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Intrinsic geometry of oriented congruences in three dimensions☆

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ABSTRACT

Starting from the classical notion of an oriented congruence (i.e. a foliation by oriented curves) in $\mathbb{R}^3$, we abstract the notion of an *oriented congruence structure*. This is a 3-dimensional CR manifold (M, H, J) with a preferred splitting of the tangent space $TM = \mathcal{V} \oplus H$. We find all local invariants of such structures using Cartan's equivalence method refining Cartan's classification of 3-dimensional CR structures. We use these invariants and perform Fefferman like constructions, to obtain interesting Lorentzian metrics in four dimensions, which include explicit Ricci-flat and Einstein metrics, as well as not conformally Einstein Bach-flat metrics.

1. Introduction

We study the local differential geometry of oriented congruences in 3-dimensional manifolds. This geometry turns out to be very closely related to local 3-dimensional *CR geometry*. The latter can be traced back to Elie Cartan's 1932 papers [3], in which he used his *equivalence method* to determine the *full set of local invariants* of locally embedded 3-dimensional strictly pseudoconvex CR manifolds.

This paper should be regarded as an extension and refinement of Cartan's work. This is because a 3-dimensional manifold with an oriented congruence on it is an abstract 3-dimensional CR manifold with an additional structure: *a preferred splitting* (see Section 3). This leads to a notion of local equivalence of such structures, which is more strict that than of Cartan. Hence the (coarse) CR equivalence classes of Cartan split into a *fine structure*; as a result we produce many *new local invariants*, corresponding to many more *nonequivalent* structures than in Cartan's situation.

From this perspective, our paper may be also placed in the realm of *special geometries*, i.e. geometries with an additional structure. These kind of geometries, such as, for example, special Riemannian geometries (hermitian, Kähler, G_2, etc.), find applications in mathematical physics (e.g. string theory). The starting point of this paper also comes from physics: a congruence in $\mathbb{R}^3$ (i.e. a foliation of $\mathbb{R}^3$ by curves) is a notion that appears in hydrodynamics (velocity flow), Newtonian gravity and electrodynamics (field strength lines). These branches of physics have distinguished the two main invariants of such foliations, which are related to the classical notions of *twist* and *shear*. One of the byproducts of our analysis is also a refinement of these physical concepts.

Contemporary physicists, because of the dimension of spacetime, have been much more interested in congruences in *four* dimensions. Such congruences live in *Lorentzian* manifolds and, as such, may be *timelike, spacelike* or *null*. It turns out

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that the *null* congruences in spacetimes, which are tangent to *unparametrized geodesics without shear*, locally define a 3-manifold, which has a CR structure on it. One of the outcomes of this paper is that we found connections between properties of *four* dimensional spacetimes admitting null and shearfree congruences, with their corresponding *three* dimensional CR manifolds, and *our new* invariants of the classical congruences in *three* dimensions. In Sections 10 and 11, in particular, we use these *three* dimensional invariants, to construct interesting families of Lorentzian metrics with shearfree congruences in *four* dimensions (including metrics which are Ricci flat or Einstein, Bach flat but not conformal to Einstein, etc.).

Throughout the paper we will always have a nondegenerate (not necessarily Riemannian) metric g_{ij} and its inverse g^{ij}. This enables us to freely raise and lower indices at our convenience. We use the Einstein summation convention. We also denote by $\omega_1\omega_2 = \frac{1}{2}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1)$ the symmetrized tensor product of two 1-forms ω_1 and ω_2. In this paper we shall be working in the smooth category; i.e., everything will be assumed to be C^∞, without mentioning it explicitly in what follows.

A large part of the paper is based on lengthy calculations, which are required by our main tool, namely *Cartan's equivalence method*. These calculations were checked by the symbolic calculation program Mathematica. The structure of the paper is reflected in the table of contents.

Contents

We have been inspired by our contact with Andrzej Trautman, Jacek Tafel and Jerzy Lewandowski, whom we thank warmly. We also thank the Mathematisches Institute der Humboldt-Universität zu Berlin, and Ilka Agricola and Jürgen Leiterer, in particular, for their kind hospitality during the preparation of this paper.

2. Classical twist and shear

In a simply connected domain U of Euclidean space $\mathbb{R}^3$, equipped with the flat metric $g_{ij} = \delta_{ij}$, we consider a smooth foliation by uniformly oriented curves. Let $\mathbf{v}$ be a vector field $\mathbf{v} = v^i\nabla_i$ tangent to the foliation, consistent with the orientation.

We denote the total symmetrization by round brackets on the indices, the total antisymetrization by square brackets on the indices, and use $\epsilon_{ijk} = \epsilon_{[ijk]}, \epsilon_{123} = 1$. We have the following classical decomposition

$$\nabla_i v_j = \alpha_{ij} + \sigma_{ij} + \frac{1}{3}\theta g_{ij}, \tag{2.1}$$

where

$$\alpha_{ij} = \nabla_{[i}\, v_{j]} = \frac{1}{2}\epsilon_{ijk}(\mathrm{curl}\,\mathbf{v})^k,$$
$$\theta = g^{ij}\nabla_i v_j = \mathrm{div}\,\mathbf{v},$$
$$\sigma_{ij} = \nabla_{(i v_j)} - \frac{1}{3}\theta g_{ij}.$$

The decomposition (2.1) defines three functions, depending on the choice of $\mathbf{v}$, which can be used to characterize the foliation. One of these functions is the divergence θ, also called the *expansion of the vector field* $\mathbf{v}$. It merely characterizes the vector field $\mathbf{v}$, hence it is *not* interesting as far the properties of the foliations are concerned. The second function is

$$\alpha = |\alpha_{ij}| = \sqrt{g^{ik}g^{jl}\alpha_{ij}\alpha_{kl}},$$

the norm of the antisymmetric part α_{ij}, called the *twist of the vector field* $\mathbf{v}$.

Vanishing of twist, the *twist-free* condition $\alpha = 0$, is equivalent to curl $\mathbf{v} = 0$. Although this condition is $\mathbf{v}$-dependent, it has a clear geometric meaning for the foliation. Indeed, a vector field $\mathbf{v}$ with vanishing twist may be represented by a gradient: $\mathbf{v} = \nabla f$ for some function $f : U \to \mathbb{R}$. In such a case the level surfaces of the function f define a foliation of U with 2-dimensional leaves orthogonal to $\mathbf{v}$. This can be rephrased by saying that the distribution $\mathcal{V}^\perp$ of 2-planes, perpendicular to $\mathbf{v}$, is integrable.

The third function obtained from the decomposition (2.1) is

$$\sigma = |\sigma_{ij}| = \sqrt{g^{ik}g^{jl}\sigma_{ij}\sigma_{kl}},$$

the norm of the trace-free symmetric part σ_{ij}, called the *shear of the vector field* $\mathbf{v}$.

Regardless of whether or not $\mathcal{V}^\perp$ is integrable, the condition of *vanishing shear* $\sigma = 0$ is equivalent to $\nabla_{[i}\, v_{j]} = \frac{1}{3}\theta g_{ij}$. Recalling that the Lie derivative $\mathcal{L}_{\mathbf{v}} g_{ij} = \nabla_{[i}\, v_{j]}$, we see that the shearfree condition for $\mathbf{v}$ is the condition that this Lie derivative be proportional to the metric. Thus $\sigma = 0$ if and only if $\mathcal{L}_{\mathbf{v}} g_{ij} = h g_{ij}$. This condition again is $\mathbf{v}$ dependent. However, it implies the following geometric property of the foliation: the metric $g_{|\mathcal{V}^\perp}$ induced by g_{ij} on the distribution $\mathcal{V}^\perp$ is conformally preserved when Lie transported along $\mathbf{v}$. To say it differently we introduce a complex structure J on each 2-plane of $\mathcal{V}^\perp$. This is possible since each such plane is equipped with a metric $g_{|\mathcal{V}^\perp}$ and the orientation induced by the orientation of $\mathbf{v}$. Knowing this, we define J on each 2-plane as a rotation by $\frac{\pi}{2}$, using the right hand rule. Now we can rephrase the statement about conformal preservation of the metric $g_{|\mathcal{V}^\perp}$ during Lie transport along $\mathbf{v}$, by saying that it is equivalent to the constancy of J under the Lie transport along $\mathbf{v}$.

The above notions of expansion, twist and shear are the classical notions of elasticity theory. As we have seen, they are not invariants of the foliation by curves, because they depend on the choice of the vector field $\mathbf{v}$. Nonetheless they do carry some invariant information. One of the main purposes of this paper is to find all of the local invariants of the intrinsic geometry associated with such foliations. With this classical motivation we now pass to the subject proper of this paper.

3. Oriented congruences

Consider a smooth oriented real 3-dimensional manifold M equipped with a Riemannian metric g. Assume that M is smoothly foliated by uniformly oriented curves. Such a foliation is called an *oriented congruence*. Note that we are *not* assuming that the curves in the congruence are geodesics for the metric g.

Our first observation is that M has the structure of a smooth abstract *CR manifold*. To see this we introduce the oriented line bundle $\mathcal{V}$, a subbundle of TM, consisting of the tangent lines to the foliation. Using the metric we also have $\mathcal{V}^\perp$, the 2-plane subbundle of TM consisting of the planes orthogonal to the congruence. These 2-planes are oriented by the right hand rule and are equipped with the induced metric $g_{|\mathcal{V}^\perp}$. Hence $\mathcal{V}^\perp$ is endowed with the complex structure operator J as we explained in the previous section. The pair $(\mathcal{V}^\perp, J)$, by the very definition, equips M with the structure of an abstract 3-dimensional CR manifold. This CR manifold has an additional structure consisting in the preferred splitting $TM = \mathcal{V}^\perp \oplus \mathcal{V}$. It also defines an equivalence class $[g]$ of *adapted* Riemannian metrics g' in which $g'(\mathcal{V}, \mathcal{V}^\perp) = 0$ and such that $g'_{|\mathcal{V}^\perp}$ is hermitian for J. Thus, an oriented congruence in (M, g) defines a whole class of Riemannian manifolds $(M, [g])$ which are adapted to it.

Conversely, given an oriented abstract 3-dimensional CR manifold (M, H, J) with a distinguished line subbundle $\mathcal{V}$ such that $\mathcal{V} \cap H = \{0\}$, we may reconstruct the oriented congruence. The curves of this congruence consist of the trajectories of $\mathcal{V}$. They are oriented by the right hand rule applied in such a way that it agrees with the orientation of H determined by J. Here $J : H \to H$ and $J^2 = -\mathrm{id}$. Since $\mathrm{T}M = H \oplus \mathcal{V}$ we recover also the equivalence class $[g]$ of adapted Riemannian metrics g' in which $g(\mathcal{V}, H) = 0$ and such that $g'_{|H}$ is hermitian for J.

We summarize with: let M be an oriented 3-dimensional manifold, then

Proposition 3.1. *There is a one to one correspondence between oriented congruences on M with a distinguished orthogonal distribution $\mathcal{V}^\perp$, and CR structures (H, J) on M with a distinguished line subbundle $\mathcal{V}$ such that $TM = H \oplus \mathcal{V}$.*

We now pass to the dual formulation. Given a CR structure (H, J) with a preferred splitting $TM = H \oplus \mathcal{V}$, we define H^0 to be the annihilator of H and $\mathcal{V}^0$ to be the annihilator of $\mathcal{V}$. Note that H^0 is a *real line* subbundle of T^*M and $\mathcal{V}^0$ is a 2-plane subbundle of T^*M. This H^0 is known as the *characteristic bundle* associated with the CR structure. $\mathcal{V}^0$ is equipped with the complex structure J^*, the adjoint of J with respect to the natural duality pairing. The complexification $\mathbb{C}\mathcal{V}^0$ splits into $\mathbb{C}\mathcal{V}^0 = \mathcal{V}^0_+ \oplus \mathcal{V}^0_-$, where $\mathcal{V}^0_\pm$ are the $\mp$i eigenspaces of J^*. Both spaces $\mathcal{V}^0_\pm$ are *complex line* subbundles of the complexification $\mathbb{C}T^*M$ of the cotangent bundle. $\mathcal{V}^0_-$ is the complex conjugate of $\mathcal{V}^0_+$, $\overline{\mathcal{V}}^0_\pm = \mathcal{V}^0_\mp$.

The reason for passing to the dual formulation is that we want to apply Cartan's method of equivalence to determine the local invariants of an oriented congruence in M. For this we need a local nonzero section λ of H^0 and a local nonzero section μ of $\mathcal{V}^0_+$. Then $\lambda \wedge \mu \wedge \bar{\mu} \neq 0$. Any other local section λ' of H^0 and any other local section μ' of $\mathcal{V}^0_+$ are related to λ and μ by $\lambda' = f\lambda$ and $\mu' = h\mu$, for some real function f and some complex function h. This motivates the following definition:

Definition 3.2. A *structure* $(M, [\lambda, \mu])$ *of an oriented congruence* on a 3-dimensional manifold M is an equivalence class of pairs of 1-forms $[\lambda, \mu]$ on M satisfying the following conditions:

(i) λ is real, μ is complex
(ii) $\lambda \wedge \mu \wedge \bar{\mu} \neq 0$ at each point of M
(iii) two pairs (λ, μ) and (λ', μ') are equivalent iff there exist nonvanishing functions f (real) and h (complex) on M such that

$$\lambda' = f\lambda, \qquad \mu' = h\mu. \tag{3.1}$$

We say that two such structures $(M, [\lambda, \mu])$ and $(M', [\lambda', \mu'])$ are (locally) *equivalent* iff there exists a (local) diffeomorphism $\phi : M \to M'$ such that

$$\phi^*(\lambda') = f\lambda, \qquad \phi^*(\mu') = h\mu \tag{3.2}$$

for some nonvanishing functions f (real) and h (complex) on M. If such a diffeomorphism is from M to M it is called an *automorphism* of $(M, [\lambda, \mu])$. The full set of automorphisms is called the *group of automorphisms* of $(M, [\lambda, \mu])$. A vector field X on M is called a *symmetry* of $(M, [\lambda, \mu])$ iff

$$\mathcal{L}_X\lambda = f\lambda, \qquad \mathcal{L}_X\mu = h\mu.$$

Here the functions f (real) and h (complex) are not required to be nonvanishing; they may even vanish identically. Observe, that if X and Y are two symmetries of $(M, [\lambda, \mu])$ then their commutator $[X, Y]$ is also a symmetry. Thus, we may speak about the *Lie algebra of symmetries*.

Remark 3.3. Note that Cartan [3] would define a 3-dimensional CR manifold as a structure $(M, [\lambda, \mu])$ as above, with the exception that condition (iii) is weakened to

(iii)$_{CR}$ two pairs (λ, μ) and (λ', μ') are equivalent iff there exist nonvanishing functions f (real) and h (complex) and a complex function p on M such that

$$\lambda' = f\lambda, \qquad \mu' = h\mu + p\lambda.$$

In this sense our structure of an oriented congruence $(M, [\lambda, \mu])$ is a CR manifold on which there is an additional structure. In particular the diffeomorphisms ϕ that provide an equivalence of our structures are special cases of CR diffeomorphisms, which for CR structures defined *a la* Cartan by (iii)$_{CR}$ are $\phi : M \to M'$ such that $\phi^*(\lambda') = f\lambda$, $\phi^*(\mu') = h\mu + p\lambda$. In terms of the nowadays definition of a CR manifold as a triple (M, H, J), this last Cartan condition is equivalent to the CR map requirement: $\mathrm{d}\phi \circ J = J \circ \mathrm{d}\phi$ and similarly for ϕ^{-1}.

Remark 3.4. Two CR structures which are not equivalent in the sense of Cartan [3] are also not equivalent, in our sense, as oriented congruences; but not vice versa. On the other hand, every symmetry of an oriented congruence $(M, [\lambda, \mu])$ is a CR symmetry of the CR structure determined by $[\lambda, \mu]$ via (iii)$_{CR}$; and not vice versa.

We omit the proof of the following easy proposition.

Proposition 3.5. *A given structure $(M, [\lambda, \mu])$ determines a CR structure (M, H, J) with the preferred splitting $TM = H \oplus \mathcal{V}$, where H is the annihilator of $Span_{\mathbb{R}}(\lambda)$ and $\mathbb{C}\mathcal{V}$ is the annihilator of $Span_{\mathbb{C}}(\mu) \oplus Span_{\mathbb{C}}(\bar{\mu})$. The class of adapted Riemannian metrics $[g]$ is parametrized by two arbitrary nonvanishing functions f (real) and h (complex) and given by*

$$g = f^2\lambda^2 + 2|h|^2\mu\bar{\mu}.$$

4. Elements of Cartan's equivalence method

Here we outline the procedure we will follow in applying Cartan's method to our particular situation.

4.1. Cartan invariants

Consider two structures $(M, [\lambda, \mu])$ and $(M', [\lambda', \mu'])$. Our aim is to determine whether they are equivalent or not, according to Definition 3.2, Eq. (3.2). This question is not easy to answer, since it is equivalent to the problem of the existence of a solution ϕ for a system (3.2) of linear first order PDEs in which the right hand side is undetermined. Elie Cartan associates with the forms $(\lambda, \mu, \bar{\mu})$ and $(\lambda', \mu', \bar{\mu}')$, representing the structures, two systems of *ordered coframes* $\{\Omega_i\}$ and $\{\Omega_i'\}$ on manifolds P and P' of the same dimension, say $n \geq 3$, which are fiber bundles over M. Then he shows that equations like (3.2) for ϕ have a solution *if and only if* a simpler system

$$\Phi^*\Omega_i' = \Omega_i, \quad i = 1, 2, \ldots, n \tag{4.1}$$

of differential equations for a diffeomorphism $\Phi : P \to P'$ has a solution. Note that derivatives of Φ still occur in (4.1), since Φ^* is the pullback of forms from P' to P.

One famous example is his original solution to the equivalence problem for 3-dimensional strictly pseudoconvex CR structures. There P and P' are 8-dimensional, and his procedure produces two systems of eight linearly independent 1-forms $\{\Omega_i\}$ and $\{\Omega_i'\}$.

In our situation, provided $n < \infty$, and if we are able to find n well defined linearly independent 1-forms $\{\Omega_i\}$ on P, then $(P, \{\Omega_i\})$ provides the *full system of local invariants* for the original structure $(M, [\lambda, \mu])$. In particular, using $(P, \{\Omega_i\})$ one introduces the *scalar invariants*, which are the coefficients $\{K_I\}$ in the decomposition of $\{d\Omega_i\}$ with respect to the *invariant* basis of 2-forms $\{\Omega_i \wedge \Omega_j\}$.

Now in order to determine if two structures $(M, [\lambda, \mu])$ and $(M', [\lambda', \mu'])$ are equivalent, it is enough to have n functionally independent $\{K_I\}$. Then the condition (4.1) becomes

$$\Phi^*K'_I = K_I, \quad I = 1, 2, \ldots, n. \tag{4.2}$$

The advantage of this condition, as compared to (4.1), is that (4.2), being the pull back of functions, does not involve derivatives of Φ. In this case the existence of Φ becomes a question involving the implicit function theorem, and the whole problem reduces to checking whether a certain Jacobian is non-degenerate.

We remark that an immediate application of the invariants obtained by Cartan's equivalence method is to use them to find *all* the homogeneous examples of the particular structure under consideration. The procedure of enumerating these examples is straightforward and algorithmic once the Cartan invariants have been determined. In our situation the homogeneous examples will often have local symmetry groups of dimension *three*. The 3-dimensional Lie groups are classified according to the Bianchi classification of 3-dimensional Lie algebras [1]. Since we will use this classification in subsequent sections, we recall it below.

4.2. Bianchi classification of 3-dimensional Lie algebras

In this section X_1, X_2, X_3 denote a basis of a 3-dimensional Lie algebra $\mathfrak{g}$ with Lie bracket $[\cdot, \cdot]$. All the nonequivalent Lie algebras fall into *Bianchi types* I, II, VI_0, VII_0, VIII, IX, V, IV, VI_h, VII_h. Apart from types VI_h and VII_h, there is always precisely one Lie algebra corresponding to a given type. For each value of the real parameter $h < 0$ there is also precisely one Lie algebra of type VI_h. Likewise for each value of the parameter $h > 0$ there is precisely one Lie algebra of type VII_h. The commutation relations for each Bianchi type are given in the following table.

Bianchi type:	I	II	VI_0	VII_0	VIII	IX
$[X_1, X_2] =$	0	0	0	0	$-X_3$	X_3
$[X_3, X_1] =$	0	0	$-X_2$	X_2	X_2	X_2
$[X_2, X_3] =$	0	X_1	X_1	X_1	X_1	X_1
Bianchi type:	V	IV	VI_h	VII_h		
$[X_1, X_2] =$	0	0	0	0		
$[X_3, X_1] =$	X_1	X_1	$-X_2 + hX_1$	$X_2 + hX_1$		
$[X_2, X_3] =$	$-X_2$	$X_1 - X_2$	$X_1 - hX_2$	$X_1 - hX_2$		

Note that Bianchi type I corresponds to the abelian Lie group, type II corresponds to the Heisenberg group; types VIII and IX correspond to the simple groups: $\mathbf{SO}(1,2)$, $\mathbf{SL}(2,\mathbb{R})$ for type VIII, and $\mathbf{SO}(3)$, $\mathbf{SU}(2)$ for type IX.

5. Basic relative invariants of an oriented congruence

We make preparations to apply the Cartan method of equivalence for finding all local invariants of the structure of an oriented congruence $(M,[\lambda,\mu])$ on a 3-manifold M.

Given a structure $(M,[\lambda,\mu])$ we take representatives λ and μ of 1-forms from the class $[\lambda,\mu]$. Since $(\lambda,\mu,\bar\mu)$ is a basis of 1-forms on M we can express the differentials $\mathrm{d}\lambda$ and $\mathrm{d}\mu$ in terms of the corresponding basis of 2-forms $(\mu\wedge\bar\mu,\mu\wedge\lambda,\bar\mu\wedge\lambda)$. We have

$$\begin{aligned}\mathrm{d}\lambda &= \mathrm{i}a\mu\wedge\bar\mu + b\mu\wedge\lambda+\bar b\bar\mu\wedge\lambda\\ \mathrm{d}\mu &= p\mu\wedge\bar\mu+q\mu\wedge\lambda+s\bar\mu\wedge\lambda\\ \mathrm{d}\bar\mu &= -\bar p\mu\wedge\bar\mu+\bar s\mu\wedge\lambda+\bar q\bar\mu\wedge\lambda,\end{aligned}\tag{5.1}$$

where a is a real valued function and b,p,q,s are complex valued functions on M. Given any function u on M we define first order linear partial differential operators acting on u by

$$\mathrm{d}u = u_\lambda\lambda+u_\mu\mu+u_{\bar\mu}\bar\mu.$$

Note that u_λ is a real vector field acting on u, u_μ is a complex vector field of type $(1,0)$ acting on u and $u_{\bar\mu}$ is a complex vector field of type $(0,1)$ acting on u. The commutators of these operators, when acting on u are

$$\begin{aligned}u_{\bar\mu\mu}-u_{\mu\bar\mu} &= -\mathrm{i}au_\lambda - pu_\mu+\bar p u_{\bar\mu}\\ u_{\lambda\mu}-u_{\mu\lambda} &= -bu_\lambda-qu_\mu-\bar s u_{\bar\mu}\\ u_{\lambda\bar\mu}-u_{\bar\mu\lambda} &= -\bar b u_\lambda - su_\mu-\bar q u_{\bar\mu}.\end{aligned}\tag{5.2}$$

A function u on a CR manifold $(M,[\lambda,\mu])$ is called a *CR function* if

$$\mathrm{d}u\wedge\lambda\wedge\mu\equiv 0.\tag{5.3}$$

In terms of the differential operators above this is the same as

$$u_{\bar\mu}\equiv 0.\tag{5.4}$$

Thus $u_{\bar\mu}$ is just the *tangential Cauchy–Riemann operator* acting on u. The Eq. (5.3) or (5.4) is called the *tangential Cauchy–Riemann equation*.

