdot \Upsilon(JX)\big) + J\big((JX) \cdot \Upsilon(X)\big)$$

and so

$$(JX) \cdot \Upsilon(JX) = -X \cdot \Upsilon(X), \quad \forall X.$$

This implies $\Upsilon = 0$. Hence ∇' has the same geodesics as $\overset{p}{\nabla}$ and we are reduced to the situation treated first. $\square$

Finally, using the notation of the Lemma 5.9, observe that if $\nabla'_X Y = \nabla_X Y + \frac{1}{2}T(X, Y)$ then $T = \operatorname{Tor}\nabla' - \operatorname{Tor}\nabla$. Thus, the space of connections with the same geodesics as $\overset{p}{\nabla}$ is an affine space through $\overset{p}{\nabla}$ and modelled on the vector space of algebraic torsion tensor fields. In particular, the connections in the class are uniquely parametrised by their torsion. Thus, we have the following result.

Corollary 5.11. *On a compatible projective almost complex structure $\overset{JP}{\nabla}$ is the unique almost complex connection with the same geodesics as $\overset{p}{\nabla}$, and torsion*

$$-\frac{1}{2}\overset{p}{G}{}^{\mathrm{skew}}.$$

TABLE 2

Conformal Gray–Hervella type	Usual name	Analogous projective condition
$\overset{c}{G}=0$	LCK	$\overset{p}{G}=0$
$\overset{c}{G}(X,X)=0$	NKW	$\overset{p}{G}(X,X)=0$
$\mathrm{Alt}(g(\cdot,J\overset{c}{G}(\cdot,\cdot)))=0$	LCAK	none available
$\overset{c}{G}=\overset{c}{G}_+$	Hermitian	$\overset{p}{G}=\overset{p}{G}_+$
$\overset{c}{G}=\overset{c}{G}_-$		$\overset{p}{G}=\overset{p}{G}_-$
$\overset{c}{G}_-(X,X)=0$		$\overset{p}{G}_-(X,X)=0$
satisfied identically		$\overset{p}{G}_+(X,X)=0$
$\mathrm{Alt}(g(\cdot,J\overset{c}{G}_-(\cdot,\cdot)))=0$		none available

Recall the abbreviations: CNK: Conformally nearly Kähler; LCAK: Locally conformally almost Kähler; LCK: Locally conformally Kähler; NKW: Near Kähler Weyl.

5.2. Projective analogues of the conformal Gray–Hervella types. Note that $\overset{p}{G}$ is trace-free, but in an obvious way this admits an $SL(m)$ type decomposition into the parts

$$\overset{p}{G}{}^{\mathrm{symm}}_+, \qquad \overset{p}{G}{}^{\mathrm{skew}}_+, \qquad \overset{p}{G}{}^{\mathrm{symm}}_-, \quad \text{and} \quad \overset{p}{G}{}^{\mathrm{skew}}_-,$$

where for example, $\overset{p}{G}{}^{\mathrm{skew}}_+(X,Y)=\frac{1}{2}(\overset{p}{G}_+(X,Y)-\overset{p}{G}_+(Y,X))$. It is easily verified that in dimensions greater than 2 these four components are functionally independent (see Section 6).

We summarise here how some of the available projectively invariant conditions we can impose, in terms of this decomposition, correspond to analogous conditions in the conformal Gray–Hervella classification. See Table 2.

Note that the condition $\overset{p}{G}{}^{\mathrm{skew}}_-=0$ is the condition for a projective almost complex structure to be integrable, that is $N_J=0$. Thus this is implied by (but weaker than) the "Hermitian" condition $\overset{p}{G}=\overset{p}{G}_+$. A projective almost complex structure (M,J,p) is compatible if and only if $\overset{p}{G}{}^{\mathrm{symm}}_-=0$. Thus for a compatible projective almost complex structure (M,J,p) the integrability condition and the "Hermitian" condition $\overset{p}{G}=\overset{p}{G}_+$ agree (as in the conformal case).

The condition $\overset{p}{G}_+(X,X)=0$ is one example of many conditions one could impose in the projective case for which there is no conformal analogue.

5.3. Higher projective invariants. On any smooth n-manifold M the highest exterior power of the tangent bundle $(\Lambda^n TM)$ is a line bundle. As discussed in Section 4.8, for any smooth n-manifold M, $(\Lambda^n TM)^2$ is an orientable line bundle. Again we choose an orientation and a root: For our subsequent discussion, it is convenient to take the positive $(2n+2)$th root of $(\Lambda^n TM)^2$ and we denote this K or $E(1)$. Then for $w \in \mathbb{R}$ we denote K^w by $E(w)$. Sections of $E(w)$ will be described as *projective densities* of weight w.

Now we consider a projective manifold (M,p). Each connection $\nabla \in p$ determines a connection (also denoted ∇) on $(\Lambda^n TM)^2$ and hence on its roots $E(w)$, $w \in \mathbb{R}$. For $\nabla \in p$ let us (temporarily) denote the connection induced on $E(1)$ by D^∇, and write $-F^\nabla$ for its curvature. It is easily verified that, under the transformation (5.1), D transforms according to

$$D_a^{\widehat{\nabla}} = D_a^\nabla + \Upsilon_a,$$

where we view Υ_a as a multiplication operator. Since the connections on $E(1)$ form an affine space modelled on $\Gamma(T^*M)$ it follows that by moving around in p we can hit any connection on $E(1)$, and conversely a choice of connection on $E(1)$ determines a connection in p.

Now $E(1)$ is a trivial bundle and any chosen trivialisation determines a flat connection on $E(1)$ in the obvious way. Such a connection will be called a *scale*. It follows that there is a special class s of connections in p: $\nabla \in s$ if and only if D^∇ is a scale; if D^∇ is a scale we shall also call ∇ a scale. Again using that $E(1)$ is a trivial bundle it follows that ∇ is a scale if and only if $F^\nabla = 0$.

Since

$$F^{\widehat{\nabla}} = F^\nabla - d\Upsilon \tag{5.10}$$

it is clear that if ∇ and $\widehat{\nabla}$ are both scales then $d\Upsilon = 0$; in fact from the definition of scales Υ is then exact. We will henceforth drop the notation D^∇ and write ∇ for any connection, induced by $\nabla \in p$, on densities, tensor bundles and so forth.

We now consider a projective almost complex manifold (M,J,p); this is oriented by J. We have then the preferred connection $\overset{p}{\nabla}$ on TM. Thus the curvature $\overset{p}{R}$ of $\overset{p}{\nabla}$ is a (projective) invariant of the structure. Consider now $\overset{p}{\nabla}$ as a connection on $E(n+1)$. From the definition of $E(n+1)$, it follows that the curvature of $\overset{p}{\nabla}$ on this is the trace of $\overset{p}{R}$, as given:

$$\overset{p}{R}_{ab}{}^c{}_c.$$

Whence the curvature of $\overset{p}{\nabla}$ on $K = E(1)$ is $-\overset{p}{F} := -F^{\overset{p}{\nabla}}$ where $-(n+1)\overset{p}{F} = \overset{p}{R}_{ab}{}^c{}_c$. For comparison with other discussions of projective geometry, such as e.g. [3] we note that since $\overset{p}{\nabla}$ is torsion free, $\overset{p}{R}_{ab}{}^c{}_d$ satisfies the first Bianchi identity and hence $\overset{p}{R}_{ab}{}^c{}_c = -2\,\mathrm{Ric}_{[ab]}$ (where $[\cdots]$ indicates the skew part). So $(n+1)\overset{p}{F} = 2\,\mathrm{Ric}_{[ab]}$ (and thus $\overset{p}{F} = -\overset{p}{\beta}$, where $\overset{p}{\beta}$ is defined in Section 3 of [3]).

The curvature $\overset{p}{F}$ is a fundamental invariant of the structure. By the Bianchi identity, this 2-form is closed $d\overset{p}{F} = 0$. An obvious question is whether it has cohomological content. Suppose we choose a scale ∇ in p and compare its curvature (which is 0) with that of $\overset{p}{\nabla}$. Using (5.10) we obtain the following.

THEOREM 5.12. *Let (M, J, p) be a projective almost complex structure. The curvature $F^{\overset{p}{\nabla}}$ of the canonical Weyl connection is an invariant of the structure. This two form is exact: for any scale $\nabla \in p$ we have*

$$F^{\overset{p}{\nabla}} = dA^{\nabla}.$$

Of course we may obtain further invariants of (M, J, p) by decomposing $F^{\overset{p}{\nabla}}$ into its Hermitian and anti-Hermitian parts:

$$F_{\pm}(\cdot,\cdot) := F(\cdot,\cdot) \pm F(J\cdot, J\cdot).$$

From Theorem 5.12, we see that the situation is analogous to the conformal almost Hermitian case. Note that the analogue of conformal transformations are *special projective transformations*, that is, those transformations (5.1) (equivalently (5.2)) where Υ is required to be exact. With these observations in mind, we see there is a natural hierarchy of curvature conditions that one can consider:

- $\overset{p}{\nabla}$, and therefore also $F^{\overset{p}{\nabla}}$, general;
- $F_{+}^{\overset{p}{\nabla}} = 0$ or $F_{-}^{\overset{p}{\nabla}} = 0$;
- $F^{\overset{p}{\nabla}} = 0$; then
  (1) $0 \neq [A^{\nabla}] \in H^1(M)$, where ∇ is any scale; or
  (2) A^{∇} exact, so $\overset{p}{\nabla}$ is a scale.

The last of these is a projective analogue of the conformally semi-Kähler (cosymplectic) condition; in case that is satisfied then $\overset{p}{\nabla}$ is the unique preferred scale that is compatible with J.

6. Examples

6.1. Example of an (M, J, c) structure with nonvanishing $F = dB$ satisfying Maxwell's equations. We consider a 4-dimensional manifold M, which is a local product of a real line $\mathbb{R}$ and a 3-dimensional Lie group G, $M = \mathbb{R} \times G$. Let θ^1, θ^2, θ^3 be a basis of the left invariant forms on G. We have

$$\begin{aligned} d\theta^1 &= a_3\theta^1 \wedge \theta^2 + a_2\theta^3 \wedge \theta^1 + a_1\theta^2 \wedge \theta^3, \\ d\theta^2 &= b_3\theta^1 \wedge \theta^2 + b_2\theta^3 \wedge \theta^1 + b_1\theta^2 \wedge \theta^3, \\ d\theta^3 &= c_3\theta^1 \wedge \theta^2 + c_2\theta^3 \wedge \theta^1 + c_1\theta^2 \wedge \theta^3, \end{aligned} \tag{6.1}$$

where a_i, b_i, c_i are real constants which satisfy relations implied by the Jacobi identity $d^2\theta^i \equiv 0$. Simple solutions for these relations are:

$$a_3 = \frac{a_2^2 c_1 - a_1 b_2 c_1 - a_1 a_2 c_2 + a_1 b_1 c_2}{a_2 b_1 - a_1 b_2},$$
$$b_3 = \frac{a_2 b_2 c_1 - b_1 b_2 c_1 + b_1^2 c_2 - a_1 b_2 c_2}{a_2 b_1 - a_1 b_2},$$
$$c_3 = \frac{-b_2 c_1^2 + a_2 c_1 c_2 + b_1 c_1 c_2 - a_1 c_2^2}{a_2 b_1 - a_1 b_2},$$

and, in the following, we will consider equations (6.1) with a_3, b_3 and c_3 as above.

We equip M with the canonical projection $\pi : M \to G$, introduce a coordinate t along the $\mathbb{R}$ factor and consider a coframe $(\omega^1 = \pi^*\theta^1, \omega^2 = \pi^*\theta^2, \omega^3 = \pi^*\theta^3, \omega^4 = dt)$ on M. Using this coframe, we define a (c, J) structure on M by representing the conformal class c via

$$c \ni g = \left(\omega^1\right)^2 + \left(\omega^2\right)^2 + \left(\omega^3\right)^2 + \left(\omega^4\right)^2$$

and the almost complex structure J via:

$$J = \omega^1 \otimes e_3 - \omega^3 \otimes e_1 + \omega^2 \otimes e_4 - \omega^4 \otimes e_2.$$

Here (e_1, e_2, e_3, e_4) is a frame of vector fields on M which is dual to the coframe $(\omega^1, \omega^1, \omega^1, \omega^1)$, $e_i \lrcorner\, \omega^j = \delta^j_i$.

Now we can find the canonical connection $\overset{c}{\nabla}$ as in (4.3). In particular, it is easy to get the explicit formula for the B-form defined in (4.4). This reads:

$$B = \frac{a_2 b_2 c_1 - b_1 b_2 c_1 + b_1^2 c_2 - a_1 b_2 c_2}{2(a_2 b_1 - a_1 b_2)}\omega^1 + \frac{(b_1 - a_2)(a_2 c_1 - a_1 c_2)}{2(a_2 b_1 - a_1 b_2)}\omega^2 - \frac{1}{2} b_1 \omega^3 - \frac{1}{2} b_2 \omega^4.$$

Its exterior differential is

$$F = dB = \frac{(a_2 c_1 - a_1 c_2)(b_2 c_1 - b_1 c_2)}{2(a_2 b_1 - a_1 b_2)}\omega^1 \wedge \omega^2 + \frac{1}{2}(a_1 c_2 - a_2 c_1)\omega^3 \wedge \omega^4$$

showing, for example, that if $(a_1 c_2 - a_2 c_1) \neq 0$ the structure (M, c, J) *not* LCAK.