It is easy to see that each of the following two conditions

$$\mathrm{d}\lambda\wedge\lambda=0,\qquad \mathrm{d}\mu\wedge\mu=0,\tag{5.5}$$

is independent of the choice of the representatives (λ,μ) from the class $[\lambda,\mu]$. Thus the identical vanishing or not of either the coefficient a, or the coefficient s, is an invariant property of the structure $(M,[\lambda,\mu])$. Using Cartan's terminology the functions a and s are the *basic relative invariants* of $(M,[\lambda,\mu])$. By definition they correspond to the identical vanishing or not of the *twist* (the function a) and of the *shear* (the function s) of the oriented congruence represented by $(M,[\lambda,\mu])$.

They are invariant versions of the classical $\mathbf{v}$-dependent notions of twist α and shear σ we considered in Section 2. Given an oriented congruence with vanishing twist a in $M=\mathbb{R}^3$ we can always find a vector field $\mathbf{v}$ tangent to the congruence such that the twist α for this vector field is zero. We also have an analogous statement for s and σ. Conversely, every vector field $\mathbf{v}$ in $\mathbb{R}^3$ which has vanishing twist α (or shear σ) defines an oriented congruence with vanishing twist a (or shear s).

We note that the twist a is just the *Levi form* of the CR structure and that the shear s is now complex; its meaning will be explained further in Section 8.

In what follows we will often use the following (see e.g. [12])

Lemma 5.1. *Let μ be a smooth complex valued 1-form defined locally in $\mathbb{R}^3$ such that $\mu\wedge\bar\mu\neq 0$. Then*

$$\mathrm{d}\mu\wedge\mu\equiv 0\quad \textit{if and only if}\ \ \mu = h\mathrm{d}\zeta$$

where ζ is a smooth complex function such that $\mathrm{d}\zeta\wedge\mathrm{d}\bar\zeta\neq 0$, and h is a smooth nonvanishing complex function.

Proof. Consider an open set $U\in\mathbb{R}^3$ in which we have μ such that $\mathrm{d}\mu\wedge\mu=0$ and $\mu\wedge\bar\mu\neq 0$. We define *real* 1-forms $\theta^1=\mathrm{Re}(\mu)$ and $\theta^2=\mathrm{Im}(\mu)$. They satisfy $\theta^1\wedge\theta^2\neq 0$ in U. Since $U\subset\mathbb{R}^3$ we trivially have $\mathrm{d}\theta^1\wedge\theta^1\wedge\theta^2\equiv 0$ and $\mathrm{d}\theta^2\wedge\theta^1\wedge\theta^2\equiv 0$. Now the real Fröbenius theorem implies that there exists a coordinate chart (x,y,u) in U such that $\theta^1=t_{11}\mathrm{d}x+t_{12}\mathrm{d}y$ and $\theta^2=t_{21}\mathrm{d}x+t_{22}\mathrm{d}y$, with some *real* functions t_{ij} in U such that $t_{11}t_{22}-t_{12}t_{21}\neq 0$. Thus in the coordinates (x,y,u) the form $\mu=\theta^1+\mathrm{i}\theta^2$ can be written as $\mu=c_1\mathrm{d}x+c_2\mathrm{d}y$, where now c_1,c_2 are *complex* functions such that $c_1\bar c_2-\bar c_1c_2\neq 0$ on U, so neither c_1 nor c_2 can be zero. The $\mathrm{d}\mu\wedge\mu\equiv 0$ condition for μ written in this representation is simply $c_2^2\mathrm{d}(\frac{c_1}{c_2})\wedge\mathrm{d}x\wedge\mathrm{d}y\equiv 0$. Thus the partial derivative $(\frac{c_1}{c_2})_u\equiv 0$, which means that the ratio $\frac{c_1}{c_2}$ does not depend on u. This ratio defines a nonvanishing *complex* function $F(x, y) = \frac{c_1}{c_2}$ of only *two* real variables x and y. Returning to μ we see that it is of the form $\mu = c_2\,(\mathrm{d}y + F(x, y)\mathrm{d}x)$. Consider the real bilinear symmetric form $G = 2\mu\bar{\mu} = |c_2|^2 \left(\mathrm{d}y^2 + 2\left(F(x, y) + \bar{F}(x, y)\right)\mathrm{d}x\mathrm{d}y + |F(x, y)|^2\mathrm{d}x^2\right)$. Invoking the classical theorem on the existence of isothermal coordinates we are able to find an open set $U' \subset U$ with new coordinates (ξ, η, u) in which $G = h^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2)$, where $h = h(\xi, \eta, u)$ is a real function in U'. This means that in these coordinates $\mu = h\mathrm{d}(\xi + \mathrm{i}\eta) = h\mathrm{d}\zeta$. The proof in the other direction is obvious. □

6. Vanishing twist and shear

Let us assume that the structure $(M, [\lambda, \mu])$ satisfies both conditions (5.5); i.e., that $a \equiv 0$ and $s \equiv 0$. It is easy to see that all such structures have no local invariants, meaning that all of them are locally equivalent. Indeed, if $\mathrm{d}\lambda \wedge \lambda \equiv 0$ then the real Fröbenius theorem guarantees that locally $\lambda = f\mathrm{d}u$. Similarly, if $\mathrm{d}\mu \wedge \mu \equiv 0$, then the Lemma 5.1 assures that $\mu = h\mathrm{d}\zeta$. Since $\mathrm{d}\zeta \wedge \lambda \wedge \mu \equiv 0$, we see that the function ζ is a *holomorphic* coordinate. Recalling the fact that $\lambda \wedge \mu \wedge \bar{\mu} \neq 0$, we conclude that if $a \equiv 0$ and $s \equiv 0$ then the CR manifold M with the preferred splitting is locally equivalent to $\mathbb{R} \times \mathbb{C}$, with local coordinates (u, ζ), such that u is real. In these coordinates the structure may be represented by $\lambda = \mathrm{d}u$ and $\mu = \mathrm{d}\zeta$. The local group of automorphisms for such structures is infinite dimensional and given in terms of two functions $U = U(u)$ and $Z = Z(\zeta)$ such that U is real, $U_u \neq 0$, Z is holomorphic and $Z_\zeta \neq 0$. The automorphism transformations are then $\tilde{u} = U(u)$, $\tilde{\zeta} = Z(\zeta)$. Note that from the point of view of Cartan's method this is the *involutive* case in which $n = \infty$. There are no local invariants in this situation.

7. Nonvanishing twist and vanishing shear

7.1. The relative invariants K_1 and K_2

Next let us assume that the structure $(M, [\lambda, \mu])$ has some twist, $a \neq 0$, but has identically vanishing shear, $s \equiv 0$. Let us interpret this in terms of the corresponding CR structure with the preferred splitting. The nonvanishing twist condition $\mathrm{d}\lambda \wedge \lambda \neq 0$ is the condition that the CR structure has nonvanishing Levi form. This means that the CR manifold is *strictly pseudoconvex* and hence is not locally equivalent to $\mathbb{R} \times \mathbb{C}$. The no shear condition, $\mathrm{d}\mu \wedge \mu \equiv 0$, by the Lemma 5.1, means that the class $[\mu]$ may be represented by a 1-form $\mu = \mathrm{d}\zeta$ with a complex function ζ on M satisfying $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \neq 0$. Note that this function trivially satisfies the *tangential Cauchy–Riemann equation* $\mathrm{d}\zeta \wedge \lambda \wedge \mu = 0$ for this CR structure, and hence is a *CR function*. If Z is *any* holomorphic function with nonvanishing derivative, then $Z = Z(\zeta)$ is again a CR function with $\mathrm{d}Z \wedge \mathrm{d}\bar{Z} \neq 0$. This gives us a distinguished class of genuinely complex CR functions $Z = Z(\zeta)$, which we denote by $[\zeta]$. Conversely if we have a *strictly pseudoconvex* 3-dimensional CR structure (M, H, J) with a distinguished class $[\zeta]$ of CR functions $Z = Z(\zeta)$, such that $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \neq 0$ and $Z' \neq 0$, then this CR structure defines a representative $(\lambda, \mu = \mathrm{d}Z)$, with λ being a nonvanishing section of the characteristic bundle H^0. This in turn defines a structure $(M, [\lambda, \mu])$ of an oriented congruence which has $a \neq 0$ and $s \equiv 0$.

Summarizing we have

Proposition 7.1. *All local structures of an oriented congruence $(M, [\lambda, \mu])$ with nonvanishing twist, $a \neq 0$, and vanishing shear, $s \equiv 0$, are in a one to one correspondence with local CR structures (M, H, J) having nonvanishing Levi form and possessing a distinguished class $[\zeta]$ of genuinely complex CR functions on M.*

Note that the proposition remains true if we drop the nonvanishing twist condition on the left and drop the nonvanishing Levi form condition on the right.

We now pass to the determination of the local invariants of $(M, [\lambda, \mu])$ with nonvanishing twist and vanishing shear. We take a representative (λ, μ). Because of our assumptions the formulae (5.1) become

$$\begin{aligned}
\mathrm{d}\lambda &= \mathrm{i}a\mu \wedge \bar{\mu} + b\mu \wedge \lambda + \bar{b}\bar{\mu} \wedge \lambda \\
\mathrm{d}\mu &= p\mu \wedge \bar{\mu} + q\mu \wedge \lambda \\
\mathrm{d}\bar{\mu} &= -\bar{p}\mu \wedge \bar{\mu} + \bar{q}\bar{\mu} \wedge \lambda.
\end{aligned} \tag{7.1}$$

For example if we were to choose μ as $\mu = \mathrm{d}\zeta$, where ζ is a particular representative of the distinguished class $[\zeta]$ of CR functions, then $\mathrm{d}\mu$ would identically vanish, so $p \equiv 0$ and $q \equiv 0$. Although this choice of μ is very convenient and quite simplifies the determination of the invariants, we will work in the most general representation (7.1) of $[\lambda, \mu]$ to get the formulae for the invariants in their full generality.

Given a choice (λ, μ) as in (7.1) we take the most general representatives

$$\omega = f\lambda, \qquad \omega_1 = h\mu, \qquad \bar{\omega}_1 = \bar{h}\bar{\mu}, \tag{7.2}$$

of the class $[\lambda, \mu]$. Here $f \neq 0$ (real) and $h \neq 0$ (complex) are arbitrary functions. Then we reexpress the differentials $\mathrm{d}\omega$, $\mathrm{d}\omega_1$ and $\mathrm{d}\bar{\omega}_1$ in terms of the general basis $(\omega, \omega_1, \bar{\omega}_1)$. We have:

$$\mathrm{d}\omega = \mathrm{i}\frac{fa}{|h|^2}\,\omega_1 \wedge \bar{\omega}_1 + \left[\mathrm{d}\log f + \frac{b}{h}\omega_1 + \frac{\bar{b}}{\bar{h}}\bar{\omega}_1\right] \wedge \omega \tag{7.3}$$

$$\mathrm{d}\omega_1 = \left[\mathrm{d}\log h - \frac{p}{\bar{h}}\bar{\omega}_1 - \frac{q}{f}\omega\right] \wedge \omega_1 \tag{7.4}$$

$$\mathrm{d}\bar{\omega}_1 = \left[\mathrm{d}\log \bar{h} - \frac{\bar{p}}{h}\omega_1 - \frac{\bar{q}}{f}\omega\right] \wedge \bar{\omega}_1. \tag{7.5}$$

Since $a \neq 0$ we can easily achieve

$$\mathrm{d}\omega \wedge \omega = \mathrm{i}\omega_1 \wedge \bar{\omega}_1 \wedge \omega \tag{7.6}$$

by taking

$$f = \frac{|h|^2}{a}. \tag{7.7}$$

Thus condition (7.6) 'fixes the gauge' in the choice of f.

Introducing the real functions $\rho > 0$ and ϕ via $h = \rho e^{\mathrm{i}\phi}$ and maintaining the condition (7.6) we may rewrite Eq. (7.3) in the form

$$\mathrm{d}\omega = \mathrm{i}\omega_1 \wedge \bar{\omega}_1 + (\Omega + \bar{\Omega}) \wedge \omega,$$

where the real valued 1-form $\Omega + \bar{\Omega}$ is

$$\Omega + \bar{\Omega} = 2\mathrm{d}\log\rho + (b - (\log a)_\mu)\mu + (\bar{b} - (\log a)_{\bar{\mu}})\bar{\mu} + t\lambda. \tag{7.8}$$

The real function t appearing in $\Omega + \bar{\Omega}$ can be determined *algebraically* from the condition that

$$(\mathrm{d}\omega_1 + \mathrm{d}\bar{\omega}_1) \wedge (\omega_1 - \bar{\omega}_1) = -\omega_1 \wedge \bar{\omega}_1 \wedge (\Omega + \bar{\Omega}). \tag{7.9}$$

If this condition is imposed then

$$t = -q - \bar{q}. \tag{7.10}$$

Now, if t is as in (7.10) and f is as in (7.7) we define $\Omega - \bar{\Omega}$ to be an imaginary 1-form such that

$$(\mathrm{d}\omega_1 + \mathrm{d}\bar{\omega}_1) \wedge (\omega_1 + \bar{\omega}_1) = \omega_1 \wedge \bar{\omega}_1 \wedge (\Omega - \bar{\Omega}). \tag{7.11}$$

This determines $\Omega - \bar{\Omega}$ to be

$$\Omega - \bar{\Omega} = 2\mathrm{i}\mathrm{d}\phi + (\bar{q} - q)\lambda + z\mu - \bar{z}\bar{\mu},$$

where z is a still undetermined function. The condition that fixes z in an algebraic fashion is the requirement that

$$\mathrm{d}\omega_1 = \Omega \wedge \omega_1, \qquad \mathrm{d}\bar{\omega}_1 = \bar{\Omega} \wedge \bar{\omega}_1. \tag{7.12}$$

If this is imposed we have

$$z = 2\bar{p} + b - (\log a)_\mu, \qquad \bar{z} = 2p + \bar{b} - (\log a)_{\bar{\mu}}. \tag{7.13}$$

Thus given a structure $(M, [\lambda, \mu])$ with nonvanishing twist and vanishing shear, the four normalization conditions (7.6), (7.9), (7.11) and (7.12) uniquely specify a 5-dimensional manifold P, which is locally $M \times \mathbb{C}$, and a well defined coframe $(\omega, \omega_1, \bar{\omega}_1, \Omega, \bar{\Omega})$ on it such that

$$\begin{aligned}
\omega &= \frac{\rho^2}{a}\lambda \\
\omega_1 &= \rho e^{\mathrm{i}\phi}\mu \\
\bar{\omega}_1 &= \rho e^{-\mathrm{i}\phi}\bar{\mu} \\
\Omega &= \mathrm{d}\log\rho + \mathrm{i}\mathrm{d}\phi + (\bar{p} + b - (\log a)_\mu)\mu - p\bar{\mu} - q\lambda \\
\bar{\Omega} &= \mathrm{d}\log\rho - \mathrm{i}\mathrm{d}\phi - \bar{p}\mu + (p + \bar{b} - (\log a)_{\bar{\mu}})\bar{\mu} - \bar{q}\lambda.
\end{aligned} \tag{7.14}$$

Here the complex coordinate along the factor $\mathbb{C}$ in $M\times\mathbb{C}$ is $h=\rho e^{i\phi}$. The coframe $(\omega,\omega_1,\bar\omega_1,\Omega,\bar\Omega)$ satisfies

$$\begin{aligned}
d\omega &= i\omega_1\wedge\bar\omega_1+(\Omega+\bar\Omega)\wedge\omega\\
d\omega_1 &= \Omega\wedge\omega_1\\
d\bar\omega_1 &= \bar\Omega\wedge\bar\omega_1\\
d\Omega &= K_1\omega_1\wedge\bar\omega_1+K_2\omega_1\wedge\omega\\
d\bar\Omega &= -K_1\omega_1\wedge\bar\omega_1+\overline{K}_2\bar\omega_1\wedge\omega,
\end{aligned}\tag{7.15}$$

where

$$K_1=\frac{1}{\rho^2}k_1,\qquad K_2=\frac{e^{-i\phi}}{\rho^3}k_2,\tag{7.16}$$

are functions on P with k_1 and k_2 given by

$$\begin{aligned}
k_1 &= \mathrm{Re}\left((\log a)_{\mu\bar\mu}-(\log a)_{\mu}p-iqa-b_{\bar\mu}+bp-2\bar p_{\bar\mu}+2|p|^2\right)\\
k_2 &= a_{\mu\lambda}-ab_{\lambda}+i(\log a)_{\mu}(b_{\bar\mu}-\bar b_{\mu}-bp+\bar b\bar p)-2a_{\mu}q-aq_{\mu}-(a\bar q)_{\mu}-ab\bar q.
\end{aligned}$$

Note that the functions k_1 and k_2 are actually defined on M. Note also that k_1 is *real* as a consequence of the commutation relations (5.2). The functions K_1 and K_2 are the *relative invariants* of the structure $(M,[\lambda,\mu])$, and (7.15) are the *structural equations* for $(M,[\lambda,\mu])$.

Theorem 7.2. *A given structure $(M,[\lambda,\mu])$ of an oriented congruence with nonvanishing twist, $a\neq 0$, and vanishing shear, $s\equiv 0$, uniquely defines a 5-dimensional manifold P, 1-forms $\omega,\omega_1,\bar\omega_1,\Omega,\bar\Omega$ and functions $K_1,K_2,\overline{K}_2$ on P such that*

- *$\omega,\omega_1,\bar\omega_1$ are as in (7.2),*
- *$\omega\wedge\omega_1\wedge\bar\omega_1\wedge\Omega\wedge\bar\Omega\neq 0$ at each point of P,*
- *the forms and functions K_1 (real), K_2 (complex) are uniquely determined by the requirement that on P they satisfy equations (7.15).*

In particular the identical vanishing, or not, of either k_1 or k_2 are invariant conditions. Also the sign of k_1 is an invariant, if $k_1\neq 0$.

7.2. Description in terms of the Cartan connection

The above theorem, stated in modern language, means the following. The manifold P is a Cartan bundle $H_2\to P\to M$, with H_2 a 2-dimensional abelian subgroup of a certain 5-dimensional Lie group G_5. The group G_5 is a subgroup of $\mathbf{SU}(2,1)$; i.e., the 8-dimensional Lie group which preserves the $(2,1)$-signature hermitian form

$$h(Z,Z)=\begin{pmatrix}Z^1, & Z^2, & Z^3\end{pmatrix}\hat h\begin{pmatrix}\bar Z^1\\ \bar Z^2\\ \bar Z^3\end{pmatrix},\qquad \hat h=\begin{pmatrix}0&0&2i\\0&1&0\\-2i&0&0\end{pmatrix}.$$

The forms $\omega,\omega_1,\bar\omega_1,\Omega,\bar\Omega$ in the theorem can be collected into a matrix of 1-forms

$$\tilde\omega=\begin{pmatrix}\frac13(2\Omega+\bar\Omega)&0&0\\ \omega_1&\frac13(\bar\Omega-\Omega)&0\\ 2\omega&2i\bar\omega_1&-\frac13(2\bar\Omega+\Omega)\end{pmatrix},$$

satisfying

$$\tilde\omega\hat h+\hat h\tilde\omega^{\dagger}=0.$$

The Lie algebra $\mathfrak{g}_5$ of the group G_5 is then

$$\mathfrak{g}_5=\left\{\begin{pmatrix}\frac13(2z_2+\bar z_2)&0&0\\ z_1&\frac13(\bar z_2-z_2)&0\\ 2x&2i\bar z_1&-\frac13(2\bar z_2+z_2)\end{pmatrix},\ x\in\mathbb{R},z_1,z_2\in\mathbb{C}\right\},$$

and as such is a real 5-dimensional Lie algebra parametrized by the parameters $x,\mathrm{Re}(z_1),\mathrm{Im}(z_1),\mathrm{Re}(z_2),\mathrm{Im}(z_2)$. It is naturally contained in $\mathfrak{su}(2,1)$. The subgroup H_2 corresponds to the subalgebra $\mathfrak{h}_2\subset\mathfrak{g}_5$ given by $x=0,z_1=0$. Now,

$\tilde{\omega}$ can be interpreted as a Cartan connection on P [7] having values in the Lie algebra $\mathfrak{g}_5 \subset \mathfrak{su}(2, 1)$. It follows from Eq. (7.15) that the curvature R of this connection is

$$R = \mathrm{d}\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} = \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & -R_1 - R_2 \end{pmatrix},$$

where

$$R_1 = -\frac{2}{3}K_2\omega \wedge \omega_1 - \frac{1}{3}\overline{K}_2\omega \wedge \bar{\omega}_1 + \frac{1}{3}K_1\omega_1 \wedge \bar{\omega}_1$$
$$R_2 = \frac{1}{3}K_2\omega \wedge \omega_1 - \frac{1}{3}\overline{K}_2\omega \wedge \bar{\omega}_1 - \frac{2}{3}K_1\omega_1 \wedge \bar{\omega}_1.$$

It yields all the invariant information about the corresponding structure $(M, [\lambda, \mu])$, very much in the same way as the Riemann curvature yields all the information about a Riemannian structure.

7.3. Conformal Lorentzian metrics

Using the matrix elements $\tilde{\omega}^i_j$ of the Cartan connection $\tilde{\omega}$ it is convenient to consider the bilinear form

$$G = -\mathrm{i}\tilde{\omega}^3_j\tilde{\omega}^j_1.$$

This form, when written explicitly in terms of $\omega, \omega_1, \bar{\omega}_1, \Omega, \bar{\Omega}$, is given by

$$G = 2\omega_1\bar{\omega}_1 + \frac{2}{3\mathrm{i}}\omega(\Omega - \bar{\Omega}).$$

Introducing the basis of vector fields $X, X_1, \bar{X}_1, Y, \bar{Y}$, the respective duals of $\omega, \omega_1, \bar{\omega}_1, \Omega, \bar{\Omega}$, one sees that G is a form of signature $(+ + + - 0)$ with the degenerate direction tangent to the vector field $Y + \bar{Y} = \rho\partial_\rho$. We may think of the Cartan bundle P as being foliated by 1-dimensional leaves tangent to this vector field. Now Eqs. (7.15) guarantee that the Lie derivative

$$\mathcal{L}_{(Y+\bar{Y})}G = 2\,G,$$

so that the bilinear form G is preserved up to a scale when Lie transported along the leaves of the foliation. Therefore the 4-dimensional leaf space $N = P/\sim$ of the foliation is naturally equipped with a conformal class of Lorentzian metrics $[g]$, the class to which the bilinear form G naturally descends. The Lorentzian metrics

$$g = 2\omega_1\bar{\omega}_1 + \frac{2}{3\mathrm{i}}\omega(\Omega - \bar{\Omega}) \tag{7.17}$$

on N are the analogs of the Fefferman metrics [5] known in CR manifold theory.

We note that N is a circle bundle above M with the fiber coordinate ϕ.

Interestingly metrics (7.17) belong to a larger conformal family, which is also well defined on N. It turns out that if we start with a bilinear form

$$G_t = 2\omega_1\bar{\omega}_1 + 2t\mathrm{i}\omega(\bar{\Omega} - \Omega)$$

where t is *any function* on P constant along the $Y + \bar{Y}$ direction, then it also projects well to a conformal Lorentzian class $[g_t]$ on N with representatives

$$g_t = 2\omega_1\bar{\omega}_1 + 2t\mathrm{i}\omega(\bar{\Omega} - \Omega) \tag{7.18}$$

parametrized by t. To see this, it is enough to look at the explicit expressions for the forms $(\omega_1, \bar{\omega}_1, \omega, \Omega, \bar{\Omega})$ in (7.14) and to note that G_t is of the form $G_t = \rho^2(\ldots)$, where the dotted terms do not depend on the coordinate ρ which is aligned with $Y + \bar{Y}$ on P.

Although t may be an arbitrary function on N, in what follows we will only be interested in the case when t is a *constant* parameter.

We return to metrics g_t in Section 10.2, where we discuss their conformal curvature F_t and provide some example of the Lorentzian metrics satisfying the so called Bach condition.

7.4. Basic examples

Example 7.3. Note that the assumption that K_1 and K_2 are *constant* on P is compatible with (7.15) iff $K_1 = K_2 = 0$. In such case the curvature R of the Cartan connection $\tilde{\omega}$ vanishes, and it follows that there is only one, modulo local equivalence, $[\lambda, \mu]$ structure with this property. It coincides with the CR structure of the *Heisenberg group*

$$M = \{(z, w) \in \mathbb{C}^2 : \mathrm{Im}(w) = |z|^2\}$$

with the preferred splitting $\mathcal{V}$ generated by the vector field $\mathbf{v} = \partial_u$, $u = \mathrm{Re}(w)$. We call this the *standard splitting* on the Heisenberg group. The resulting oriented congruence has the maximal possible group of symmetries isomorphic to the group G_5.

Example 7.4. We recall that a 3-dimensional CR manifold M embedded in $\mathbb{C}^2$ via

$$M = \left\{(z, w = u + \mathrm{i}v) \in \mathbb{C}^2 : v = \frac{1}{2}H(z)\right\},$$

where H is a real-valued function of the variable $z \in \mathbb{C}$, is called *rigid*. It can be given a structure of an oriented congruence by choosing the splitting to be spanned by the vector field ∂_u. As in the above special case of the Heisenberg group we call this preferred splitting on M the *standard splitting* on a rigid CR structure. Intrinsically this CR-manifold with the preferred splitting may be described in terms of the forms λ and μ given by

$$\lambda = \mathrm{d}u + \frac{\mathrm{i}}{2}(H_{\bar{z}}\mathrm{d}\bar{z} - H_z\mathrm{d}z), \qquad \mu = \mathrm{d}z. \tag{7.19}$$

Via (3.1), these forms define a structure $(M, [\lambda, \mu])$ of an oriented congruence on M. In the following we assume that

$$H_{z\bar{z}} \neq 0$$

at every point of M. It means that M is strictly pseudoconvex.

Definition 7.5. A structure $(M, [\lambda, \mu])$ of an oriented congruence with vanishing shear and nonvanishing twist on a manifold M is called (locally) *flat* iff (locally) it has vanishing curvature R for its Cartan connection $\tilde{\omega}$. The necessary and sufficient conditions for that are $K_1 \equiv 0$ and $K_2 \equiv 0$.

A short calculation leads to the following proposition.

Proposition 7.6. *Let $(M, [\lambda, \mu])$ be a structure of an oriented congruence associated with the rigid CR-manifold M via the forms λ and μ of (7.19). Then for any real-valued function $H = H(z)$ such that $H_{z\bar{z}} \neq 0$ this structure has vanishing shear and non-vanishing twist. Its relative invariant K_2 is identically vanishing, $K_2 \equiv 0$; the relative invariant K_1 is given by $K_1 = \frac{1}{\rho^2}[\log(H_{z\bar{z}})]_{z\bar{z}}$. When it vanishes the structure is flat.*

Example 7.7. We remark that the Heisenberg group CR structure may have various splittings that endow M with nonequivalent structures of an oriented congruence. To see this we perturb the standard splitting on the Heisenberg group given by the vector field ∂_u. This is accomplished by choosing a 2-parameter family of CR-functions on M given by

$$\zeta_{\epsilon_1\epsilon_2} = \epsilon_1 z + \epsilon_2(u + \mathrm{i}|z|^2), \tag{7.20}$$

and defining the structure of an oriented congruence on M via (3.1) with the forms

$$\lambda = \mathrm{d}u + \mathrm{i}(z\mathrm{d}\bar{z} - \bar{z}\mathrm{d}z), \qquad \mu_{\epsilon_1\epsilon_2} = \mathrm{d}\zeta_{\epsilon_1\epsilon_2}.$$

Note that since λ is a section of the characteristic bundle H^0 of the Heisenberg group CR-structure, and $\mu_{\epsilon_1\epsilon_2}$ is the differential of a CR-function, the structure $(M, [\lambda, \mu_{\epsilon_1\epsilon_2}])$ is *twisting* and *without shear* for all values of the real parameters ϵ_1 and ϵ_2. The real vector field $\mathbf{v}$ which gives the splitting on M is given by

$$\mathbf{v} = \partial_u + \frac{\epsilon_2}{\epsilon_1}\left[\frac{\mathrm{i}\epsilon_1 + 2\epsilon_2 z}{-\mathrm{i}\epsilon_1 + \epsilon_2(\bar{z} - z)}\partial_z + \frac{-\mathrm{i}\epsilon_1 + 2\epsilon_2\bar{z}}{\mathrm{i}\epsilon_1 + \epsilon_2(z - \bar{z})}\partial_{\bar{z}}\right],$$

if $\epsilon_1 \neq 0$, and

$$\mathbf{v} = \mathrm{i}(z\partial_z - \bar{z}\partial_{\bar{z}})$$

otherwise. A short calculation shows that the relative invariants $K_{1\epsilon_1\epsilon_2}$ and $K_{2\epsilon_1\epsilon_2}$ for this 2-parameter family of structures are

$$K_{1\epsilon_1\epsilon_2} = \frac{8\epsilon_2^2}{\rho^2|2\epsilon_2 z + \mathrm{i}\epsilon_1|^4}, \qquad K_{2\epsilon_1\epsilon_2} \equiv 0.$$

This proves that the structures with $\epsilon_2 = 0$ and $\epsilon_2 \neq 0$ are *not* locally equivalent. To analyse if the structures with $\epsilon_2 \neq 0$ are equivalent or not we need to apply further the Cartan equivalence method. We will perform it in a more general setting than this example.

7.5. The case $K_1 \neq 0$, $K_2 \equiv 0$

Let $(M, [\lambda, \mu])$ be an arbitrary structure of an oriented congruence which has nonvanishing twist, vanishing shear, and in addition has the relative invariants K_1 and K_2 such that

$$K_1 \neq 0 \quad \text{and} \quad K_2 \equiv 0.$$

Given such a structure, using the system (7.15) and the assumption $K_2 \equiv 0$, we observe that the corresponding structural form Ω has closed real part,

$$\mathrm{d}(\Omega + \bar{\Omega}) \equiv 0. \tag{7.21}$$

The assumption that $K_1 \neq 0$ enables us to make a further reduction of the Cartan system (7.15) defining the invariants. Indeed since $K_1 = \frac{1}{\rho^2} k_1 \neq 0$, we may restrict ourselves to a (possibly double-sheeted) hypersurface N_0 in P on which

$$K_1 = \pm 1,$$

where the sign is determined by the sign of the function k_1. Recall that this sign is an invariant of the structure.

Locally N_0 is a circle bundle over M defined by the condition

$$\rho^2 = |k_1|.$$

Now the system (7.15) when pulled back to N_0 locally reduces to

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{i}\omega_1 \wedge \bar{\omega}_1 + 2\mathrm{d}A \wedge \omega \\
\mathrm{d}\omega_1 &= \mathrm{d}A \wedge \omega_1 + \mathrm{i}\Sigma \wedge \omega_1 \\
\mathrm{d}\bar{\omega}_1 &= \mathrm{d}A \wedge \bar{\omega}_1 - \mathrm{i}\Sigma \wedge \bar{\omega}_1 \\
\mathrm{d}\Sigma &= \mp \mathrm{i}\omega_1 \wedge \bar{\omega}_1.
\end{aligned} \tag{7.22}$$

Here the real 1-form Σ is the pullback of the form $\frac{1}{2\mathrm{i}}(\Omega - \bar{\Omega})$ from P to N_0. According to our choice of Σ, the *minus* sign in (7.22) corresponds to $K_1 = +1$. The differential $\mathrm{d}A$ of the real function A on N_0 is determined by the condition that $2\mathrm{d}A$ is locally equal to the pullback of the $\Omega + \bar{\Omega}$ from P to N_0. Note that this pullback must be closed due to (7.21). Looking at the explicit expression for $\Omega + \bar{\Omega}$ in (7.8) and (7.10) and the integrability conditions for (7.22) we find that locally we have

$$2\mathrm{d}A = A_1\omega_1 + \bar{A}_1\bar{\omega}_1, \tag{7.23}$$

with

$$A_1 = \frac{\mathrm{e}^{-\mathrm{i}\phi}}{\sqrt{|k_1|}}\left(\left(\log\frac{|k_1|}{a}\right)_\mu + b\right). \tag{7.24}$$

The function A_1 gives a new relative invariant for the structures $(M, [\lambda, \mu])$ with $K_1 \neq 0$ and $K_2 \equiv 0$. It follows from the construction that two such structures $(M, [\lambda, \mu])$ and $(M', [\lambda', \mu'])$ are (locally) equivalent if there exists a (local) diffeomorphism of the corresponding manifolds N_0 and $N_0{}'$ which transforms the corresponding forms $(\omega, \omega_1, \bar{\omega}_1, \Sigma)$ to $(\omega', \omega'_1, \bar{\omega}'_1, \Sigma')$. This in turn implies that the relative invariant A_1 must be transformed to A'_1.

Remark 7.8. We note that among all the structures with $K_1 \neq 0$ and $K_2 \equiv 0$ the simplest have $A_1 \equiv 0$. Modulo local equivalence there are only two such structures, corresponding to the $\mp$ sign in (7.22) with $A_1 \equiv 0$. These are the *'flat cases' for the subtree* in which $K_1 \neq 0$ and $K_2 \equiv 0$.

The function A defining the relative invariant A_1 is defined only up to the addition of a constant, $A \to A + t$. Given a family of functions $A(t) = A + t$ we consider the family of bilinear forms $G_A(t)$ on N_0 defined by

$$G_{A(t)} = \mathrm{e}^{-2(A+t)}\omega_1\bar{\omega}_1.$$

The forms $G_{A(t)}$ are clearly degenerate on N_0. Denoting by $(X, X_1, \bar{X}_1, Y)$ the dual vector fields to the basis of 1-forms $(\omega, \omega_1, \bar{\omega}_1, \Sigma)$ on N_0, we see that the signature of $G_{A(t)}$ is $(+, +, 0, 0)$ with the degenerate directions aligned with the real vector fields X and Y. Next we observe that the system (7.22) implies that $[X, Y] \equiv 0$, hence the distribution spanned by X and Y is integrable. Thus N_0 is foliated by real 2-dimensional leaves. Locally the leaf space S of this foliation is a 2-dimensional real manifold, which is a Riemann surface, since the pullback to S of the 1-form ω_1 gives a basis for the $(1, 0)$ forms. Now the formula (7.23) implies that $X(A) = Y(A) \equiv 0$. Using this and the system (7.22), a calculation shows that

$$\mathcal{L}_X G_{A(t)} \equiv 0, \qquad \mathcal{L}_Y G_{A(t)} \equiv 0.$$

This means that the bilinear forms $G_{A(t)}$ descend to Riemannian homothetic metrics $g_{A(t)}$ on the Riemann surface S. We have the following theorem.

Theorem 7.9. *The Riemann surface S naturally associated with the structure of an oriented congruence having $K_1 \neq 0$, $K_2 \equiv 0$ possesses Riemannian homothetic metrics $g_{A(t)}$ whose Gaussian curvatures $\kappa(t)$ are related to the relative invariant A_1 via:*

$$\kappa(t) = \mp e^{2(A+t)}, \qquad \text{i.e. } 2dA = d\log\kappa.$$

Example 7.7 (*Continued*). Calculating A_1 for the structures $(M, [\lambda, \mu_{\epsilon_1\epsilon_2}])$ of Example 7.7, assuming that $\epsilon_2 \neq 0$, we easily find that for all ϵ_1, and $\epsilon_2 \neq 0$, we have $A_1 \equiv 0$. Thus for all nonzero values of ϵ_2, and all values of ϵ_1, the structures *are* locally equivalent. Hence the apparent 2-parameter family of the structures $(M, [\lambda, \mu_{\epsilon_1\epsilon_2}])$ includes *only two nonequivalent* cases; isomorphic to those with $(\epsilon_1, \epsilon_2) = (1, 0)$, and e.g. to those with $(\epsilon_1, \epsilon_2) = (0, 1)$. The first case is the flat case $K_1 \equiv 0$, $K_2 \equiv 0$, corresponding to the Heisenberg group with the standard splitting. The second case is considerably different, being one of the 'flat cases' for the subtree $K_1 \neq 0$ and $K_2 \equiv 0$, corresponding to $A_1 \equiv 0$ and the *minus* sign in (7.22). In particular the (0, 1) case has only a 4-dimensional symmetry group, as opposed to the 5-dimensional symmetry group of the (1, 0) case.

We would like to point out that if we were to choose a more complicated CR function than the $\zeta_{\epsilon_1\epsilon_2}$ of (7.20), for example

$$\zeta = \epsilon_1 z + \epsilon_2(u + i|z|^2)^m,$$

with $m \neq 0$ and $m \neq 1$, we would produce an oriented congruence $(M, [du + i(zd\bar{z} - \bar{z}dz), d\zeta])$, still twisting and without shear, again based on the Heisenberg group, but not equivalent to either of the two structures above. The reason for this is that the condition $m \neq 0$, $m \neq 1$ makes $(M, [du + i(zd\bar{z} - \bar{z}dz), d\zeta])$ have the relative invariant K_2 nonvanishing.

We now give a local representation for an arbitrary structure $(M, [\lambda, \mu])$ with vanishing shear, nonvanishing twist, and with $K_1 \neq 0$, $K_2 \equiv 0$. This can be done by integration of the system (7.22). Interestingly this integration can be performed explicitly, leading to the following theorem.

Theorem 7.10. *If $(M, [\lambda, \mu])$ is a structure of an oriented congruence with vanishing shear, nonvanishing twist, and with the relative invariants $K_1 \neq 0$, $K_2 \equiv 0$ then there exists a coordinate system $(u, z, \bar{z})$ on M such that the forms λ and μ representing the structure can be chosen to be*

$$\lambda = du + \frac{i}{2}(H_{\bar{z}}d\bar{z} - H_z dz), \qquad \mu = dz,$$

where the real functions $A = A(z)$ and $H = H(z)$ satisfy the system of PDEs

$$h_{z\bar{z}} = \mp e^{2A}e^{-h} \tag{7.25}$$

$$H_{z\bar{z}} = e^{-h} \tag{7.26}$$

with a real function $h = h(z)$. The structure corresponding to such λ and μ satisfies the system

$$\begin{aligned}
d\omega &= i\omega_1 \wedge \bar{\omega}_1 + 2dA \wedge \omega\\
d\omega_1 &= dA \wedge \omega_1 + i\Sigma \wedge \omega_1\\
d\bar{\omega}_1 &= dA \wedge \bar{\omega}_1 - i\Sigma \wedge \bar{\omega}_1\\
d\Sigma &= \mp i\omega_1 \wedge \bar{\omega}_1
\end{aligned}$$

with forms

$$\omega = e^{2A}\lambda, \qquad \omega_1 = e^{A}e^{-h/2}e^{i\phi}\mu, \qquad \bar{\omega}_1 = e^{A}e^{-h/2}e^{-i\phi}\bar{\mu}$$

$$\Sigma = d\phi + \frac{i}{2}(h_{\bar{z}}d\bar{z} - h_z dz).$$

The relative invariant A_1 of this structure is given by

$$A_1 = 2e^{-A}e^{h/2}e^{-i\phi}A_z.$$

Note that the system of PDEs (7.25) and (7.26) is underdetermined. To see that it always has solutions, choose a real function $H = H(z)$ on the complex plane. Define the real function $h = h(z)$ via Eq. (7.26), insert it into Eq. (7.25) and solve this *real* PDE for a real function $A = A(z)$. Since the function H can be chosen arbitrarily, returning to Example 7.4, we see that this theorem characterizes the oriented congruences which are locally equivalent to those defined on rigid CR manifolds with the standard splitting.

Corollary 7.11. *Every structure $(M, [\lambda, \mu])$ of an oriented congruence with vanishing shear, nonvanishing twist, and with the relative invariants $K_1 \neq 0$, $K_2 \equiv 0$ admits one symmetry.*

Proof. To prove this it is enough to check that in the local representation (7.25) and (7.26) the symmetry is generated by $X_0 = \partial_u$. □

Starting with a structure $(M,[\lambda,\mu])$ having $K_1\neq 0$ and $K_2\equiv 0$ we constructed its associated circle bundle $\mathbb{S}^1\to N_0\to M$ equipped with the invariant forms $(\omega,\omega_1,\bar\omega_1,\Sigma)$. Using the dual basis $(X,X_1,\bar X_1,Y)$ and the system (7.22) we see that the symmetry X_0 lifts to a vector field $\tilde X=e^{2A}X$ with the property that

$$\mathcal{L}_{\tilde X}\Sigma=0,\qquad \mathcal{L}_{\tilde X}\omega_1=2\tilde X(A)\omega_1.$$

We now introduce a quotient 3-dimensional manifold M_Σ whose points are the integral curves of $\tilde X$. Then the forms Σ and ω_1 descend from N_0 to a class of forms $[\Sigma,\omega_1]$ on M_Σ given up to the transformations $\Sigma\to\Sigma,\omega_1\to h\omega_1$. Thus they can be used to define a structure of an oriented congruence $(M_\Sigma,[\Sigma,\omega_1])$. This structure naturally associated with $(M,[\lambda,\mu])$ may be locally represented by the coordinates $(\phi,z,\bar z)$ of Theorem 7.10 with the representatives Σ and ω_1 given by

$$\Sigma=d\phi+\frac{i}{2}(h_{\bar z}d\bar z-h_z dz),\qquad \omega_1=dz.$$

Here the real function $h=h(z)$ is related to the original structure $(M,[\lambda,\mu])$ via Eqs. (7.25) and (7.26). In particular $(M_\Sigma,[\Sigma,\omega_1])$ is again based on a rigid CR structure with the standard splitting.

Now we use Theorem 7.10 to describe all the structures with $K_1\neq 0$ and $K_2\equiv 0$ which have a 4-dimensional transitive symmetry group. It turns out that they must be equivalent to those with $dA\equiv 0$. This is because the existence of a 4-dimensional transitive symmetry group implies that A_1 must be a constant. But since A and h depend only on z and $\bar z$, and A_1 has nontrivial $e^{i\phi}$ dependence, it is possible iff $A_z\equiv 0$; hence $A_1\equiv 0$. Thus according to Remark 7.8 there are only two such structures. One of them, the one with the *upper* sign in (7.22), is equivalent to the structure $(\epsilon_1,\epsilon_2)=(0,1)$ of Example 7.7. To find the second one we use Theorem 7.10 and integrate Eqs. (7.25) and (7.26) for $A=0$. Modulo equivalence we get two solutions

$$h_\mp=2\log\left(1\mp\frac{1}{2}z\bar z\right),\qquad H_\mp=\mp 2\log\left(1\mp\frac{1}{2}z\bar z\right),\qquad A=0$$

which lead to the two nonequivalent 'flat models' with $K_1=\pm1, A_1\equiv 0$. These are generated by the forms

$$\lambda_\mp=du+\frac{i}{2}\frac{zd\bar z-\bar z dz}{1\mp\frac{1}{2}z\bar z},\qquad \mu=dz.\tag{7.27}$$

Obviously the structure corresponding to the upper sign is isomorphic to the structure $(\epsilon_1,\epsilon_2)=(0,1)$ of Example 7.7. Interestingly, in either of the two nonequivalent cases the forms (λ,μ) can be used to intrinsically define a *flat* CR structure (in the sense of Cartan's paper [3]) on M parametrized by $(u,z,\bar z)$. Another feature of these two nonequivalent structures is that their Riemann surface $S_\mp$ described by Theorem 7.9 is equipped with metrics $g_{A(t)}$ which may be represented by

$$g_\mp=\frac{2dzd\bar z}{(1\mp\frac{1}{2}z\bar z)^2}.$$

Thus these Riemann surfaces are either locally homothetic to the Poincaré disc (in the upper sign case) or to the 2-dimensional sphere $\mathbb{S}^2$ (in the lower sign case). This leads to the following definition.

Definition 7.12. The two structures of an oriented congruence $(M,[\lambda_\mp,\mu])$ generated by the forms $\lambda_\mp,\mu$ of (7.27) are called the *Poincaré disc structure* (in the upper sign case) and the *spherical structure* (in the lower sign case).

We further note that the natural structures $(M_\Sigma,[\Sigma_\mp,\omega_1])$ associated with the structures (7.27) are locally isomorphic to the original structures $(M,[\lambda_\mp,\mu])$. Finally we note that the forms λ_+,μ are identical with the forms which appear in the celebrated vacuum Taub-NUT solution of the Lorentzian Einstein field equations (see formulae (11.1) and (11.2) with $K-1=m=a=0$ and with the coordinate z replaced by $2/z$). We summarize the considerations of this paragraph in the following theorem.