The Hodge star of F is:

$$*F = \frac{(a_2 c_1 - a_1 c_2)(b_2 c_1 - b_1 c_2)}{2(a_2 b_1 - a_1 b_2)}\omega^3 \wedge \omega^4 + \frac{1}{2}(a_1 c_2 - a_2 c_1)\omega^1 \wedge \omega^2,$$

and

$$d * F = \frac{1}{2}(b_1 - a_2)(a_2 c_1 - a_1 c_2)\omega^1 \wedge \omega^2 \wedge \omega^3 + \frac{(a_2 c_1 - a_1 c_2)(b_2 c_1 - b_1 c_2)(-b_2 c_1^2 + a_2 c_1 c_2 + b_1 c_1 c_2 - a_1 c_2^2)}{2(a_2 b_1 - a_1 b_2)^2}$$

$$\times\,\omega^1\wedge\omega^2\wedge\omega^4$$
$$-\frac{c_2(a_2c_1-a_1c_2)(b_2c_1-b_1c_2)}{2(a_2b_1-a_1b_2)}\omega^1\wedge\omega^3\wedge\omega^4$$
$$+\frac{c_1(a_2c_1-a_1c_2)(b_2c_1-b_1c_2)}{2(a_2b_1-a_1b_2)}\omega^2\wedge\omega^3\wedge\omega^4.$$

Thus, the structure satisfies Maxwell's equations $d*F=0$, for example, when:

$$b_1=a_2,\qquad b_2=sa_2,\qquad c_2=sc_1,\quad\text{and}\quad s=\text{const}.$$

6.2. (M,p,J) **in dimension 2 with** $G_-(X,X)=0$**.** Recall that an oriented 2-dimensional conformal manifold (M,c) is the same as a complex 2-manifold (M,J), and indeed the same as a conformal Hermitian manifold (M,J,c).

THEOREM 6.1. *Let (M,p) be a projective structure on a 2-dimensional manifold M. Assume that (M,p) is compatible with the natural complex structure J on M in the sense of Definition 5.6. Then $\overset{p}{G}\equiv 0$. Furthermore, the torsion free connection $\overset{p}{\nabla}$ preserves the canonical conformal class c in M, that is $\overset{p}{\nabla}_i g_{ij}=2B_ig_{ij}$ for any representative g of c.*

Proof. In the following, we will represent connections ∇ by the connection 1-forms ${\Gamma^i}_j$ in a frame.

Working locally, we start with a coframe (θ^1,θ^2) on M, which may be defined by means of its structure equations:

$$d\theta^1=\alpha\theta^1\wedge\theta^2,\qquad d\theta^2=\beta\theta^1\wedge\theta^2,\tag{6.2}$$

with functions a and b on M, such that $d^2\equiv 0$. Then we consider connection one forms ${\Gamma^i}_j$ defined by the equation

$$d\theta^i+{\Gamma^i}_j\wedge\theta^j=0.\tag{6.3}$$

This equation is stating that the corresponding connection ∇ is torsion free. The relation between the 1-forms ${\Gamma^i}_j$ and the connection ∇ is given by

$${\Gamma^i}_j={\Gamma^i}_{jk}\theta^k,\qquad {\Gamma^i}_{jk}=\theta^i(\nabla_{e_k}e_j),$$

where (e_1,e_2) is the frame of vector fields on M such that $\theta^j(e_i)={\delta^j}_i$.

Given (6.2), the most general solution to the equations (6.3) is:

$$\begin{aligned}{\Gamma^1}_1&=a\theta^1+(\alpha+b)\theta^2,\\ {\Gamma^1}_2&=b\theta^1+c\theta^2,\\ {\Gamma^2}_1&=f\theta^1+p\theta^2,\\ {\Gamma^2}_2&=(p-\beta)\theta^1+q\theta^2\end{aligned}\tag{6.4}$$

with arbitrary functions a,b,c,f,p,q on M. At a given point of M the values of these functions parametrise the space of torsion free connections. They define projective structures on M by the following:

The projective transformations (5.1) rewritten in the coframe (θ^1, θ^2) mean that the connection 1-forms ${\Gamma^i}_j$ are in the same projective class as

$$\hat{\Gamma}^i{}_j = {\Gamma^i}_j + {\delta^i}_j \Upsilon + \Upsilon_j \theta^i,$$

where Υ is a 1-form, $\Upsilon = \Upsilon_1 \theta^1 + \Upsilon_2 \theta^2$.