Theorem 7.13. *All structures $(M,[\lambda,\mu])$ of an oriented congruence with vanishing shear, nonvanishing twist, having the relative invariants $K_1\neq 0, K_2\equiv 0$ and possessing a 4-dimensional transitive symmetry group are locally isomorphic to either the Poincaré disc structure $(M,[\lambda_-,\mu])$ or the spherical structure $(M,[\lambda_+,\mu])$, i.e. they are isomorphic to one of the 'flat models' for the $K_1\neq 0$ and $K_2\equiv 0$ case.*

We now pass to the determination of all local invariants for the structures with $A_1\neq 0$. Let $(M,[\lambda,\mu])$ be such a structure with the corresponding circle bundle N_0 and the system of invariants (7.22). Looking at the explicit form (7.24) of the relative invariant A_1, we see that we may always choose a section of the bundle N_0 such that A_1 is *real* and *positive*. Locally this corresponds to the choice of ϕ as a function on the manifold M such that

$$\frac{e^{-i\phi}}{\sqrt{|k_1|}}\left(\left(\log\frac{|k_1|}{a}\right)_\mu+b\right)=\frac{e^{i\phi}}{\sqrt{|k_1|}}\left(\left(\log\frac{|k_1|}{a}\right)_{\bar\mu}+\bar b\right)>0.\tag{7.28}$$

If ϕ satisfies (7.28) then

$$A_1 > 0,$$

and all the structural objects defined by the system (7.22) may be uniquely pulled back to M. As the result of this pullback the real 1-form Σ becomes dependent on the pulled back forms $(\omega, \omega_1, \bar{\omega}_1)$. Since these three 1-forms constitute a coframe on M we may write $\Sigma = B_0\omega + B_1\omega_1 + \bar{B}_1\bar{\omega}_1$ where B_0 (real) and B_1 (complex) are functions on M. Now using the fact that these structures admit a symmetry (Corollary 7.11), we get $B_0 \equiv 0$. Hence

$$\Sigma = B_1\omega_1 + \bar{B}_1\bar{\omega}_1.$$

With this notation the pulled back system (7.22) becomes

$$\begin{aligned} d\omega &= i\omega_1 \wedge \bar{\omega}_1 + 2A_1(\omega_1 + \bar{\omega}_1) \wedge \omega \\ d\omega_1 &= -(A_1 + i\bar{B}_1)\omega_1 \wedge \bar{\omega}_1 \\ d\bar{\omega}_1 &= (A_1 - iB_1)\omega_1 \wedge \bar{\omega}_1, \end{aligned} \tag{7.29}$$

with the fourth equation given by

$$d(B_1\omega_1 + \bar{B}_1\bar{\omega}_1) = \mp i\omega_1 \wedge \bar{\omega}_1. \tag{7.30}$$

Remark 7.14. Note that since on N_0 the complex function A_1 was constrained by $d(A_1\omega_1 + \bar{A}_1\bar{\omega}_1) = 0$, because of (7.23), the Eqs. (7.29) and (7.30) should be supplemented by the equation $d[A_1(\omega_1 + \bar{\omega}_1)] = 0$ for $A_1 > 0$. This however is equivalent to

$$dA_1 \wedge (\omega_1 + \bar{\omega}_1) = [iA_1(B_1 + \bar{B}_1)]\omega_1 \wedge \bar{\omega}_1,$$

and turns out to follow from the integrability conditions for (7.29) and (7.30).

Writing these integrability conditions explicitly we have:

$$\begin{aligned} dA_1 &= \left[a_{11} + \frac{i}{2}A_1(B_1 + \bar{B}_1)\right]\omega_1 + \left[a_{11} - \frac{i}{2}A_1(B_1 + \bar{B}_1)\right]\bar{\omega}_1 \\ dB_1 &= B_{11}\omega_1 + \left[b_{12} + \frac{1}{2}A_1(\bar{B}_1 - B_1) + i\left(\pm\frac{1}{2} - |B_1|^2\right)\right]\bar{\omega}_1 \\ d\bar{B}_1 &= \left[b_{12} - \frac{1}{2}A_1(\bar{B}_1 - B_1) - i\left(\pm\frac{1}{2} - |B_1|^2\right)\right]\omega_1 + \bar{B}_{11}\bar{\omega}_1, \end{aligned} \tag{7.31}$$

where the real functions a_{11}, b_{12} are the scalar invariants of the next higher order than A_1 and B_1.

Theorem 7.15. *The functions $A_1 > 0$ and B_1 (complex) constitute the full system of basic scalar invariants for the structures $(M, [\lambda, \mu])$ with $K_1 \neq 0$, $K_2 \equiv 0$ and $A_1 \neq 0$. It follows from the construction that two such structures $(M, [\lambda, \mu])$ and $(M', [\lambda', \mu'])$ are (locally) equivalent iff there exists a (local) diffeomorphism between M and M' which transforms the corresponding forms $(\omega, \omega_1, \bar{\omega}_1)$ to $(\omega', \omega'_1, \bar{\omega}'_1)$. This in particular implies that the invariants A_1 and B_1 must be transformed to A'_1 and B'_1.*

The system (7.29)–(7.31) and the above theorem can be used to find all structures with $K_1 \neq 0$ and $K_2 \equiv 0$ having a strictly 3-dimensional transitive symmetry group. These are the structures described by the system (7.29)–(7.31) with *constant* basic invariants $A_1 > 0$, B_1. It follows that it is possible only if $B_1 = i\tau$, $A_1 = \frac{\pm 1 - 2\tau^2}{2\tau} > 0$ and $\tau \neq 0$ is a real parameter. This leads to *only* two quite different cases, which are described by Propositions 7.16 and 7.17.

Proposition 7.16. (i) *All locally nonequivalent structures $(M, [\lambda, \mu])$ of oriented congruences having vanishing shear, nonvanishing twist, $K_1 \neq 0$, $K_2 \equiv 0$, and possessing a strictly 3-dimensional transitive group G_h of symmetries of Bianchi type* VI$_h$, *$h \leq 0$, may be locally represented by*

$$\lambda = y^b du - y^{-1}dx, \qquad \mu = y^{-1}(dx + idy).$$

Here $(u, z, \bar{z})$ with $z = x + iy$ are coordinates on M and

$$b = -2(1 \mp 2\tau^2).$$

The real parameter τ is related to the invariants B_1 and A_1 via

$$B_1 = i\tau, \qquad A_1 = -\frac{\mp 1 + 2\tau^2}{2\tau} > 0,$$

and as such enumerates nonequivalent structures.

(ii) *Regardless of the values of* τ *the structures corresponding to the upper and lower signs in the expressions above are nonequivalent. In the case of the lower signs the real parameter* $\tau < 0$. *In the case of the upper signs* $\tau < -\frac{1}{\sqrt{2}}$ *or* $0 < \tau < \frac{1}{2}$ *or* $\frac{1}{2} < \tau < \frac{1}{\sqrt{2}}$.

(iii) *The structures are locally CR equivalent to the Heisenberg group CR structure only in the case of the upper signs with* $\tau = \frac{\sqrt{3}}{2\sqrt{2}}$.

(iv) *The symmetry group is of Bianchi type* VI_h, *with the parameter* $h \le 0$ *related to* τ *via*

$$h = -\left(\frac{3 \mp 4\tau^2}{1 \mp 4\tau^2}\right)^2.$$

In the lower sign case the possible values of h *are* $-9 < h < -1$, *and for each value of* h *we always have one structure with the symmetry group* G_h. *In the upper sign case* h *may assume all values* $h \le 0$, $h \ne -1$. *In this case, we always have*

- *two nonequivalent structures with symmetry group* G_h *with* $h < -9$;
- *one structure with symmetry group* G_h *with* $-9 \le h < -1$; *if the parameter* τ *is* $\tau = \frac{\sqrt{3}}{2\sqrt{2}}$ *then* $h = -9$ *and the structure is based on the Heisenberg group with a particular nonstandard splitting;*
- *two nonequivalent structures with symmetry group* G_h *with* $-1 < h < 0$;
- *one structure with symmetry group of Bianchi type* VI_0.

Proposition 7.17. *Modulo local equivalence there exists only one structure* $(M, [\lambda, \mu])$ *of an oriented congruence having vanishing shear, nonvanishing twist,* $K_1 \ne 0$, $K_2 \equiv 0$, *and possessing a strictly 3-dimensional transitive group of symmetries of Bianchi type* IV . *Locally it may be represented by the forms*

$$\lambda = y^{-1}(\mathrm{d}u + \log y\mathrm{d}x), \qquad \mu = y^{-1}(\mathrm{d}x + \mathrm{i}\mathrm{d}y).$$

Here $(u, z, \bar{z})$ *with* $z = x + \mathrm{i}y$ *are coordinates on* M. *The structure has the basic local invariants* $A_1 = \frac{1}{2}$ *and* $B_1 = \frac{\mathrm{i}}{2}$.

Summarizing we have the following theorem.

Theorem 7.18. *All locally nonequivalent structures* $(M, [\lambda, \mu])$ *of oriented congruences having vanishing shear, nonvanishing twist,* $K_1 \ne 0$, $K_2 \equiv 0$, *and possessing a strictly 3-dimensional transitive group of symmetries are locally equivalent to one of the structures defined in Propositions 7.16 and 7.17.*

Remark 7.19. Example 7.3, Theorems 7.13 and 7.18 describe *all* locally nonequivalent *homogeneous* structures of an oriented congruence having vanishing shear, nonvanishing twist and with the invariant $K_2 \equiv 0$. They may have

- Maximal symmetry group of dimension 5, and then they are locally isomorphic to the Heisenberg group with the standard splitting.
- Symmetry group of exact dimension 4, and then they are locally isomorphic to one of the two nonequivalent structures of Theorem 7.13.
- Symmetry group of exact dimension 3 which must be of either Bianchi type VI_h or IV; in this case they are given by Propositions 7.16 and 7.17.

7.6. *The case* $K_2 \ne 0$

Looking at the explicit expression for K_2 in (7.16) we see that in this case we may fix both ρ and ϕ by the requirement that

$$K_2 = 1. \tag{7.32}$$

Indeed this normalization forces ρ and ϕ to be

$$\rho = |k_2|^{\frac{1}{3}}, \qquad \phi = \mathrm{Arg}(k_2).$$

This provides an embedding of M into P. Using it (technically speaking, by inserting ρ and ϕ in the definitions of the invariant coframe (7.14)) we pullback the forms $(\omega_1, \bar{\omega}_1, \omega, \Omega, \bar{\Omega})$ on P to M. Also K_1 is pulled back to M, so that

$$K_1 = \frac{k_1}{|k_2|^{\frac{2}{3}}}.$$

Since M is 3-dimensional the pulled back forms are no longer linearly independent, and the pullback of the derived form Ω decomposes onto the invariant coframe $(\omega_1, \bar{\omega}_1, \omega)$ on M. We denote the coefficients of this decomposition by (Z_1, Z_2, Z_0) so that:

$$\Omega = Z_1\omega_1 + Z_2\bar{\omega}_1 + Z_0\omega$$
$$\bar{\Omega} = \bar{Z}_2\omega_1 + \bar{Z}_1\bar{\omega}_1 + \bar{Z}_0\omega.$$

These coefficients constitute the basic scalar invariants of the structures under consideration. They satisfy the following differential system:

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{i}\omega_1\wedge\bar\omega_1+(Z_1+\bar Z_2)\omega_1\wedge\omega+(Z_2+\bar Z_1)\bar\omega_1\wedge\omega\\ \mathrm{d}\omega_1 &= -Z_2\omega_1\wedge\bar\omega_1-Z_0\omega_1\wedge\omega\\ \mathrm{d}\bar\omega_1 &= \bar Z_2\omega_1\wedge\bar\omega_1-\bar Z_0\bar\omega_1\wedge\omega \end{aligned} \tag{7.33}$$

with

$$\begin{aligned} \mathrm{d}[Z_1\omega_1+Z_2\bar\omega_1+Z_0\omega] &= K_1\omega_1\wedge\bar\omega_1+\omega_1\wedge\omega\\ \mathrm{d}[\bar Z_2\omega_1+\bar Z_1\bar\omega_1+\bar Z_0\omega] &= -K_1\omega_1\wedge\bar\omega_1+\bar\omega_1\wedge\omega. \end{aligned}$$

Instead of considering the last two equations above, it is convenient to replace them by the integrability conditions for the system (7.33). These are:

$$\begin{aligned} \mathrm{d}Z_1 &= Z_{11}\omega_1+(-K_1+\mathrm{i}Z_0-Z_1Z_2+Z_2\bar Z_2+Z_{21})\bar\omega_1+(Z_0\bar Z_2+Z_{01}-1)\omega\\ \mathrm{d}\bar Z_1 &= (-K_1-\mathrm{i}\bar Z_0-\bar Z_1\bar Z_2+Z_2\bar Z_2+\bar Z_{21})\omega_1+\bar Z_{11}\bar\omega_1+(\bar Z_0Z_2+\bar Z_{01}-1)\omega\\ \mathrm{d}Z_2 &= Z_{21}\omega_1+Z_{22}\bar\omega_1+(Z_{02}+Z_0\bar Z_1+Z_0Z_2-\bar Z_0Z_2)\omega\\ \mathrm{d}\bar Z_2 &= \bar Z_{22}\omega_1+\bar Z_{21}\bar\omega_1+(\bar Z_{02}+\bar Z_0Z_1+\bar Z_0\bar Z_2-Z_0\bar Z_2)\omega\\ \mathrm{d}Z_0 &= Z_{01}\omega_1+Z_{02}\bar\omega_1+Z_{00}\omega\\ \mathrm{d}\bar Z_0 &= \bar Z_{02}\omega_1+\bar Z_{01}\bar\omega_1+\bar Z_{00}\omega\\ \mathrm{d}K_1 &= K_{11}\omega_1+\bar K_{11}\bar\omega_1+K_{10}\omega, \end{aligned} \tag{7.34}$$

where, in addition to the basic scalar invariants Z_0, Z_1, Z_2, K_1, we have introduced the scalar invariants of the next higher order: $Z_{00}, Z_{01}, Z_{02}, Z_{11}, Z_{21}, Z_{22}$ (complex) and K_{10} (real). Note that if the basic scalar invariants Z_0, Z_1, Z_2, K_1 were constants, all the higher order invariants such as $Z_{00}, Z_{01}, Z_{02}, Z_{11}, Z_{21}, Z_{22}, K_{10}$ would be identically vanishing.

Theorem 7.20. *All locally nonequivalent structures $(M, [\lambda, \mu])$ of oriented congruences having vanishing shear, nonvanishing twist, and with $K_2 \neq 0$ are described by the invariant system (7.33) with the integrabilty conditions (7.34).*

Now we pass to the determination of all nonequivalent structures with $K_2 \neq 0$ which have a strictly 3-dimensional transitive group of symmetries. They correspond to the structures of Theorem 7.20 with all the scalar invariants being constants. It turns out that there are two families of such structures. The first family is described by the following invariant system:

$$\begin{aligned} \mathrm{d}\omega_1 &= \mathrm{e}^{\mathrm{i}\alpha}[-(2\sin\alpha)^{-1/3}\omega_1\wedge\bar\omega_1-(2\sin\alpha)^{1/3}\omega_1\wedge\omega],\\ \mathrm{d}\bar\omega_1 &= \mathrm{e}^{-\mathrm{i}\alpha}[(2\sin\alpha)^{-1/3}\omega_1\wedge\bar\omega_1-(2\sin\alpha)^{1/3}\bar\omega_1\wedge\omega],\\ \mathrm{d}\omega &= \mathrm{i}\omega_1\wedge\bar\omega_1+(2\sin\alpha)^{-1/3}(\mathrm{e}^{\mathrm{i}\alpha}\omega_1\wedge\omega+\mathrm{e}^{-\mathrm{i}\alpha}\bar\omega_1\wedge\omega). \end{aligned}$$

All the nonvanishing scalar invariants here are:

$$K_1=(2\sin\alpha)^{-2/3}$$

and

$$Z_1=\mathrm{i}(2\sin\alpha)^{2/3},\qquad Z_2=\mathrm{e}^{\mathrm{i}\alpha}(2\sin\alpha)^{-1/3},\qquad Z_0=\mathrm{e}^{\mathrm{i}\alpha}(2\sin\alpha)^{1/3}.$$

Two different values α and α' of the parameter yield different respective quadruples (K_1, Z_0, Z_1, Z_2) and (K_1', Z_0', Z_1', Z_2'), and hence correspond to nonequivalent structures.

The second family of nonequivalent structures with a strictly 3-dimensional group of symmetries corresponds to the following invariant system:

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{i}\omega_1\wedge\bar\omega_1+\mathrm{i}\beta^{-1}\omega\wedge(\omega_1-\bar\omega_1)\\ \mathrm{d}\omega_1 &= -\mathrm{i}(\beta\omega+\beta^{-1}\bar\omega_1)\wedge\omega_1\\ \mathrm{d}\bar\omega_1 &= \mathrm{i}(\beta\omega+\beta^{-1}\omega_1)\wedge\bar\omega_1. \end{aligned} \tag{7.35}$$

The nonvanishing scalar invariants here are:

$$K_1=(\beta^3+3)\beta^{-2},\qquad Z_1=-2\mathrm{i}\beta^{-1},\qquad Z_2=-\mathrm{i}\beta^{-1},\qquad Z_0=-\mathrm{i}\beta. \tag{7.36}$$

The corresponding structures of an oriented congruence are parametrized by a real parameter $\beta \neq 0$. This means that each $\beta \neq 0$ defines a distinct structure.

A further analysis of this system shows that the congruence structures described by it have a transitive symmetry group of Bianchi type VII_0 (iff $\beta = -2^{\frac{1}{3}}$), Bianchi type VIII (iff $\beta > -2^{\frac{1}{3}}$), and of Bianchi type IX (iff $\beta < -2^{\frac{1}{3}}$).

If we parametrize the 3-dimensional manifold M by $(u, z, \bar{z})$, the structures (M, λ, μ) corresponding to the system (7.35) may be locally represented by:

$$\begin{aligned}\lambda &= \mathrm{d}u + \frac{2\beta e^{-i\beta u} + i\bar{z}}{\beta(z\bar{z} - 2\beta^2(2+\beta^3))}\mathrm{d}z + \frac{2\beta e^{i\beta u} - iz}{\beta(z\bar{z} - 2\beta^2(2+\beta^3))}\mathrm{d}\bar{z}\\ \mu &= -\frac{2\beta^2 e^{-i\beta u}}{z\bar{z} - 2\beta^2(2+\beta^3)}\mathrm{d}z, \qquad \bar{\mu} = -\frac{2\beta^2 e^{i\beta u}}{z\bar{z} - 2\beta^2(2+\beta^3)}\mathrm{d}\bar{z}.\end{aligned} \tag{7.37}$$

Note that the above (λ, μ) can be also used to define a *CR structure* on M, and that different $\beta \neq 0$ correspond to different CR structures in the sense of Cartan.

Three particular values of $\beta \neq 0$ in (7.37) are worthy of mention. These are:

$$\beta = \beta_B = -2^{\frac{1}{3}},$$

when the local symmetry group (both the CR and the oriented congruence symmetry) changes the structure from Bianchi type IX, with $\beta < \beta_B$; through Bianchi type VII_0, with $\beta = \beta_B$; to Bianchi type VIII, with $\beta > \beta_B$.

Next is:

$$\beta = \beta_H = -1,$$

when the lowest order *Cartan* invariant of the CR structure associated with λ_{β_H} and μ_{β_H} is identically vanishing [15]; in this case the CR structure becomes locally equivalent to the Heisenberg group CR structure, and the 3-dimensional transitive CR symmetry group of Bianchi type IX is extendable, from the local $\mathbf{SO}(3)$ group, to the 8-dimensional local CR symmetry group $\mathbf{SU}(2, 1)$.

The third distinguished β is:

$$\beta = \beta_K = -3^{\frac{1}{3}}.$$

Note that for $\beta = \beta_K$ *our* invariant K_1 of the congruence structure $(\lambda_\beta, \mu_\beta)$ vanishes, $K_1 \equiv 0$, as in (7.36). This case is of some importance, since it will be shown in Section 10.2 that the congruence structures with $K_1 \equiv 0$ and $K_2 \neq 0$ have very nice properties.

8. Vanishing twist and nonvanishing shear

Now we assume the opposite of Section 7, namely that $(M, [\lambda, \mu])$ has some shear, $s \neq 0$, but has identically vanishing twist, $a \equiv 0$. As in Section 6 the no twist condition $\mathrm{d}\lambda \wedge \lambda \equiv 0$ yields $\lambda = f\mathrm{d}t$ for some real function t on M. Thus in this case we again have a foliation of M by the level surfaces $t = \text{const}$. Each leaf $\mathcal{C}$ of this foliation is a 2-dimensional real submanifold which is equipped with a complex structure J determined by the requirement that its holomorphic vector bundle $H^{1,0} = \{X - iJX, X \in \Gamma(T\mathcal{C})\}$ coincides with the anihilator of $Span_{\mathbb{C}}(\lambda) \oplus Span_{\mathbb{C}}(\bar{\mu})$. But the simple situation of M being locally equivalent to $\mathbb{R} \times \mathbb{C}$ is no longer true. If $s \neq 0$ the manifold M gets equipped with the structure of a fiber bundle $\mathcal{C} \to M \to V$, with fibers $\mathcal{C}$ being 1-dimensional complex manifolds—the leaves of the foliation given by $t = \text{const}$, and with the base V being 1-dimensional, and parametrized by t. This can be rephrased by saying that we have a 1-parameter family of complex curves $\mathcal{C}(t)$, with complex structure tensors $J_{\mathcal{C}(t)}$, which are *not* invariant under Lie transport along the vector field ∂_t. Recall that having a complex structure in a real 2-dimensional vector space is equivalent to having a conformal metric and an orientation in the space. Thus the condition of having $s \neq 0$ means that, under Lie transport along ∂_t, the metrics on the 2-planes tangent to the surfaces $t = \text{const}$ change in a fashion more general than conformal. This means that small circles on these two planes do not go to small circles when Lie transported along ∂_t. They may, for example, be distorted into small ellipses, which intuitively means that the congruence generated by ∂_t has shear. This explains the name of the complex parameter s, as was promised in Section 5.

We now pass to a more explicit description of this situation. We start with an arbitrary structure $(M, [\lambda, \mu])$ with $\mathrm{d}\lambda \wedge \lambda = 0$. This guarantees that the 2-dimensional distribution annihilating λ defines a foliation in M, and M is additionally equipped with a transversal congruence of curves. Note that a foliation of a 3-space by 2-surfaces equipped with a congruence *locally* can either be described in terms of coordinates (t, x, y) such that the tangent vector to the congruence is ∂_t (in such case the surfaces are in general curved for each value of the parameter t), or in terms of coordinates $(u, z, \bar{z})$ such that locally the surfaces are 2-planes (in such case the congruence is tangent to a vector field with a more complicated representation $X = \partial_u + S\partial_z + \bar{S}\partial_{\bar{z}}$. Regardless of the descriptions, the leaves of the foliation are given by the level surfaces of the real parameters $t = \text{const}$ (in the first case, as in the beginning of this Section) or $u = \text{const}$ (as it will be used in this Section from now on). Having this in mind and recalling the allowed transformations (3.1) we conclude that our $(M, [\lambda, \mu])$ with $\mathrm{d}\lambda \wedge \lambda = 0$ may be represented by a pair of 1-forms

$$\lambda = \mathrm{d}u, \qquad \mu = \mathrm{d}z + H\mathrm{d}\bar{z} + G\mathrm{d}u,$$

where $H = H(u, z, \bar{z})$ and $G = G(u, z, \bar{z})$ are complex-valued functions on M, with coordinates $(u, z, \bar{z})$, such that $|H| < 1$. The foliation has leaves tangent to the vector fields ∂_z, $\partial_{\bar{z}}$. Each leaf is equipped with a complex structure, which may be described by saying that its $T^{(1,0)}$ space is spanned by the vector field

$$Z = \partial_z - \bar{H}\partial_{\bar{z}}; \tag{8.1}$$

consequently the $T^{(0,1)}$ space is spanned by the complex conjugate vector field

$$\bar{Z} = \partial_{\bar{z}} - H\partial_z.$$

The congruence on M which gives the preferred splitting is tangent to the real vector field

$$X = \partial_u + \frac{\bar{G}H - G}{1 - H\bar{H}}\partial_z + \frac{G\bar{H} - \bar{G}}{1 - H\bar{H}}\partial_{\bar{z}}. \tag{8.2}$$

Thus we have the following proposition.

Proposition 8.1. *All structures $(M, [\lambda, \mu])$ with vanishing twist, $a \equiv 0$, may be locally represented by*

$$\lambda = \mathrm{d}u, \qquad \mu = \mathrm{d}z + H\mathrm{d}\bar{z} + G\mathrm{d}u, \tag{8.3}$$

where $H = H(u, z, \bar{z})$ and $G = G(u, z, \bar{z})$ are complex-valued functions on M, with coordinates $(u, z, \bar{z})$, such that $|H| < 1$. They have nonvanishing shear $s \neq 0$ iff

$$H_u - GH_z + HG_z - G_{\bar{z}} \neq 0.$$

The following two cases are of particular interest:

- $H \equiv 0$. In this case all surfaces $u =$ const are equipped with the standard complex structure. The coordinate z is the holomorphic coordinate for it, but the congruence is tangent to a complicated real vector field $X = \partial_u - G\partial_z - \bar{G}\partial_{\bar{z}}$.
- $G \equiv 0$. Here each surface $u =$ const has its own complex structure J, for which z is not a holomorphic coordinate; J is determined by specifying a complex function H. A nice feature of this case is that the congruence is now tangent to the very simple vector field $X = \partial_u$, which enables us to identify coordinates t and u.

Note that in Proposition 8.1 we made an assumption about the modulus of the function H. The modulus equal to one is excluded because it violates the condition that the forms λ, μ, $\bar{\mu}$ are independent. We excluded also the $H > 1$ case, since because of the coordinate transformation $z \to \bar{z}$ followed by $H \to 1/H$, such structures are in one to one equivalence with those having $|H| < 1$. We now turn to the question about nonequivalent structures among those covered by Proposition 8.1.

8.1. The invariant T_0 and the relative invariants T_1, K_0, K_1

To answer this we have to go back to the beginning of Section 5 and again perform the Cartan analysis on the system (5.1), but now with $a \equiv 0$, $s \neq 0$. In this case the formulae (5.1) become

$$\begin{aligned} \mathrm{d}\lambda &= b\mu \wedge \lambda + \bar{b}\bar{\mu} \wedge \lambda \\ \mathrm{d}\mu &= p\mu \wedge \bar{\mu} + q\mu \wedge \lambda + s\bar{\mu} \wedge \lambda \\ \mathrm{d}\bar{\mu} &= -\bar{p}\mu \wedge \bar{\mu} + \bar{s}\mu \wedge \lambda + \bar{q}\bar{\mu} \wedge \lambda. \end{aligned} \tag{8.4}$$

It is convenient to write the complex shear function s as

$$s = |s|\mathrm{e}^{\mathrm{i}\psi}.$$

Now for a chosen pair (λ, μ) representing the structure, using (8.4), we find that the differentials of the Cartan frame

$$(\omega, \omega_1, \bar{\omega}_1) = (f\lambda, \rho\mathrm{e}^{\mathrm{i}\phi}\mu, \rho\mathrm{e}^{-\mathrm{i}\phi}\bar{\mu}) \tag{8.5}$$

are:

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}\log f \wedge \omega + \frac{b}{\rho}\mathrm{e}^{-\mathrm{i}\phi}\omega_1 \wedge \omega + \frac{\bar{b}}{\rho}\mathrm{e}^{\mathrm{i}\phi}\bar{\omega}_1 \wedge \omega \\ \mathrm{d}\omega_1 &= \mathrm{i}\mathrm{d}\phi \wedge \omega_1 + \mathrm{d}\log\rho \wedge \omega_1 + \frac{p}{\rho}\mathrm{e}^{\mathrm{i}\phi}\omega_1 \wedge \bar{\omega}_1 + \frac{q}{f}\omega_1 \wedge \omega + \frac{|s|}{f}\mathrm{e}^{\mathrm{i}(2\phi+\psi)}\bar{\omega}_1 \wedge \omega \\ \mathrm{d}\bar{\omega}_1 &= -\mathrm{i}\mathrm{d}\phi \wedge \bar{\omega}_1 + \mathrm{d}\log\rho \wedge \bar{\omega}_1 - \frac{\bar{p}}{\rho}\mathrm{e}^{-\mathrm{i}\phi}\omega_1 \wedge \bar{\omega}_1 + \frac{|s|}{f}\mathrm{e}^{-\mathrm{i}(2\phi+\psi)}\omega_1 \wedge \omega + \frac{\bar{q}}{f}\bar{\omega}_1 \wedge \omega. \end{aligned}$$

Because of $s \neq 0$, we can gauge the structure so that

$$\mathrm{d}\omega_1 \wedge \omega_1 = \omega_1 \wedge \bar{\omega}_1 \wedge \omega. \tag{8.6}$$

This requirement defines f modulo sign to be $f=\pm|s|$. Writing f as

$$f=\mathrm{e}^{\mathrm{i}\epsilon\pi}|s|,$$

where $\epsilon=0,1$, and still requiring the normalization (8.6), we get

$$\phi=-\frac{1}{2}\psi+\epsilon\frac{\pi}{2}.$$

Thus the functions f and ϕ are fixed modulo ϵ.

After this normalization we introduce a *real* 1-form Ω such that

$$(\mathrm{d}\omega_1-\mathrm{d}\bar{\omega}_1)\wedge(\omega_1+\bar{\omega}_1)=2\Omega\wedge\omega_1\wedge\bar{\omega}_1. \tag{8.7}$$

This equation defines Ω to be

$$\Omega=\mathrm{d}\log\rho+z\omega_1+\bar{z}\bar{\omega}_1+\left(1-\mathrm{e}^{\mathrm{i}\epsilon\pi}\frac{q+\bar{q}}{2|s|}\right)\omega,$$

where z is an auxiliary complex parameter. The condition that fixes z in an algebraic fashion is:

$$\mathrm{d}\omega_1\wedge\omega=\Omega\wedge\omega_1\wedge\omega,\qquad \mathrm{d}\bar{\omega}_1\wedge\omega=\Omega\wedge\bar{\omega}_1\wedge\omega. \tag{8.8}$$

It uniquely specifies z to be

$$z=\frac{(\mathrm{i}\psi_\mu-2\bar{p})}{2\rho}\mathrm{e}^{\frac{\mathrm{i}}{2}(\psi-\epsilon\pi)},\qquad \bar{z}=\frac{(-\mathrm{i}\psi_{\bar{\mu}}-2p)}{2\rho}\mathrm{e}^{-\frac{\mathrm{i}}{2}(\psi-\epsilon\pi)}.$$

Thus given a structure $(M,[\lambda,\mu])$ with vanishing twist and nonvanishing shear, the three normalization conditions (8.6)–(8.8) uniquely specify a 4-dimensional manifold P, which is locally $M\times\mathbb{R}_+$, and a well defined coframe $(\omega,\omega_1,\bar{\omega}_1,\Omega)$ on it such that

$$\begin{aligned}
\omega&=\mathrm{e}^{\mathrm{i}\epsilon\pi}|s|\lambda\\
\omega_1&=\rho\mathrm{e}^{-\frac{\mathrm{i}}{2}(\psi-\epsilon\pi)}\mu\\
\bar{\omega}_1&=\rho\mathrm{e}^{\frac{\mathrm{i}}{2}(\psi-\epsilon\pi)}\bar{\mu}\\
\Omega&=\mathrm{d}\log\rho+\frac{(\mathrm{i}\psi_\mu-2\bar{p})}{2\rho}\mathrm{e}^{\frac{\mathrm{i}}{2}(\psi-\epsilon\pi)}\omega_1+\frac{(-\mathrm{i}\psi_{\bar{\mu}}-2p)}{2\rho}\mathrm{e}^{-\frac{\mathrm{i}}{2}(\psi-\epsilon\pi)}\bar{\omega}_1+\left(1-\mathrm{e}^{\mathrm{i}\epsilon\pi}\frac{q+\bar{q}}{2|s|}\right)\omega.
\end{aligned} \tag{8.9}$$

Here the positive coordinate along the factor $\mathbb{R}_+$ in the fibration $\mathbb{R}_+\to P\to M$ is ρ. The coframe $(\omega,\omega_1,\bar{\omega}_1,\Omega)$ satisfies

$$\begin{aligned}
\mathrm{d}\omega&=T_1\omega_1\wedge\omega+\bar{T}_1\bar{\omega}_1\wedge\omega\\
\mathrm{d}\omega_1&=\Omega\wedge\omega_1+(\omega_1+\bar{\omega}_1)\wedge\omega+\mathrm{i}T_0\omega_1\wedge\omega\\
\mathrm{d}\bar{\omega}_1&=\Omega\wedge\bar{\omega}_1+(\omega_1+\bar{\omega}_1)\wedge\omega-\mathrm{i}T_0\bar{\omega}_1\wedge\omega\\
\mathrm{d}\Omega&=\mathrm{i}K_0\omega_1\wedge\bar{\omega}_1+K_1\omega_1\wedge\omega+\bar{K}_1\bar{\omega}_1\wedge\omega
\end{aligned} \tag{8.10}$$

where

$$T_0=\frac{\psi_\lambda+\mathrm{i}(\bar{q}-q)}{2|s|}\mathrm{e}^{\mathrm{i}\epsilon\pi},\qquad T_1=\frac{t_1}{\rho},\qquad K_0=\frac{k_0}{2\rho^2},\qquad K_1=\frac{k_1}{2\rho} \tag{8.11}$$

and

$$\begin{aligned}
t_1&=(b|s|+|s|_\mu)\frac{\mathrm{e}^{\frac{\mathrm{i}}{2}(\psi-\epsilon\pi)}}{|s|}\\
k_0&=-\psi_{\mu\bar{\mu}}-\psi_{\bar{\mu}\mu}+p\psi_\mu+\bar{p}\psi_{\bar{\mu}}+2\mathrm{i}(p_\mu-\bar{p}_{\bar{\mu}})\\
k_1&=2(t_1-\bar{t}_1)+\mathrm{e}^{\frac{\mathrm{i}}{2}\epsilon\pi}[(b\bar{q}-bq-q_\mu+\bar{q}_\mu+\mathrm{i}q\psi_\mu-\mathrm{i}\psi_{\mu\lambda})\mathrm{e}^{\frac{\mathrm{i}}{2}\psi}+\mathrm{i}\psi_{\bar{\mu}}|s|\mathrm{e}^{-\frac{\mathrm{i}}{2}\psi}]|s|^{-1}.
\end{aligned} \tag{8.12}$$

Note that functions T_0, T_1, K_0 and K_1 are invariants of the structure on the *bundle* $\mathbb{R}_+\to P\to M$, with the fiber coordinate ρ. They are defined modulo the parameter $\epsilon=0,1$. Thus two structures which differ only by the value of ϵ are equivalent.

If we want to look for the invariants on the *original* manifold M we must examine the fiber coordinate dependence of the structural functions T_0, T_1, K_0 and K_1. Since the last three functions T_1, K_0, K_1 have a nontrivial ρ dependence they do not project to invariant functions on M. However, since in all these cases this dependence is just *scaling* by ρ we conclude that they lead to the *relative* invariants on M. Thus the vanishing or not of any of the functions t_1, k_1 (complex), k_0 (real) is an invariant property of the structure on M. The situation is quite different for the real function T_0. Although originally defined on P it is *constant* along the fibers. Thus it projects to a well defined invariant on the original manifold M. Thus T_0 is an invariant of the structure on M. We summarize the above discussion in the following theorem.

Theorem 8.2. *A given structure* $(M, [\lambda, \mu])$ *of an oriented congruence with vanishing twist,* $a \equiv 0$*, and nonvanishing shear,* $s \neq 0$*, uniquely defines a 4-dimensional manifold* P*, 1-forms* $\omega, \omega_1, \bar{\omega}_1, \Omega$ *and functions* T_0, K_0 *(real)* T_1, K_1 *(complex) on* P *such that*

- $\omega, \omega_1, \bar{\omega}_1, \Omega$ *are as in (8.9),*
- $\omega \wedge \omega_1 \wedge \bar{\omega}_1 \wedge \Omega \neq 0$ *at each point of* P*,*
- *the forms and functions* T_0, T_1, K_0, K_1 *are uniquely determined by the requirement that on* P *they satisfy Eqs. (8.10).*

In particular T_0 *is an invariant of the structure on* M*; the identical vanishing, or not, of either of the functions* t_1*,* k_0 *or* k_1 *defined in (8.12) is an invariant condition on* M*.*

The structures covered by Theorem 8.2 admit symmetry groups of *at most* four dimensions. Those for which the symmetry group is strictly 4-dimensional have *all* the relative invariants t_1, k_0, k_1 equal to *zero* and *constant* invariant T_0. When finding such structures it is enough to consider $T_0 = \alpha = \text{const} \geq 0$ since, due to the fact that T_0 is defined modulo sign ($e^{i\epsilon\pi} = \pm 1$), each structure with $T_0 = \alpha < 0$ is equivalent to the one with $T_0 = |\alpha|$. Inspecting all the possibilities we get the following theorem.

Theorem 8.3. *All locally nonequivalent structures* $(M, [\lambda, \mu])$ *of oriented congruences having vanishing twist, nonvanishing shear, and possessing a strictly 4-dimensional transitive group of symmetries are parametrized by a real constant* $\alpha \geq 0$ *as follows.*

- *if* $0 \leq \alpha < 1$ *they can be locally represented by*

$$\lambda = \mathrm{d}u, \qquad \mu = \mathrm{d}x + e^{2u\sqrt{1-\alpha^2}}(\alpha + i\sqrt{1-\alpha^2})\mathrm{d}y$$

- *if* $\alpha = 1$ *they can be locally represented by*

$$\lambda = \mathrm{d}u, \qquad \mu = \mathrm{d}x + (i + 2u)\mathrm{d}y$$

- *if* $\alpha > 1$ *they can be locally represented by*

$$\begin{aligned} &\lambda = \mathrm{d}u, \\ &\mu = [(i+\alpha)\cos(u\sqrt{\alpha^2-1}) - i\sqrt{\alpha^2-1}\sin(u\sqrt{\alpha^2-1})]\mathrm{d}x \\ &\quad + [(i+\alpha)\sin(u\sqrt{\alpha^2-1}) + i\sqrt{\alpha^2-1}\cos(u\sqrt{\alpha^2-1})]\mathrm{d}y. \end{aligned}$$

Here (u, x, y) *are coordinates on* M*. The real parameter* $\alpha \geq 0$ *is just the invariant* $T_0 = \alpha$ *and as such enumerates nonequivalent structures.*

8.2. Description in terms of the Cartan connection

Eq. (8.10) can be better understood in terms of the matrix $\tilde{\omega}$ of 1-forms defined by

$$\tilde{\omega} = \begin{pmatrix} 2(\Omega - \omega) & 0 & 0 \\ \omega_1 & \Omega - \omega & \omega \\ \bar{\omega}_1 & \omega & \Omega - \omega, \end{pmatrix}$$

where the 1-forms $(\omega_1, \bar{\omega}_1, \omega, \Omega)$ are as in (8.10) or as in (8.9).

This matrix has values in the 4-dimensional Lie algebra $\mathfrak{g}_4$ which is a semidirect product of two 2-dimensional Abelian Lie algebras

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} 2x & 0 & 0 \\ 0 & x & y \\ 0 & y & x \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$$

and

$$\mathfrak{h}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ u + iv & 0 & 0 \\ u - iv & 0 & 0 \end{pmatrix} \mid u, v \in \mathbb{R} \right\},$$

for which the commutator is the usual commutator of 3×3 matrices. Thus

$$\mathfrak{g}_4 = \mathfrak{h}_0 \oplus \mathfrak{h}_1, \tag{8.13}$$

as the direct sum of vector spaces $\mathfrak{h}_0$ and $\mathfrak{h}_1$, with the commutator between $\mathfrak{h}_0$ and $\mathfrak{h}_1$ given by

$$[\mathfrak{h}_0, \mathfrak{h}_1] \subset \mathfrak{h}_1.$$

It turns out that due to the relations (8.10), $\tilde{\omega}$ is a Cartan connection on the principal fibre bundle $\mathbb{R}_+ \to P \to M$, which has as its structure group a 1-parameter Lie group generated by the vector field $\rho\partial_\rho$ dual to Ω.

Remark 8.4. It is worthwhile to note that the fiber bundle $\mathbb{R}_+ \to P \to M$ has some additional structure. Indeed, Eq. (8.10) guarantee that P is foliated by 2-dimensional leaves of the *integrable* 2-dimensional real distribution $\mathcal{D}$ annihilating forms ω_1 and $\bar{\omega}_1$. Thus, locally, P has also the structure of a fiber bundle over the leaf space $P/\mathcal{D}$. This is actually a *principal* fiber bundle $H_0 \to P \to P/\mathcal{D}$, with the structure group H_0 having $\mathfrak{h}_0$ as its Lie algebra.

Eqs. (8.10) imply that the curvature R of the Cartan connection $\tilde{\omega}$ is

$$R = \mathrm{d}\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} = \begin{pmatrix} 2R_1 & 0 & 0 \\ R_3 & R_1 & R_2 \\ \bar{R}_3 & R_2 & R_1 \end{pmatrix},$$

where

$$R_1 = \mathrm{i}K_0\omega_1 \wedge \bar{\omega}_1 + (K_1 - T_1)\omega_1 \wedge \omega + (\bar{K}_1 - \bar{T}_1)\bar{\omega}_1 \wedge \omega$$
$$R_2 = T_1\omega_1 \wedge \omega + \bar{T}_1\bar{\omega}_1 \wedge \omega$$
$$R_3 = \mathrm{i}T_0\omega_1 \wedge \omega.$$

In particular the absence of vertical $\Omega\wedge$ terms in the curvature confirms our interpretation of $\tilde{\omega}$ as a $\mathfrak{g}_4$-valued Cartan connection on P over M.

The Cartan connection $\tilde{\omega}$ yields all the invariant information about the corresponding structures $(M, [\lambda, \mu])$ and can be used in an invariant description of various examples of such structures. In particular, the invariant decomposition (8.13) may be used to distinguish two large classes $(M, [\lambda, \mu])_0$ and $(M, [\lambda, \mu])_1$ of nonequivalent structures $(M, [\lambda, \mu])$. These are defined by the requirement that the curvature R of their Cartan connection $\tilde{\omega}$ has values in the respective parts $\mathfrak{h}_0$ for $(M, [\lambda, \mu])_0$, and $\mathfrak{h}_1$ for $(M, [\lambda, \mu])_1$.

8.2.1. Curvature $R \in \mathfrak{h}_0$

The curvature R of the Cartan connection $\tilde{\omega}$ resides in $\mathfrak{h}_0$ iff it is of the form

$$R = \begin{pmatrix} 2R_1 & 0 & 0 \\ 0 & R_1 & R_2 \\ 0 & R_2 & R_1 \end{pmatrix}.$$

An example of a structure $(M, [\lambda, \mu])$ with such R is given by the following forms $(\omega_1, \bar{\omega}_1, \omega, \Omega)$:

$$\omega_1 = \mathrm{e}^r(\mathrm{d}x + \mathrm{i}\mathrm{e}^{2(u+f)}\mathrm{d}y),$$
$$\bar{\omega}_1 = \mathrm{e}^r(\mathrm{d}x - \mathrm{i}\mathrm{e}^{2(u+f)}\mathrm{d}y),$$
$$\omega = \mathrm{d}u,$$
$$\Omega = \mathrm{d}r + 2\mathrm{d}u + 2f_x\mathrm{d}x,$$

with a real function $f = f(x, y)$ of real variables x and y. These two variables, supplemented with the real u and r, constitute a coordinate system (u, x, y, r) on $R_+ \to P \to M$. The triple (u, x, y) parametrizes M, and r is related to the positive fiber coordinate ρ via $\rho = \mathrm{e}^r$.

For each choice of a twice differentiable function $f = f(x, y)$ the forms $(\omega_1, \bar{\omega}_1, \omega, \Omega)$ satisfy the differential system (8.10) with

$$K_1 \equiv 0, \qquad T_1 \equiv 0, \qquad T_0 \equiv 0,$$

and the relative invariant K_0 being

$$K_0 = -\mathrm{e}^{-2(r+u+f)}f_{xy}.$$

A special case here is $f_{xy} \equiv 0$, in particular $f \equiv 0$. If this happens the corresponding structures $(M, [\lambda, \mu])$ are all equivalent to the structure with 4-dimensional transitive symmetry group having $\alpha = 0$ in Theorem 8.3. If $f_{xy} \neq 0$, then $K_0 \neq 0$, and the corresponding structures have the curvature of the Cartan connection $\tilde{\omega}$ in the form

$$R = -\mathrm{e}^{-2(r+u+f)} \begin{pmatrix} 2\mathrm{i}\omega_1 \wedge \bar{\omega}_1 & 0 & 0 \\ 0 & \mathrm{i}\omega_1 \wedge \bar{\omega}_1 & 0 \\ 0 & 0 & \mathrm{i}\omega_1 \wedge \bar{\omega}_1 \end{pmatrix} f_{xy}.$$

As such they are *special cases* of structures with $R \in \mathfrak{h}_0$. We will return to them in Section 8.3.1, where we further analyze the case $K_0 \neq 0$, $T_1 = 0$ and $K_1 = 0$.

8.2.2. Curvature $R \in \mathfrak{h}_1$

The case of $R \in \mathfrak{h}_1$ is entirely characterized by the requirement that all the relative invariants t_1, k_0, k_1 identically vanish. Examples of such structures are structures with a 4-dimensional transitive group of symmetries given in Theorem 8.3. However these examples do not exhaust the list of nonequivalent structures having $R \in \mathfrak{h}_1$. To find them *all* we proceed as follows.