In two dimensions locally, and up to diffeomorphism, there are only *two* complex structures, differing by a sign. In the coframe (θ^1, θ^2), these can be written as:

$$J = \varepsilon\big(\theta^1 \otimes e_2 - \theta^2 \otimes e_1\big),$$

where $\varepsilon = \pm 1$. So given a projective structure p, represented by ∇ as above (or in our frame by ${\Gamma^i}_j$ as in (6.4)), then J determines the unique connection $\overset{p}{\nabla}$ in the projective class. We will write this connection in our coframe in terms of the corresponding connection 1-forms $\overset{p}{\Gamma}{}^i{}_j$. These read:

$$\begin{aligned}
\overset{p}{\Gamma}{}^1{}_1 &= (-\beta - c)\theta^1 + \frac{1}{2}(3\alpha + 2b - f - q)\theta^2,\\
\overset{p}{\Gamma}{}^1{}_2 &= \frac{1}{2}(\alpha + 2b - f - q)\theta^1 + c\theta^2,\\
\overset{p}{\Gamma}{}^2{}_1 &= f\theta^1 + \frac{1}{2}(-a - \beta - c + 2p)\theta^2,\\
\overset{p}{\Gamma}{}^2{}_2 &= \frac{1}{2}(-a - 3\beta - c + 2p)\theta^1 + (\alpha - f)\theta^2,
\end{aligned} \tag{6.5}$$

with the A as in (5.3) given by

$$A = -\frac{1}{2}(a + \beta + c)\theta^1 + \frac{1}{2}(\alpha - f - q)\theta^2.$$

Now we restrict to the situation of a projective class of connections ${\Gamma^i}_j$ which are compatible with J in the sense of Definition 5.6. This means that we impose the restriction $\overset{p}{G}_-(X,X) \equiv 0$. It follows that, in two dimensions, this condition is equivalent to $\overset{p}{G} \equiv 0$,

$$\text{in two dimensions:} \quad \overset{p}{G}_-(X,X) \equiv 0 \quad \Longleftrightarrow \quad \overset{p}{G} \equiv 0.$$

Actually the $\overset{p}{G}_-(X,X) \equiv 0$ condition, in terms of the functions a, b, c, f, p, q defined above, is equivalent to

$$c = a + \beta - 2p, \qquad f = -\alpha - 2b + q.$$

Then the connection $\overset{c}{\nabla}$ is represented by

$$
\begin{aligned}
\overset{p}{\Gamma}{}^1{}_1 &= (-a - 2\beta + 2p)\theta^1 + (2\alpha + 2b - q)\theta^2, \\
\overset{p}{\Gamma}{}^1{}_2 &= (\alpha + 2b - q)\theta^1 + (a + \beta - 2p)\theta^2, \\
\overset{p}{\Gamma}{}^2{}_1 &= -(\alpha + 2b - q)\theta^1 - (a + \beta - 2p)\theta^2, \\
\overset{p}{\Gamma}{}^2{}_2 &= (-a - 2\beta + 2p)\theta^1 + (2\alpha + 2b - q)\theta^2,
\end{aligned}
\tag{6.6}
$$

and A is given by

$$A = (-a - \beta + p)\theta^1 + (\alpha + b - q)\theta^2.$$

Now, since $\overset{p}{G} \equiv 0$, by the general theory, the connection $\overset{p}{\nabla}$ not only is compatible with J, but it actually preserves J. Since J defines c, it must conformally preserve the metric

$$g = g_{ij}\theta^i\theta^j = \left(\theta^1\right)^2 + \left(\theta^2\right)^2,$$

representing the conformal class on M. Indeed, a short calculation shows that

$$\overset{p}{\nabla}_i g_{jk} = 2B_i g_{jk}$$

with

$$B = B_i\theta^i = (a + 2\beta - 2p)\theta^1 + (-2\alpha - 2b + q)\theta^2.$$

This finishes the proof. $\square$

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