We want to find all structures with

$$R = \begin{pmatrix} 0 & 0 & 0 \\ R_3 & 0 & 0 \\ \bar{R}_3 & 0 & 0 \end{pmatrix},$$

i.e. those for which *all* the relative invariants T_1, K_1, K_0, as in (8.10), vanish:

$$T_1 \equiv 0, \qquad K_0 \equiv 0, \qquad K_1 \equiv 0. \tag{8.14}$$

Assuming (8.14), Eqs. (8.10) guarantee that real coordinates u and r may be introduced on P such that

$$\omega = \mathrm{d}u, \qquad \Omega = \mathrm{d}r.$$

Then, taking the exterior derivatives of both sides of Eqs. (8.10), we see that (8.14) forces T_0 to be a real function of u only. Denoting this function by $\alpha = \alpha(u)$ we have

$$T_0 = \alpha(u).$$

Integrating the system for such T_0, and denoting the u-derivatives by primes, we get the following theorem.

Theorem 8.5. *A structure $(M, [\lambda, \mu])$ of an oriented congruence with vanishing twist, $a \equiv 0$, nonvanishing shear, $s \neq 0$, and having the curvature of its corresponding Cartan connection $\tilde{\omega}$ of the pure $\mathfrak{h}_1$ type, $R \in \mathfrak{h}_1$, can be locally represented by*

$$\lambda = \mathrm{d}u, \qquad \mu = \mathrm{d}z - \left(\frac{\bar{h}'}{h} + \frac{\bar{h}}{h} - \mathrm{i}\alpha\frac{\bar{h}}{h}\right)\mathrm{d}\bar{z},$$

where the complex function $h = h(u) \neq 0$ satisfies a second order ODE:

$$h'' + 2h' + (\alpha^2 + \mathrm{i}\alpha')h = 0. \tag{8.15}$$

Here the nonequivalent structures are distinguished by the real invariant $T_0 = \alpha(u)$.

Note that if $\alpha(u) = \mathrm{const}$ we recover the structures from Theorem 8.3.

8.3. The case $T_1 \equiv 0$

Now we pass to the general case $T_1 \equiv 0$. To proceed we have to distinguish two subcases:

- $K_1 \equiv 0$
- $K_1 \neq 0$.

8.3.1. The case $K_1 \equiv 0$

In this situation we have

$$\mathrm{d}\Omega = \mathrm{i}K_0\omega_1 \wedge \bar{\omega}_1,$$

with K_0 given by (8.11) and (8.12). Since K_0 is not identically equal to zero, because this corresponds to the case $t_1 \equiv 0$, $k_0 \equiv 0$, $k_1 \equiv 0$ already studied, we use it to fix ρ by the requirement

$$K_0 = \mathrm{sign}(k_0) = \pm 1. \tag{8.16}$$

We note that this sign is an invariant of the structures under consideration. This implies that the structures with different signs are nonequivalent.

After the normalization (8.16) the forms $(\omega_1, \bar{\omega}_1, \omega, \Omega)$ are defined as forms on M. Performing the standard Cartan analysis on the system (8.10), we verified that after pullback to M it reads:

$$\begin{aligned} &\mathrm{d}\omega = 0, \\ &\mathrm{d}\omega_1 = (\mathrm{i}B - A)\omega_1 \wedge \bar{\omega}_1 + \mathrm{i}T_0\omega_1 \wedge \omega + \bar{\omega}_1 \wedge \omega, \\ &\mathrm{d}\bar{\omega}_1 = (\mathrm{i}B + A)\omega_1 \wedge \bar{\omega}_1 + \omega_1 \wedge \omega - \mathrm{i}T_0\bar{\omega}_1 \wedge \omega, \\ &\mathrm{d}[(A + \mathrm{i}B)\omega_1 + (A - \mathrm{i}B)\bar{\omega}_1 + \omega] = \pm\mathrm{i}\omega_1 \wedge \bar{\omega}_1. \end{aligned} \tag{8.17}$$

Here the real functions A, B, T_0 are the scalar invariants on M. They satisfy the following integrability conditions

$$\begin{aligned} dA &= \left[A_1 + \frac{i}{2}(B_1 + \bar{B}_1 \pm 1)\right]\omega_1 + \left[A_1 - \frac{i}{2}(B_1 + \bar{B}_1 \pm 1)\right]\bar{\omega}_1 + (A - BT_0)\omega \\ dB &= B_1\omega_1 + \bar{B}_1\bar{\omega}_1 + (AT_0 - B)\omega \\ dT_0 &\wedge \omega = 0, \end{aligned} \tag{8.18}$$

with the functions A_1 (real) and B_1 (complex) being the scalar invariants of the next higher order. In principle, we could have written the explicit formulae for all these scalar invariants in terms of the defining variables b, q, p and s of (8.4). We refrain from doing this, because the formulae are quite complicated, and not enlightening.

We summarize these considerations in the following theorem.

Theorem 8.6. *All locally nonequivalent structures $(M, [\lambda, \mu])$ of oriented congruences having vanishing twist, nonvanishing shear, with $T_1 \equiv 0$ and $K_1 \equiv 0$, are described by the invariant forms $(\omega, \omega_1, \bar{\omega}_1)$ satisfying the system (8.17) and (8.18) on M.*

Thus having a representative (λ, μ) of a structure with vanishing twist, nonvanishing shear and with $T_1 \equiv 0$, we can always gauge it to the invariant forms satisfying system (8.17) and (8.18). Conversely, given two 1-forms ω and ω_1 satisfying the system (8.17) and (8.18), we may consider them as a representative pair $(\lambda = \omega, \mu = \omega_1)$ of a certain structure with vanishing twist, nonvanishing shear and with $T_1 \equiv 0$.

The immediate consequence of the integrability conditions (8.18) is the *nonexistence* of structures (8.17) with a strictly 3-dimensional transitive group of symmetries. This is because, if such structures existed, they would have *constant* invariants A, B and T_0. Thus, for such structures the right hand sides of all the equations (8.18) would be zero. But this is impossible, since in such a situation the second equation (8.18) implies $B_1 \equiv 0$ which, when compared with equating to zero the r.h.s of the first equation (8.18), gives contradiction.

A family of nonequivalent structures $(M, [\lambda, \mu])$ from this branch of the classification is given in Section 8.2.1. Indeed, consider the examples of this section for which

$$f_{xy} \neq 0.$$

Since this guarantees that $K_1 \neq 0$, and since we have $T_1 = 0$ and $K_1 = 0$ (and, what is less important for us here $T_0 = 0$) for them, we may perform the above described normalization procedure on the invariant forms $(\omega_1, \bar{\omega}_1, \omega, \Omega)$ defined in Section 8.2.1. A simple calculation, based on the normalization

$$-e^{-2(r+u+f)}f_{xy} = \pm 1, \tag{8.19}$$

leads to the reduction to M, where the invariant forms read:

$$\begin{aligned} \omega &= du, \\ \omega_1 &= e^{-(u+f)}\left(\mp f_{xy}\right)^{\frac{1}{2}}(dx + ie^{2(u+f)}dy), \\ \bar{\omega}_1 &= e^{-(u+f)}\left(\mp f_{xy}\right)^{\frac{1}{2}}(dx - ie^{2(u+f)}dy). \end{aligned}$$

They satisfy the system (8.17) and (8.18) with the functions A and B given by:

$$\begin{aligned} A &= \frac{1}{4}\left(\mp f_{xy}\right)^{-\frac{3}{2}}\left(2f_x f_{xy} + f_{xxy}\right)e^{u+f} \\ B &= \frac{1}{4}\left(\mp f_{xy}\right)^{-\frac{3}{2}}\left(2f_y f_{xy} - f_{xyy}\right)e^{-u-f}. \end{aligned}$$

These structures can thus be represented on M by

$$\lambda = du, \qquad \mu = dx + ie^{2(u+f(x,y))}dy.$$

The only scalar invariants for them are the functions A and B as above since, as we already noticed, the scalar invariant T_0 identically vanishes, $T_0 \equiv 0$.

Note in particular that, given a function $f = f(x, y)$, two structures $(M, [\lambda, \mu])$ with λ, μ as above, corresponding to two different signs of f_{xy} are nonequivalent. This is because the sign $\pm$ in (8.19) is an invariant of such structures.

Remark 8.7. The structures described above belong to a subclass of structures for which the curvature R is much more restricted than to $\mathfrak{h}_0$. Since, in addition to $T_0 \equiv 0$, we have here $T_1 \equiv 0$, the curvature R is actually contained in the diagonal 1-dimensional subalgebra of $\mathfrak{h}_0$. Moreover, since also $K_1 \equiv 0$, the curvature R does not involve $\omega\wedge$ terms. This means that in this example, similarly as in all examples with $T_0 \equiv T_1 \equiv K_1 \equiv 0$, the curvature of the Cartan connection $\tilde{\omega}$ is *horizontal from the point of view of the principal fiber bundle $H_0 \to P \to P/\mathcal{D}$* discussed in Remark 8.4. Thus here, the Cartan connection $\tilde{\omega}$ can be reinterpreted as a $\mathfrak{g}_4$-valued Cartan connection on the bundle $H_0 \to P \to P/\mathcal{D}$.

8.3.2. The case $K_1 \neq 0$

If $K_1 \neq 0$ we can use definition (8.11) to scale it in such a way that it has values on the unit circle

$$K_1 = e^{i\gamma}.$$

This fixes ρ uniquely, and the system (8.10) is again reduced to an invariant system on M. This reads (with new A and B):

$$\begin{aligned}
&d\omega = 0,\\
&d\omega_1 = (iB - A)\omega_1 \wedge \bar\omega_1 + (1 - C + iT_0)\omega_1 \wedge \omega + \bar\omega_1 \wedge \omega,\\
&d\bar\omega_1 = (iB + A)\omega_1 \wedge \bar\omega_1 + \omega_1 \wedge \omega + (1 - C - iT_0)\bar\omega_1 \wedge \omega,\\
&d[(A + iB)\omega_1 + (A - iB)\bar\omega_1 + C\omega] = iK_0\omega_1 \wedge \bar\omega_1 + e^{i\gamma}\omega_1 \wedge \omega + e^{-i\gamma}\bar\omega_1 \wedge \omega.
\end{aligned} \tag{8.20}$$

Here, all the real invariants are T_0, A, B, C, γ and K_0 are well defined functions on M. They are expressible in terms of the original variables defining the structure and the functions k_0, k_1 of (8.12). In particular,

$$K_0 = 2\frac{k_0}{|k_1|^2}.$$

To discuss the integrability conditions for the system (8.20) we have to distinguish two cases:

- either $K_1 = e^{i\gamma} \neq \pm 1$,
- or $K_1 = e^{i\gamma} \equiv \pm 1$.

In the first case:

$$\begin{aligned}
&dT_0 = i(e^{i\gamma}\omega_1 - e^{-i\gamma}\bar\omega_2) + T_{00}\omega\\
&dA = \frac{1}{2}\left[i\left(\frac{K_0}{2} + A_1\right) + A_2\right]\omega_1 + \frac{1}{2}\left[-i\left(\frac{K_0}{2} + A_1\right) + A_2\right]\bar\omega_1 + A_0\omega\\
&dB = \frac{1}{2}\left[-\frac{K_0}{2} + A_1 + iB_1\right]\omega_1 + \frac{1}{2}\left[-\frac{K_0}{2} + A_1 - iB_1\right]\bar\omega_1 + B_0\omega\\
&dC \ = \ [-2A + AC + A_0 + BT_0 + i(BC - AT_0 + B_0) + e^{i\gamma}]\omega_1\\
&\qquad + [-2A + AC + A_0 + BT_0 - i(BC - AT_0 + B_0) + e^{-i\gamma}]\bar\omega_1 + C_0\omega\\
&d\gamma = [B + (A + \gamma_1)\cot\gamma + i\gamma_1]\omega_1 + [B + (A + \gamma_1)\cot\gamma - i\gamma_1]\bar\omega_1 + \gamma_0\omega\\
&dK_0 = K_{01}\omega_1 + \bar K_{01}\bar\omega_1 + 2[(A + \gamma_1)\csc\gamma + (1 - C)K_0]\omega,
\end{aligned} \tag{8.21}$$

and in addition to the the basic scalar invariants K_0, γ, A, B, C, we have higher order scalar invariants $A_0, A_1, A_2, B_0, B_1, C_0, \gamma_0, \gamma_1$ (all real) and K_{01} (complex).

In the second case, when $e^{i\gamma} \equiv \pm 1$, one of the integrability conditions is the vanishing of the scalar invariant A of (8.20),

$$A \equiv 0.$$

The rest of the integrability conditions are

$$\begin{aligned}
&dT_0 = \pm i(\omega_1 - \bar\omega_2) + T_{00}\omega\\
&dB = \left[-\frac{K_0}{2} + iB_1\right]\omega_1 + \left[-\frac{K_0}{2} - iB_1\right]\bar\omega_1 + B_0\omega\\
&dC = [BT_0 + i(BC + B_0) \pm 1]\omega_1 + [BT_0 - i(BC + B_0) \pm 1]\bar\omega_1 + C_0\omega\\
&dK_0 = K_{01}\omega_1 + \bar K_{01}\bar\omega_1 + 2[\mp B + (1 - C)K_0]\omega,
\end{aligned} \tag{8.22}$$

with the new higher order scalar invariants B_0, B_1, C_0 (all real) and K_{01} (complex).

Theorem 8.8. *All locally nonequivalent structures $(M, [\lambda, \mu])$ of oriented congruences, having vanishing twist, nonvanishing shear, with $T_1 \equiv 0$ and $K_1 \neq 0$, are described by the invariant forms $(\omega, \omega_1, \bar\omega_1)$ satisfying*

- *either the system (8.20) and (8.21) on M, in which case $K_1 = e^{i\gamma} \neq \pm 1$,*
- *or the system (8.20) and (8.22) on M, in which case $K_1 \equiv \pm 1$ and $A \equiv 0$.*

As it is readily seen from the integrability conditions (8.21) and (8.22) neither of these cases admits structures with a strictly 3-dimensional transitive symmetry group (look at the equations for dT_0 in (8.21) and (8.22), and observe that $T_0 = \text{const}$, which implies $dT_0 = 0$, is forbidden!).

8.4. The case $T_1 \neq 0$

To analyze this case we again start with the basic system (8.10) and we assume that $t_1 \neq 0$. This assumption enables us to normalize T_1 so that its modulus is equal to one. Thus now we require

$$|T_1| = 1,$$

which uniquely fixes ρ to be

$$\rho = |t_1|.$$

After such normalization all the forms become forms on M and, depending on the location of T_1 on the unit circle, we have to consider two cases:

- either $T_1 = e^{i\delta} \neq \pm 1$,
- or $T_1 = \pm 1$.

We analyze the $T_1 \neq \pm 1$ case first. Here we easily reduce the system (8.10) to the following system on M:

$$\begin{aligned} d\omega &= (e^{i\delta}\omega_1 + e^{-i\delta}\bar{\omega}_1) \wedge \omega, \\ d\omega_1 &= (iB - A)\omega_1 \wedge \bar{\omega}_1 + (1 - C + iT_0)\omega_1 \wedge \omega + \bar{\omega}_1 \wedge \omega, \\ d\bar{\omega}_1 &= (iB + A)\omega_1 \wedge \bar{\omega}_1 + \omega_1 \wedge \omega + (1 - C - iT_0)\bar{\omega}_1 \wedge \omega. \end{aligned} \tag{8.23}$$

It has the following integrability conditions:

$$\begin{aligned} d\delta &= [\delta_1 + i((B - \delta_1)\cot\delta - A)]\omega_1 + [\delta_1 - i((B - \delta_1)\cot\delta - A)]\bar{\omega}_1 + \delta_0\omega \\ dT_0 \wedge \omega \ &= \ \{[B_0 + BC - AT_0 + 2\sin\delta + i(2A - AC - BT_0 - A_0 + C_1) - e^{i\beta}(T_0 - iC)]\omega_1 \\ &\quad + [B_0 + BC - AT_0 + 2\sin\delta - i(2A - AC - BT_0 - A_0 + \bar{C}_1) - e^{-i\beta}(T_0 + iC)]\bar{\omega}_1\} \wedge \omega. \end{aligned} \tag{8.24}$$

Here, the new scalar invariants are: T_0, δ, A, B, C (real), and the higher order scalar invariants are: δ_0, δ_1, B_0 (real) and C_1 (complex).

In the $T_1 \equiv \pm 1$ case the Eqs. (8.23) are still valid, provided that we put

$$B \equiv 0.$$

This condition is implied by $T_1 \equiv \pm 1$. Thus in this case the invariant forms satisfy

$$\begin{aligned} d\omega &= \pm(\omega_1 + \bar{\omega}_1) \wedge \omega, \\ d\omega_1 &= -A\omega_1 \wedge \bar{\omega}_1 + (1 - C + iT_0)\omega_1 \wedge \omega + \bar{\omega}_1 \wedge \omega, \\ d\bar{\omega}_1 &= A\omega_1 \wedge \bar{\omega}_1 + \omega_1 \wedge \omega + (1 - C - iT_0)\bar{\omega}_1 \wedge \omega. \end{aligned} \tag{8.25}$$

The integrability conditions for this system are:

$$dT_0 = T_{00}\omega + ((\mp 1 - A)T_0 + i(2A - AC - A_0 + C_1 \pm C))\omega_1 + ((\mp 1 - A)T_0 - i(2A - AC - A_0 + \bar{C}_1 \pm C))\bar{\omega}_1, \tag{8.26}$$

with the invariant sign equal to ± 1, the new scalar invariants being: T_0, A, C (real), and the higher order scalar invariants being: B_0, T_{00} (real) and C_1 (complex).

We summarize with the following theorem.

Theorem 8.9. *All locally nonequivalent structures $(M, [\lambda, \mu])$ of oriented congruences having vanishing twist, nonvanishing shear, with $T_1 \neq 0$, are described by the invariant forms $(\omega, \omega_1, \bar{\omega}_1)$ satisfying*

- *either the system (8.23) and (8.24) on M, in which case $T_1 = e^{i\delta} \neq \pm 1$,*
- *or the system (8.25) and (8.26) on M, in which case $T_1 \equiv \pm 1$.*

We pass to the determination of the structures with strictly 3-dimensional transitive group of symmetries.

Using the system (8.23) and (8.24) we easily establish that in the case $T_1 \neq \pm 1$ the structures are governed by the following system of invariant forms:

$$\begin{aligned} d\omega &= (e^{i\delta}\omega_1 + e^{-i\delta}\bar{\omega}_1) \wedge \omega, \\ d\omega_1 &= -\frac{1 - C - \cos 2\delta}{1 - C + \cos 2\delta} e^{-i\delta}\omega_1 \wedge \bar{\omega}_1 + (1 - C + i\sin 2\delta)\omega_1 \wedge \omega + \bar{\omega}_1 \wedge \omega, \\ d\bar{\omega}_1 &= \frac{1 - C - \cos 2\delta}{1 - C + \cos 2\delta} e^{i\delta}\omega_1 \wedge \bar{\omega}_1 + \omega_1 \wedge \omega + (1 - C - i\sin 2\delta)\bar{\omega}_1 \wedge \omega. \end{aligned} \tag{8.27}$$

In a similar way, if $T_1 \equiv \pm 1$, using the system (8.25) and (8.26), we see that the structures with 3-dimensional symmetry groups are governed by the following system:

$$\begin{aligned} d\omega &= \pm(\omega_1 + \bar{\omega}_1) \wedge \omega, \\ d\omega_1 &= \pm\omega_1 \wedge \bar{\omega}_1 + iT_0\omega_1 \wedge \omega + \bar{\omega}_1 \wedge \omega, \\ d\bar{\omega}_1 &= \mp\omega_1 \wedge \bar{\omega}_1 + \omega_1 \wedge \omega - iT_0\bar{\omega}_1 \wedge \omega. \end{aligned} \tag{8.28}$$

9. Nonvanishing twist and nonvanishing shear

The Cartan procedure applied to this case is very similar to the one in Section 8 concerned with $a \equiv 0$ and $s \neq 0$. There, before the final reduction to three dimensions, the procedure stopped at the intermediate 4-dimensional manifold $M \times \mathbb{R}_+$ parametrized by the points of M and the positive coordinate ρ. In the present case, in addition to $s \neq 0$, we also have $a \neq 0$, which enables us to make an immediate reduction to three dimensions and thus to produce invariants on M. Explicitly this reduction is achieved as follows.

We start with the general system (5.1) of Section 5. We have

$$a \neq 0, \qquad s \neq 0$$

and we again write the complex shear function s as

$$s = |s|e^{i\psi}.$$

Now, for a chosen pair (λ, μ) representing the structure, we impose the conditions

$$d\omega \wedge \omega = i\omega_1 \wedge \bar{\omega}_1 \wedge \omega \qquad (9.1)$$

$$d\omega_1 \wedge \omega_1 = \omega_1 \wedge \bar{\omega}_1 \wedge \omega \qquad (9.2)$$

on the Cartan frame

$$\omega = f\lambda, \qquad \omega_1 = \rho e^{i\phi}\mu, \qquad \bar{\omega}_1 = \rho e^{-i\phi}\bar{\mu}.$$

Note that (9.1) is possible because of $a \neq 0$ and (9.2) is possible because of $s \neq 0$. It is a matter of straightforward calculation to show that these two conditions uniquely specify the choice of f, ρ and ϕ. To write the relevant formulae for f, ρ and ϕ we denote the sign of a by $e^{i\epsilon\pi}$, where $\epsilon = 0$ or 1. Then having $e^{i\epsilon\pi} = \text{sign}(a)$, these formualae are:

$$f = e^{i\epsilon\pi}|s|, \qquad \rho e^{i\phi} = \sqrt{|a|}\sqrt{|s|}e^{-\frac{i}{2}(\psi-\epsilon\pi)}$$

and the forms $(\omega, \omega_1, \bar{\omega}_1)$ satisfy

$$\begin{aligned} d\omega &= i\omega_1 \wedge \bar{\omega}_1 + k_1\omega_1 \wedge \omega + \bar{k}_1\bar{\omega}_1 \wedge \omega \\ d\omega_1 &= k_2\omega_1 \wedge \bar{\omega}_1 + k_3\omega_1 \wedge \omega + \bar{\omega}_1 \wedge \omega \\ d\bar{\omega}_1 &= -\bar{k}_2\omega_1 \wedge \bar{\omega}_1 + \omega_1 \wedge \omega + \bar{k}_3\bar{\omega}_1 \wedge \omega. \end{aligned} \qquad (9.3)$$

Here the complex functions k_1, k_2, k_3 are defined on M and:

$$\begin{aligned} k_1 &= \frac{(b|s| + |s|_\mu)}{\sqrt{|a|}\sqrt{|s|^3}}e^{\frac{i}{2}(\psi-\epsilon\pi)} \\ k_2 &= \frac{-(\log|a|)_{\bar{\mu}} + 2p - (\log|s|)_{\bar{\mu}} + i\psi_{\bar{\mu}}}{2\sqrt{|a|}\sqrt{|s|}}e^{-\frac{i}{2}(\psi-\epsilon\pi)} \\ k_3 &= \frac{ib_{\bar{\mu}} - i\bar{b}_\mu - ibp + i\bar{b}\bar{p} + e^{-i\epsilon\pi}|a|(q - \bar{q} - (\log|s|)_\lambda + i\psi_\lambda)}{2|a||s|}. \end{aligned}$$

These functions constitute the full system of invariants of $(M, [\lambda, \mu])$ for $a \neq 0, s \neq 0$.

Theorem 9.1. *A given structure $(M, [\lambda, \mu])$ of an oriented congruence with nonvanishing twist, $a \neq 0$, and nonvanishing shear, $s \neq 0$, uniquely defines the frame of invariant 1-forms $\omega, \omega_1, \bar{\omega}_1$ and invariant complex functions k_1, k_2, k_3 on M. The forms and the functions are determined by the requirement that they satisfy the system (9.3). Starting with an arbitrary representative (λ, μ) of the structure $[\lambda, \mu]$, the forms are given by*

$$\omega = e^{i\epsilon\pi}|s|\lambda, \qquad \omega_1 = \sqrt{|a|}\sqrt{|s|}e^{-\frac{i}{2}(\psi-\epsilon\pi)}\mu, \qquad \bar{\omega}_1 = \sqrt{|a|}\sqrt{|s|}e^{\frac{i}{2}(\psi-\epsilon\pi)}\bar{\mu},$$

where the shear function is $s = |s|e^{i\psi}$. Here $e^{i\epsilon\pi}$, $\epsilon = 0, 1$, denotes the sign of the twist function a. The system (9.3) encodes all the invariant information of the structure $(M, [\lambda, \mu])$.

We pass to the determination of all homogeneous examples with $a \neq 0, s \neq 0$. Now the maximal dimension of a group of transitive symmetries is three. The structures with 3-dimensional groups of symmetries correspond to those satisfying system (9.3) with all the functions k_1, k_2, k_3 being constants. Applying the exterior differential to the system (9.3) with k_1, k_2, k_3 constants we arrive at the following theorem.

Theorem 9.2. *All homogeneous structures $(M, [\lambda, \mu])$ with nonvanishing twist, $a \neq 0$, and nonvanishing shear, $s \neq 0$, have a strictly 3-dimensional symmetry group and fall into four main types characterized by:*

I: $k_3 = 1$. *In this case there is a 2-real parameter family of nonequivalent structures distinguished by real constants x and y related to the invariants k_1 and k_2 via:*

$$k_1 = x, \qquad k_2 = \mathrm{i}y.$$

II: $k_3 = \mathrm{e}^{\mathrm{i}\phi}$, $0 < \phi < 2\pi$. *In this case there is a 2-real parameter family of nonequivalent structures distinguished by real constants x, y which together with the parameter ϕ are constrained by the equation*

$$\cos\phi(1 - 2xy + \cos\phi) = 0.$$

The invariants k_1, k_2, k_3 are then given by

$$k_1 = x\left(\cot\frac{\phi}{2} + \mathrm{i}\right), \qquad k_2 = -\mathrm{i}y\left(\cot\frac{\phi}{2} + \mathrm{i}\right), \qquad k_3 = \cos\phi + \mathrm{i}\sin\phi.$$

III: $k_3 + \bar{k}_3 = 0$, $k_3 \neq \pm\mathrm{i}$. *In this case there is a 3-real parameter family of nonequivalent structures distinguished by real constants $y' \neq \pm 1$, x, y related to the invariants k_1, k_2, k_3 via:*

$$k_1 = x + \mathrm{i}y, \qquad k_2 = \bar{k}_1 = x - \mathrm{i}y, \qquad k_3 = \mathrm{i}y'.$$

IV: $|k_3| \neq 1$, $k_3 + \bar{k}_3 \neq 0$. *In this case there is a 3-real parameter family of nonequivalent structures distinguished by real constants $x' \neq 0$, y', x, y constrained by the equation*

$$x'^2 + y'^2 + 2y'(x^2 + y^2) - 4xy = 1.$$

The invariants k_1, k_2, k_3 are then given by

$$k_1 = x + \mathrm{i}y, \qquad k_3 = x' + \mathrm{i}y', \qquad k_2 = \frac{\bar{k}_1(1 + k_3^2) - k_1(k_3 + \bar{k}_3)}{1 - |k_3|^2}.$$

Among all the structures covered by the above theorem, the simplest have $k_1 = k_2 = k_3 \equiv 0$. This unique structure belongs to the case III above and is the *flat case* for the branch $a \neq 0$, $s \neq 0$. We describe it in the following proposition.

Proposition 9.3. *A structure of an oriented congruence $(M, [\lambda, \mu])$ with nonvanishing twist, $a \neq 0$, nonvanishing shear $s \neq 0$ and having $k_1 = k_2 = k_3 \equiv 0$, may be locally represented by forms*

$$\lambda = \mathrm{d}u + \frac{\sqrt{2}\mathrm{e}^{\mathrm{i}u} - \mathrm{i}\bar{z}}{z\bar{z} - 1}\mathrm{d}z + \frac{\sqrt{2}\mathrm{e}^{-\mathrm{i}u} + \mathrm{i}z}{z\bar{z} - 1}\mathrm{d}\bar{z}, \qquad \mu = \frac{2\mathrm{e}^{\mathrm{i}u}}{z\bar{z} - 1}\mathrm{d}z - \sqrt{2}\lambda, \tag{9.4}$$

where $(u, z, \bar{z})$ are coordinates on M. This structure has the local symmetry group of Bianchi type VIII, locally isomorphic to the group $\mathbf{SL}(2, \mathbb{R})$.

Remark 9.4. There are more structures with $a \neq 0$, $s \neq 0$, which have a 3-dimensional transitive symmetry group of Bianchi type VIII. It is quite complicated to write them all here. For example, among them, there is a 1-parameter family of nonequivalent structures with $k_1 = k_2 \equiv 0$. They may be represented by

$$\lambda = \mathrm{d}u + \frac{\kappa\mathrm{e}^{\mathrm{i}u} - \mathrm{i}\bar{z}}{z\bar{z} - 1}\mathrm{d}z + \frac{\kappa\mathrm{e}^{-\mathrm{i}u} + \mathrm{i}z}{z\bar{z} - 1}\mathrm{d}\bar{z}, \qquad \mu = (\kappa^2 - 1)\frac{2\mathrm{e}^{\mathrm{i}u}}{z\bar{z} - 1}\mathrm{d}z - \kappa\lambda, \tag{9.5}$$

where $\kappa > 0$, $\kappa \neq 1$. The only nonvanishing invariant for this 1-parameter family is $k_3 = -\mathrm{i}(1 - \frac{2}{\kappa^2})$. It may be considered as a deformation of the flat case above, which corresponds to $\kappa = \sqrt{2}$.

Remark 9.5. In a similar way, among all the structures with $a \neq 0$, $s \neq 0$, which have a 3-dimensional transitive symmetry group of Bianchi type IX, we may easily characterize those with $k_1 = k_2 \equiv 0$. They may be represented by

$$\lambda = \mathrm{d}u + \frac{\kappa\mathrm{e}^{\mathrm{i}u} - \mathrm{i}\bar{z}}{z\bar{z} + 1}\mathrm{d}z + \frac{\kappa\mathrm{e}^{-\mathrm{i}u} + \mathrm{i}z}{z\bar{z} + 1}\mathrm{d}\bar{z}, \qquad \mu = (\kappa^2 + 1)\frac{2\mathrm{e}^{\mathrm{i}u}}{z\bar{z} + 1}\mathrm{d}z - \kappa\lambda, \tag{9.6}$$

where $\kappa > 0$. Here the only nonvanishing invariant is $k_3 = -\mathrm{i}(1 + \frac{2}{\kappa^2})$.

Remark 9.6. It is interesting to remark which of the structures (9.5) and (9.6) correspond to the flat CR-structure in the sense of Cartan. According to [15], they correspond to $\kappa = 0, \sqrt{2}$ in the (9.5) case, and $\kappa = 0$ in the (9.6) case. Thus in these cases the corresponding structures of an oriented congruence are locally CR-equivalent to the hyperquadric CR structure of Example 7.3, with a nonstandard splitting, which causes the shear $s \neq 0$.

It is a rather complicated matter to describe which Bianchi types having a 3-dimensional transitive symmetry group correspond to a given homogeneous structure with $a \neq 0$, $s \neq 0$. We remark that the groups of Bianchi types I and V *are excluded* for such structures. We also fully describe the situation for Bianchi types II and IV. This is summarized in the following theorem.

Theorem 9.7. *There are only two nonequivalent structures of an oriented congruence $(M, [\lambda, \mu])$ with $a \neq 0$, $s \neq 0$, which have a local transitive symmetry group of Bianchi type II. They may be locally represented by*

$$\lambda = \mathrm{d}u + \frac{\mathrm{i}}{2}(z\mathrm{d}\bar{z} - \bar{z}\mathrm{d}z), \qquad \mu = \mathrm{d}z \pm \sqrt{2}(1-\mathrm{i})\lambda,$$

where $(u, z, \bar{z})$ are coordinates on M. The constant invariants are

$$k_1 = \pm\frac{1-\mathrm{i}}{\sqrt{2}}, \qquad k_2 = \pm\frac{1+\mathrm{i}}{\sqrt{2}}, \qquad k_3 = -\mathrm{i},$$

and the sign ± 1 distinguishes between the nonequivalent structures.

There are also only two 2-parameter families of nonequivalent structures of an oriented congruence $(M, [\lambda, \mu])$ with $a \neq 0$, $s \neq 0$, which have a local transitive symmetry group of Bianchi type IV. They may be locally represented by

$$\lambda = y^{-1}(\mathrm{d}u - \log y\mathrm{d}x), \qquad \mu = y^{-1}\mathrm{d}(x + \mathrm{i}y) \pm \sqrt{2}(1-\mathrm{i})w\lambda,$$

where (u, x, y) are coordinates on M and $w = \mathrm{Re}(w) + \mathrm{i}\,\mathrm{Im}(w) \neq 0$ is a complex parameter. The constant invariants are

$$k_1 = \pm\frac{1-\mathrm{i}}{\sqrt{2}} + \frac{\mathrm{i}}{2\bar{w}}, \qquad k_2 = \pm\frac{1+\mathrm{i}}{\sqrt{2}} + \frac{\mathrm{i}}{2\bar{w}}, \qquad k_3 = -\mathrm{i} \pm \left(\frac{1+\mathrm{i}}{\bar{w}} + \frac{1-\mathrm{i}}{w}\right),$$

and the two real parameters $\mathrm{Re}(w)$ and $\mathrm{Im}(w)$, together with the sign ± 1 distinguish between the nonequivalent structures.

Remark 9.8. We remark that the structures with a symmetry group of Bianchi type II are in a sense the limiting case of the two families of structures with Bianchi type IV. They correspond to the limit $|w| \to \infty$.

10. Application 1: Lorentzian metrics in four dimensions

In this section we use our results about oriented congruence structures to construct Lorentzian metrics in 4-dimensions.

10.1. Vanishing twist—Nonvanishing shear case and pp-waves

Since our oriented congruence structures are 3-dimensional objects, we concentrate only on those structures, which in some *natural* manner define an associated 4-dimensional manifold. As we noted in the sections devoted to the Cartan analysis of the oriented congruence structures, in some cases, such as those described in Section 8, the Cartan bundle P encoding the basic invariants of the structures is 4-dimensional. So in this case, i.e. when the twist $a \equiv 0$ and the shear $s \neq 0$, we have a 4-dimensional manifold naturally associated with the oriented congruence structure. Moreover, in such case the Cartan procedure provides us also with a rigid coframe of invariant forms $(\omega_1, \bar{\omega}_1, \omega, \Omega)$ on P. Using these forms we may define

$$g = 2\omega_1\bar{\omega}_1 + 2\omega\Omega, \tag{10.1}$$

or, as suggested by the form of the associated Cartan connection,

$$g = 2\omega_1\bar{\omega}_1 + 2\omega(\Omega - \omega). \tag{10.2}$$

These both are well defined Lorentzian metrics on P, which are built only from the objects naturally and invariantly associated with the oriented congruence structure.

To be more specific, let us consider the structures with the curvature of the Cartan connection $R \in \mathfrak{h}_1$, as described in Theorem 8.5. In this case the bundle P is parametrized by $(z, \bar{z}, u, r)$ and the invariant forms are:

$$\Omega = \mathrm{d}r, \qquad \omega = \mathrm{d}u$$
$$\omega_1 = \mathrm{e}^r\left(h\mathrm{d}z - (\bar{h}' + \bar{h} - \mathrm{i}\alpha\bar{h})\mathrm{d}\bar{z}\right)$$
$$\bar{\omega}_1 = \mathrm{e}^r\left(\bar{h}\mathrm{d}\bar{z} - (h' + h + \mathrm{i}\alpha h)\mathrm{d}z\right),$$

with functions $\alpha = \alpha(u)$ (real) and $h = h(u)$ (complex) satisfying the ordinary differential equation (8.15). Inserting these forms in the formulae (10.1) and (10.2), we get the respective 4-dimensional Lorentzian metrics

$$g_0 = 2\mathrm{e}^{2r}\left(h\mathrm{d}z - (\bar{h}' + \bar{h} - \mathrm{i}\alpha\bar{h})\mathrm{d}\bar{z}\right)\left(\bar{h}\mathrm{d}\bar{z} - (h' + h + \mathrm{i}\alpha h)\mathrm{d}z\right) + 2\mathrm{d}u\mathrm{d}r,$$

and

$$g_{-1} = 2e^{2r}\left(h\mathrm{d}z - (\bar{h}' + \bar{h} - \mathrm{i}\alpha\bar{h})\mathrm{d}\bar{z}\right)\left(\bar{h}\mathrm{d}\bar{z} - (h' + h + \mathrm{i}\alpha h)\mathrm{d}z\right) + 2\mathrm{d}u(\mathrm{d}r - \mathrm{d}u).$$

It turns out that both these metrics have quite nice properties.

Actually, introducing a still bigger class of metrics

$$g_c = 2e^{2r}\left(h\mathrm{d}z - (\bar{h}' + \bar{h} - \mathrm{i}\alpha\bar{h})\mathrm{d}\bar{z}\right)\left(\bar{h}\mathrm{d}\bar{z} - (h' + h + \mathrm{i}\alpha h)\mathrm{d}z\right) + 2\mathrm{d}u(\mathrm{d}r - c\mathrm{d}u),$$

with $c = \mathrm{const} \in \mathbb{R}$, one checks that they all are of type N in the Petrov classification of 4-dimensional Lorentzian metrics. This means that their Weyl tensor is expressed in terms of only one nonvanishing complex function Ψ_4, called the Weyl spin coefficient, which reads

$$\Psi_4 = 2(\mathrm{i}\alpha - c - 1).$$

All the other Weyl coefficients $(\Psi_0, \Psi_1, \Psi_2, \Psi_3)$, which together with Ψ_4 totally encode the Weyl tensor of g_c, are identically zero.

Looking at the spin coefficient Ψ_4 we see that there is a distinguished metric in the class g_c. This corresponds to $c = -1$. In such case the Weyl tensor of g is just proportional to $\Psi_4 = 2\mathrm{i}\alpha$ and we have a Lorentz-geometric interpretation of the invariant $\alpha = \alpha(u)$ of the corresponding structure of the oriented congruence. Confronting these considerations with the results of Section 8.2.2 we get the following

Theorem 10.1. *Every structure of an oriented congruence (M, λ, μ) with vanishing twist, $a \equiv 0$, nonvanishing shear $s \neq 0$, and having the curvature R of its corresponding Cartan connection in $\mathfrak{h}_1$, defines a Lorentzian metric*

$$g_{-1} = 2\omega_1\bar{\omega}_1 + 2\omega(\Omega - \omega),$$

which is of Petrov type N or conformally flat. The nonequivalent metrics correspond to different structures of the oriented congruence, and the metric is conformally flat if and only if $R \equiv 0$.

Interestingly metrics g_{-1} are conformal to Ricci flat metrics. The Ricci flat metric in the conformal class of g_{-1} is given by

$$\hat{g}_{-1} = \frac{2e^{4u}}{(t + e^{2u})^2}\left(\left(h\mathrm{d}z - (\bar{h}' + \bar{h} - \mathrm{i}\alpha\bar{h})\mathrm{d}\bar{z}\right)\left(\bar{h}\mathrm{d}\bar{z} - (h' + h + \mathrm{i}\alpha h)\mathrm{d}z\right) + e^{-2r}\mathrm{d}u(\mathrm{d}r - \mathrm{d}u)\right),$$

where t is a real constant. For each $\alpha = \alpha(u)$ and for each solution $h = h(u)$ of (8.15), the corresponding Ricci flat metric is the so called *linearly polarized pp-wave* from General Relativity Theory (see [10], p. 385).

10.2. Nonvanishing twist—Vanishing shear case and the Bach metrics

Another example of 4-dimensional Lorentzian manifolds naturally associated with the structures of oriented congruences appears in the nonvanishing twist—vanishing shear case, as we explained in Section 7.3. Actually in Section 7.3 we defined *conformal* Lorentzian 4-manifolds equipped with the *conformal* class of Lorentzian metrics $[g_t]$, which are naturally associated with a congruence structure with twist and without shear. Here we study the conformal properties of these metrics.

10.2.1. The Cotton and Bach conditions for conformal metrics

We recall [4] that a Lorentzian metric g on a manifold M is called *conformal to Einstein* iff there exists a real function Υ on M such that the rescaled metric $\hat{g} = e^{2\Upsilon}g$ satisfies the Einstein equations $Ric(\hat{g}) = \Lambda\hat{g}$. In the case of an oriented M with $\dim M = 4$ there are two *necessary* conditions [2,8] for g to be conformal to Einstein (in algebraically generic cases [4] these necessary conditions are sufficient). To describe these conditions we denote by F the curvature 2-form of the Cartan normal conformal connection $\omega_{[g]}$ associated with a conformal class $[g]$ (see [7] for definitions). The curvature F is horizontal. Thus, choosing a representative g of the conformal class $[g]$, we can calculate its Hodge dual $*F$ and calculate the 6×6 matrix of 3-forms

$$D * F = \mathrm{d} * F + \omega_{[g]} \wedge *F - *F \wedge \omega_{[g]} \tag{10.3}$$

for the connection $\omega_{[g]}$. This matrix has a remarkably simple form

$$D * F = \begin{pmatrix} 0 & *j^\mu & 0 \\ 0 & 0 & *j_\mu \\ 0 & 0 & 0 \end{pmatrix},$$

where $*j^\mu$ is a vector-valued 3-form, the Hodge dual of the so called *Yang–Mills current* j^μ for the conformal connection $\omega_{[g]}$. Having said this we introduce the vacuum Yang–Mills equation for the conformal connection $\omega_{[g]}$

$$D * F = 0 \tag{10.4}$$

i.e. the condition that the Yang–Mills current j^μ vanishes. It turns out that in $\dim M = 4$ Eq. (10.4) are *conformally invariant*. They are equivalent to the requirement that the *Bach tensor* of g identically vanishes [2,4]. This condition is known [9] to constitute a first system of equations which a 4-dimensional metric g must satisfy to be conformal to Einstein.

Another independent condition can be obtained by decomposing F into $F = F^+ \oplus F^-$, where $*F^\pm = \pm iF^\pm$ are its self-dual and anti-self-dual parts (note that i appears here as a consequence of the assumed Lorentzian signature). Decomposing the curvatures $F^\pm$ onto a basis of 2-forms $\{\theta^i \wedge \theta^j\}$ associated with a coframe $\{\theta^i\}$ in which g takes the form $g = g_{ij}\theta^i\theta^j$, we recall that the second necessary condition for a 4-metric g to be conformal to Einstein is

$$[F^+_{ij}, F^-_{kl}] = 0 \qquad \forall i, j, k, l = 1, 2, 3, 4. \tag{10.5}$$

Here [,] is the commutator of the 6×6 matrices F^+_{ij} and F^-_{kl}. We term (10.4) the *Bach condition* and (10.5) the *Cotton condition* [4].

10.2.2. Conformal curvature of the associated metrics

Now we calculate the Cartan normal conformal connection and its curvature for the conformal metrics (7.18). We recall the setting from Sections 7.2 and 7.3. The structure of an oriented congruence (M, λ, μ) with vanishing shear and nonvanishing twist defines a 5-dimensional principal fiber bundle $H_2 \to P \to M$, on which the invariant forms $(\omega_1, \bar\omega_1, \omega, \Omega, \bar\Omega)$, satisfying the system (7.15) reside. There is another fiber bundle associated with such a situation. This is the bundle $P \to N$ with a 4-dimensional base N and with 1-dimensional fibers. The manifold N is in addition fibered over M also with 1-dimensional fibers. The forms

$$\{\theta^1, \theta^2, \theta^3, \theta^4\} = \{\omega_1, \bar\omega_1, \omega, t\mathrm{i}(\bar\Omega - \Omega)\}$$

on P are used to define a bilinear form $G_t = 2(\theta^1\theta^2 + \theta^3\theta^4)$ on P. Although this is degenerate on P, it projects to a well defined conformal class $[g_t]$ of *Lorentzian* metrics

$$g_t = 2(\theta^1\theta^2 + \theta^3\theta^4) \tag{10.6}$$

on N, see (7.18).

One can try to calculate the Cartan normal conformal connection for the metrics g_t on N itself, but we prefer to do this on the 5-dimensional bundle P instead. This is more convenient, since in such an approach we can directly use the coframe derivatives (7.15) of the forms $(\omega_1, \bar\omega_1, \omega, \Omega, \bar\Omega)$ on P, without the necessity of projecting them from P to N.

Thus, in the following, we associate the dual set of vector fields $(E_1, \bar E_1, E_0, E_2, \bar E_2)$ to $(\omega_1, \bar\omega_1, \omega, \Omega, \bar\Omega)$, and we will use them to denote the derivatives of the functions, such as the invariants K_1, K_2 and $\bar K_2$. The conventions will be as follows: the symbols $K_{11} = E_1(K_1)$ and $K_{1\bar 1} = \bar E_1(K_1)$ will denote the directional derivatives of K_1 in the respective directions of the vector fields E_1 and $\bar E_1$. In particular $K_{2\bar 10}$ will denote $E_0(\bar E_1(K_2))$.

A (rather tedious) calculation gives the following expressions for the Cartan normal conformal connection ω_t for the metrics g_t on P:

$$\omega_t = \begin{pmatrix} \frac12(\Omega+\bar\Omega) & \tau^1 & \tau^2 & \tau^3 & \tau^4 & 0 \\ \theta^1 & -\mathrm{i}\Omega_1 & 0 & -\Omega_2 & \frac{\mathrm{i}}{2}\theta^1 & \tau^2 \\ \theta^2 & 0 & \mathrm{i}\Omega_1 & -\bar\Omega_2 & -\frac{\mathrm{i}}{2}\theta^2 & \tau^1 \\ \theta^3 & \frac{\mathrm{i}}{2}\theta^2 & -\frac{\mathrm{i}}{2}\theta^1 & -\frac12(\Omega+\bar\Omega) & 0 & \tau^4 \\ \theta^4 & \bar\Omega_2 & \Omega_2 & 0 & \frac12(\Omega+\bar\Omega) & \tau^3 \\ 0 & \theta^2 & \theta^1 & \theta^4 & \theta^3 & -\frac12(\Omega+\bar\Omega) \end{pmatrix}. \tag{10.7}$$

Here the 1-forms Ω_1 (real) and Ω_2 (complex) are

$$\Omega_1 = tK_1\theta^3 + \frac{1-t}{2t}\theta^4, \qquad \Omega_2 = \mathrm{i}tK_1\theta^1 + \mathrm{i}t\bar K_2\theta^3, \qquad \bar\Omega_2 = -\mathrm{i}tK_1\theta^2 - \mathrm{i}tK_2\theta^3$$

and the 1-forms $\{\tau^1, \tau^2, \tau^3, \tau^4\}$ are:

$$\tau^1 = -\frac16(5t-2)K_1\theta^2 + \frac14(2\mathrm{i}tK_{11} + K_2(1-t))\theta^3$$

$$\tau^2 = \bar\tau^1 = -\frac16(5t-2)K_1\theta^1 + \frac14(-2\mathrm{i}tK_{1\bar 1} + \bar K_2(1-t))\theta^3$$

$$\tau^3 = \frac14(2\mathrm{i}tK_{11} - K_2(t+1))\theta^1 - \frac14(2\mathrm{i}t\bar K_{1\bar 1} + \bar K_2(t+1))\theta^2 - t^2K_1^2\theta^3 + \frac16(4t-1)K_1\theta^4$$

$$\tau^4 = \frac16(4t-1)K_1\theta^3 - \frac14\theta^4.$$

The next step, namely the calculation of the curvature $F_t = \mathrm{d}\omega_t + \omega_t \wedge \omega_t$ of ω_t, is really tedious, but achievable with the help of symbolic calculation programs such as, e.g. Mathematica. The resulting formulae are too complicated to display here, but the $\mathfrak{so}(1,3)$-part of the curvature, which is just the Weyl tensor of g_t, is worth quoting. We present it in terms of the (lifted to P) Weyl spinors Ψ_0, Ψ_1, Ψ_2, Ψ_3 and Ψ_4. These read:

$$\begin{aligned}
&\Psi_0 = 0, \qquad \Psi_1 = 0,\\
&\Psi_2 = \frac{1}{6}(1-4t)K_1,\\
&\Psi_3 = \frac{1}{4}\left(2\mathrm{i}tK_{1\bar{1}} + (3t-1)\bar{K}_2\right),\\
&\Psi_4 = -\mathrm{i}t\bar{K}_{2\bar{1}}.
\end{aligned} \tag{10.8}$$

We have the following

Proposition 10.2. *Every metric g_t with $K_1 \equiv 0$ or $t = \frac{1}{4}$ is of Petrov type III or its specializations. If $t = \frac{1}{3}$ and $K_1 \equiv 0$, then the conformal class $[g_{1/3}]$ of the metric $g_{1/3}$ is of Petrov type N.*

Calculation of the Yang–Mills current $j = j_\mu \theta^\mu$ for ω_t is also possible. Since the covariant derivative of the Hodge dual of the curvature F_t is *horizontal* with respect to the bundle $P \to N$, the current components j_μ, as viewed on P or on N, differ only by nonvanishing scales. The result of our calculation on P reads:

$$\begin{aligned}
&j^1 = \bar{j}^2 = \frac{1}{3}(1-4t)[K_{111}\theta^1 - 2\mathrm{i}K_{11}\theta^4] + \frac{1}{6}j^1_2\theta^2 - \frac{1}{6}j^1_3\theta^3\\
&j^3 = -\frac{1}{6}j^1_3\theta^1 - \frac{1}{6}\bar{j}^1_3\theta^2 - \frac{1}{6}j^3_3\theta^3 - \frac{1}{6}j^1_2\theta^4\\
&j^4 = \frac{2}{3}(4t-1)[K_1\theta^4 + iK_{11}\theta^1 - iK_{1\bar{1}}\theta^2] - \frac{1}{6}j^1_2\theta^3,
\end{aligned}$$

where

$$\begin{aligned}
j^1_2 &= (1-4t)(1-12t)K_1^2 + (7t-1)(K_{11\bar{1}} + K_{1\bar{1}1})\\
j^1_3 &= 16\mathrm{i}t(4t-1)K_1K_{11} - 2(1-2t)(1-4t)K_1K_2 + (1-4t)K_{2\bar{1}1} + 3\mathrm{i}t(K_{11\bar{1}1} + K_{1\bar{1}11})\\
j^3_3 &= 16t^2(1-4t)K_1^3 - 36t^2K_{11}K_{1\bar{1}} + 3(1-t)(1+3t)|K_2|^2 + 2(t+2)K_{2\bar{1}3}\\
&\quad - 24t^2K_1(K_{11\bar{1}} + K_{1\bar{1}1}) + 2\mathrm{i}t(4-7t)(K_{1\bar{1}}K_2 - K_{11}\bar{K}_2).
\end{aligned}$$

We have also calculated the Cotton matrices $[F^+_{tij}, F^-_{tij}]$ for each value of the real parameter t. We obtained formulae which are too complicated to write here. However we observed, that among all the parameter values for t, there are a few preferred ones for which the formulae simplify significantly. These special parameter values are:

$$t = \pm\frac{1}{3}, \qquad t = \frac{1}{4}, \qquad t = 1.$$

Here we focus on $t = -\frac{1}{3}$ and $t = 1$, for which we have the following theorem.

Theorem 10.3. *If $t = -\frac{1}{3}$ or $t = 1$ and the relative invariant $K_1 \equiv 0$, then the conformal metrics $[g_t]$ satisfy the Bach condition. If in addition the relative invariant $K_2 \neq 0$, the metrics are not conformally flat and do not satisfy the Cotton condition. If $K_1 \equiv K_2 \equiv 0$ the conformal metrics $g_{-1/3}$ and g_1 have $F_t \equiv 0$, i.e. they are conformally flat.*

The theorem can be verified by using the explicit formulae for the Yang–Mills current j^μ, the matrices $[F^+_{tij}, F^-_{tij}]$, and the integrability conditions for the system (7.15) with $K_1 = 0$. These integrability conditions, in particular, imply that $K_{2\bar{1}} = 0$.

We shall return to the other two interesting values $t = 1/4$ and $t = 1/3$ for g_t below, where we consider examples.

10.2.3. Examples

As noted above a particularly interesting class of structures (M, λ, μ) corresponds to $K_1 \equiv 0$ and $K_2 \neq 0$. Looking at the list of our examples presented in Section 7 we find such a structure in Section 7.6. This corresponds to a special value of the parameter $\beta_K = -3^{\frac{1}{3}}$ in the family of structures described by the invariant system (7.35), and is locally represented by forms λ, μ as in (7.37) with $\beta_K = -3^{\frac{1}{3}}$. Actually it is worthwhile to write the metrics g_t for *all* the structures covered by (7.37). These metrics read:

$$\begin{aligned}
g_t = g_t(\beta) &= 2\mathrm{d}z\mathrm{d}\bar{z} + t\left(\mathrm{d}u + \frac{2\beta e^{-\mathrm{i}\beta u} + \mathrm{i}\bar{z}}{\beta(z\bar{z} - 2\beta^2(2+\beta^3))}\mathrm{d}z + \frac{2\beta e^{\mathrm{i}\beta u} - \mathrm{i}z}{\beta(z\bar{z} - 2\beta^2(2+\beta^3))}\mathrm{d}\bar{z}\right)\\
&\times \frac{(z\bar{z} - 2\beta^2(2+\beta^3))^2}{2\beta^4}\left(2\mathrm{d}r + \frac{2(\beta e^{-\mathrm{i}\beta u} - \mathrm{i}\bar{z})}{z\bar{z} - 2\beta^2(2+\beta^3)}\mathrm{d}z + \frac{2(\beta e^{\mathrm{i}\beta u} + \mathrm{i}z)}{z\bar{z} - 2\beta^2(2+\beta^3)}\mathrm{d}\bar{z}\right),
\end{aligned}$$

and in addition to the real parameter t, they are parametrized by the real parameter $\beta \neq 0$ which enumerates nonequivalent structures (M, λ, μ).

These are quite interesting conformal Lorentzian metrics for the following reasons.

First, if

$$\beta = \beta_K = -3^{\frac{1}{3}},$$

we have $K_1 \equiv 0$, and according to Theorem 10.3, the metrics

$$g_{-1/3}(-3^{\frac{1}{3}}) = 2\mathrm{d}z\mathrm{d}\bar{z} - \left(\mathrm{d}u + \frac{2\cdot 3^{\frac{1}{3}}\mathrm{e}^{3^{\frac{1}{3}}\mathrm{i}u} - \mathrm{i}\bar{z}}{3^{\frac{1}{3}}(z\bar{z}+2\cdot 3^{\frac{2}{3}})}\mathrm{d}z + \frac{2\cdot 3^{\frac{1}{3}}\mathrm{e}^{-3^{\frac{1}{3}}\mathrm{i}u} + \mathrm{i}z}{3^{\frac{1}{3}}(z\bar{z}+2\cdot 3^{\frac{2}{3}})}\mathrm{d}\bar{z}\right)$$
$$\times \frac{(z\bar{z}+2\cdot 3^{\frac{2}{3}})^2}{18\cdot 3^{\frac{1}{3}}}\left(2\mathrm{d}r - \frac{2(3^{\frac{1}{3}}\mathrm{e}^{3^{\frac{1}{3}}\mathrm{i}u} + \mathrm{i}\bar{z})}{z\bar{z}+2\cdot 3^{\frac{2}{3}}}\mathrm{d}z - \frac{2(3^{\frac{1}{3}}\mathrm{e}^{-3^{\frac{1}{3}}\mathrm{i}u} - \mathrm{i}z)}{z\bar{z}+2\cdot 3^{\frac{2}{3}}}\mathrm{d}\bar{z}\right),$$

and

$$g_{1}(-3^{\frac{1}{3}}) = 2\mathrm{d}z\mathrm{d}\bar{z} + \left(\mathrm{d}u + \frac{2\cdot 3^{\frac{1}{3}}\mathrm{e}^{3^{\frac{1}{3}}\mathrm{i}u} - \mathrm{i}\bar{z}}{3^{\frac{1}{3}}(z\bar{z}+2\cdot 3^{\frac{2}{3}})}\mathrm{d}z + \frac{2\cdot 3^{\frac{1}{3}}\mathrm{e}^{-3^{\frac{1}{3}}\mathrm{i}u} + \mathrm{i}z}{3^{\frac{1}{3}}(z\bar{z}+2\cdot 3^{\frac{2}{3}})}\mathrm{d}\bar{z}\right)$$
$$\times \frac{(z\bar{z}+2\cdot 3^{\frac{2}{3}})^2}{6\cdot 3^{\frac{1}{3}}}\left(2\mathrm{d}r - \frac{2(3^{\frac{1}{3}}\mathrm{e}^{3^{\frac{1}{3}}\mathrm{i}u} + \mathrm{i}\bar{z})}{z\bar{z}+2\cdot 3^{\frac{2}{3}}}\mathrm{d}z - \frac{2(3^{\frac{1}{3}}\mathrm{e}^{-3^{\frac{1}{3}}\mathrm{i}u} - \mathrm{i}z)}{z\bar{z}+2\cdot 3^{\frac{2}{3}}}\mathrm{d}\bar{z}\right),$$

are *Bach flat*. Since the invariant K_2 of the corresponding structures (M, λ, μ) is nonvanishing, they are also *not* conformal to any Einstein metric. Note that, again because of $K_1 \equiv 0$ and $K_2 \neq 0$, both metrics $g_1(-3^{\frac{1}{3}})$ and $g_{-1/3}(-3^{\frac{1}{3}})$ are of general Petrov type *III* (see Proposition 10.2). As far as we know, they both provide the first *explicit* examples of conformally non Einstein Bach metrics which are of this Petrov type (compare e.g. with [16]).

Second, note also that, since $K_1 \equiv 0$ for $\beta_K = -3^{\frac{1}{3}}$, the metric $g_{1/3}(\beta_K)$, with now $t = +1/3$, is also quite interesting. According to Proposition 10.2 this metric is of Petrov type N. In gravitation theory it would be also termed *twisting* type N (see [10]). It is not conformal to any Einstein metric, since for all metrics $g_t(\beta_K)$ the Bach tensor $B_t(\beta_k)$, when expressed in terms of the coframe $(\theta^1, \theta^2, \theta^3, \theta^4)$, reads

$$B_t(-3^{\frac{1}{3}}) = 2^5\cdot 3^4\frac{(t-1)(1+3t)}{(z\bar{z}+2\cdot 3^{\frac{2}{3}})^6}\,\theta^3\odot\theta^3.$$

This obviously does not vanish, when $t = 1/3$, hence the metrics $g_{1/3}(\beta_K)$ are examples of twisting type N metrics, which are not conformally Einstein.

Third, suggested by the structure of the Weyl tensor (10.8) for all the metrics g_t we specialize the metrics $g_t(\beta)$ to the case when $t = \frac{1}{4}$. The Yang–Mills current for this special case may be read off from the general formulae from the previous section. Here however we prefer to give the explicit formulae for the Bach tensor for $g_{1/4}(\beta)$. Here again the Bach tensor $B_{1/4}(\beta)$ for these metrics has a very simple form

$$B_{1/4}(\beta) = 6\frac{\beta^6(\beta^6+36\beta^3+36)}{(z\bar{z}-2\beta^2(2+\beta^3))^6}\theta^3\odot\theta^3.$$

As is readily seen this vanishes for the following two real values of β:

$$\beta_{S_1} = -\left(6(3+2\sqrt{2})\right)^{\frac{1}{3}}, \qquad \beta_{S_2} = -\left(6(3-2\sqrt{2})\right)^{\frac{1}{3}}.$$

Thus the two corresponding metrics $g_{1/4}(\beta_{S_1})$, and $g_{1/4}(\beta_{S_2})$ are further examples of Bach Lorentzian metrics, which are again of Petrov type III. One can check by direct calculation that they are also not conformal to any Einstein metric.

Motivated by this last example we calculated the Bach tensor for *all* the metrics $g_{1/4}$ (not necessarily those associated with the β-parametrized-structures (7.37)). This calculation leads to the following

Theorem 10.4. *If* $t = \frac{1}{4}$ *and a structure* (M, λ, μ) *with nonvanishing twist and vanishing shear has the relative invariant* K_1 *satisfying*

$$K_{11\bar{1}} + K_{1\bar{1}1} \equiv 0,$$

then the Bach tensor B_t *of the metrics* g_t *corresponding to the structure* (M, λ, μ)*, as defined in* (10.6)*, has a very simple form*

$$B_{1/4} = \frac{3}{32}\left(4K_{11}K_{1\bar{1}} + 2\mathrm{i}(K_{11}\bar{K}_2 - K_{1\bar{1}}K_2) - 7K_2\bar{K}_2 - 4(K_{2\bar{1}0} + \bar{K}_{210})\right)\theta^3\odot\theta^3,$$

in which nine out of the a priori ten components, identically vanish.

Apart from the structures with β_{S_1} and β_{S_2} we do not know examples of structures satisfying condition $K_{11\bar{1}} + K_{1\bar{1}1} \equiv 0$.

11. Application 2: Algebraically special spacetimes

All the metrics discussed in Section 10 are examples of *algebraically special spacetimes*. These are 4-dimensional Lorentzian metrics, whose Weyl tensor is *degenerate* in an open region of the spacetime. The algebraically special *vacuum* (or in other words: *Ricci flat*) metrics have the interesting property that they define a congruence of shearfree and null geodesics in the underlying *spacetime*. At this stage we must emphasize that the congruence associated with such metrics lives in *four* dimensions and the vanishing *shear* and the *geodesic* condition is a *four* dimensional notion here. Nevertheless we observe that the 3-dimensional oriented congruences in our sense are related, at least at the level of the Lorentzian metrics discussed so far, to an analogous notion in $3+1$ dimensions, where the metric is of Lorentzian signature. In this section we discuss this relationship more closely. Note that in *all* the examples of Section 10 the *four*-dimensional congruence of shearfree null geodesics was always tangent to the vector field $k=\partial_r$.

Before passing to the main subject of this section we remark that the algebraically special Lorentzian metrics are very important in physics. To be more specific we consider the metric

$$g=2\left(\mathcal{P}^2\mu\bar{\mu}+\lambda(\mathrm{d}r+\mathcal{W}\mu+\bar{\mathcal{W}}\bar{\mu}+\mathcal{H}\lambda)\right), \tag{11.1}$$

where

$$\begin{aligned}
&\lambda=\mathrm{d}u+\frac{\mathrm{i}\,(2M+(a+M)z\bar{z})}{z(1+\frac{K}{2}z\bar{z})^2}\mathrm{d}z-\frac{\mathrm{i}\,(2M+(a+M)z\bar{z})}{\bar{z}(1+\frac{K}{2}z\bar{z})^2}\mathrm{d}\bar{z}, \qquad \mu=\mathrm{d}z,\\
&\mathcal{P}^2=\frac{r^2}{(1+\frac{K}{2}z\bar{z})^2}+\frac{\left(KM-a+(KM+a)\frac{K}{2}z\bar{z}\right)^2}{(1+\frac{K}{2}z\bar{z})^4},\\
&\mathcal{W}=\frac{\mathrm{i}Ka\bar{z}}{(1+\frac{K}{2}z\bar{z})^2},\\
&\mathcal{H}=-\frac{K}{2}+\frac{mr+KM^2-aM\frac{1-\frac{K}{2}z\bar{z}}{1+\frac{K}{2}z\bar{z}}}{r^2+\frac{\left(KM-a+(KM+a)\frac{K}{2}z\bar{z}\right)^2}{(1+\frac{K}{2}z\bar{z})^2}},
\end{aligned} \tag{11.2}$$

and m, a, M, K are *real* constants.

This scary-looking metric has very interesting properties. First, it admits a 4-dimensional congruence of null and shearfree geodesics, which is tangent to the vector field $k=\partial_r$. Second, if $K=1$, it is *algebraically special*, actually of Petrov type *D*, and more importantly, it is *Ricci flat*. The parameter values $K-1=M=0$, correspond to the celebrated *Kerr metric*, describing a gravitational field outside a *rotating black hole*, with *mass m* and *angular momentum* parameter *a*. In this case the angular momentum parameter *a* measures the *twist* of the congruence tangent to *k*. If in addition $a=0$, the twist of the congruence vanishes, and the metric becomes the *Schwarzschild metric*. Third, in the $K-1=a=m=0$ case the metric is the *Taub-NUT vacuum metric*, which is important in Relativity Theory because its serves as a 'counterexample for almost everything' [13]. Fourth, it should be also noted that if $M=0$ and the other parameters, including K, are arbitrary, the metric is again type *D* and *Ricci flat*. Finally, we should mention that for general values of $K\neq 1$ and $M\neq 0$ the metric is algebraically *general* and *neither* Ricci flat *nor* Einstein.

From the point of view of our paper the relevance of the metric (11.1) and (11.2) is self evident. The four dimensional spacetime $\mathcal{M}$ on which the metric is defined, locally parametrized by $(u,z,\bar{z},r)$, is locally a product $\mathcal{M}=M\times\mathbb{R}$, with M being parametrized by $(u,z,\bar{z})$. The 3-dimensional manifold M is then naturally equipped with the oriented congruence structure (M,λ,μ), defined in terms of the 1-forms λ,μ from (11.2). Note that these forms, although defined on $\mathcal{M}$, do not depend on the r coordinate, and as such project to M. Note also that the oriented congruence structure defined by these forms has always vanishing shear $s\equiv 0$. It has nonvanishing twist, with the exception of the Schwarzschild metric $a=M=0$, or the case when $K=0$ and $M+a=0$. In this last case the metric is of Petrov type *D*, but is *neither* Ricci flat *nor* Einstein.

Since in the case of *Ricci flat* metrics (11.1) and (11.2) only the Schwarzschild metric has the corresponding structure of an oriented congruence with vanishing twist, in the next sections we decided to make a systematic study of the Lorentzian metrics (11.1) (not necessarily of the form (11.2)), with forms λ,μ defining an oriented congruence structure in *three* dimensions which have vanishing shear, but nonvanishing twist, only. Actually, for the sake of brevity, we only discuss the case when the structural invariants K_1 and K_2 of the congruence structures, as defined in Section 7.1, satisfy $K_1\neq 0$, $K_2\equiv 0$.

11.1. Reduction of the Einstein equations

As we know from Section 7.5 every structure $(M,[\lambda,\mu])$ having $K_1\neq 0, K_2\equiv 0$ defines an invariant coframe $(\omega,\omega_1,\bar{\omega}_1)$ on M which satisfies the system (7.29) and (7.31). Given such a structure we consider a 4-manifold $\mathcal{M}=\mathbb{R}\times M$ with a distinguished class of Lorentzian metrics. These metrics can be written using any representative of a class $[\lambda, \mu]$. Since the invariant forms (ω, ω_1) provide us with such a representative it is natural to use them, rather than a randomly chosen pair (λ, μ). Thus, given a structure $(M, [\lambda, \mu])$ having $K_1 \neq 0, K_2 \equiv 0$, we write a metric on

$$\mathcal{M} = \mathbb{R} \times M \tag{11.3}$$

as

$$g = P^2\,[2\omega_1\bar{\omega}_1 + 2\omega(\mathrm{d}r + W\omega_1 + \bar{W}\bar{\omega}_1 + H\omega)]. \tag{11.4}$$

Here the forms $(\omega, \omega_1, \bar{\omega}_1)$ satisfy the system (7.29) and (7.31), r is a coordinate along the $\mathbb{R}$ factor in $\mathcal{M}$, and $P \neq 0$, H (real) and W (complex) are arbitrary functions on $\mathcal{M}$.

The null vector field $k = \partial_r$ is tangent to a congruence of twisting and shearfree null geodesics in $\mathcal{M}$. This is a distinguished geometric structure on $\mathcal{M}$.

Now we pass to the question if the metrics (11.4) may be Einstein. To discuss this we need to specify what is the interesting energy momentum tensor that will constitute the r.h.s. of the Einstein equations. Since the only geometrically distinguished structure on $\mathcal{M}$ is the shearfree congruence generated by $k = \partial_r$ it is natural to consider the Einstein equations in the form

$$Ric(g) = \Phi k \odot k. \tag{11.5}$$

If the real function Φ satisfies $\Phi > 0$ the above equations have the physical interpretation of a gravitational field of 'pure radiation' type in which the gravitational energy is propagated with the speed of light along the congruence k. If $\Phi \equiv 0$ we have just Ricci-flat metrics, which correspond to vacuum gravitational fields. This last possibility is not excluded by our Einstein equations. In the following analysis we will not insist on the condition $\Phi \equiv 0$.

At this point it is worthwhile to mention that a similar problem was studied by one of us some years ago in [14]; see also the more modern treatment in [6]. Using the results of [6,14] and the symbolic calculation program Mathematica, we reduced the Einstein equation (11.5) to the following form:

First, it turns out that the Einstein equation (11.5) can be fully integrated along k, so that the r dependence of the functions P, H, W is explicitly determined. Actually we have:

$$\begin{aligned} P &= \frac{p}{\cos\frac{r}{2}}\\ W &= \mathrm{i}\alpha \mathrm{e}^{-\mathrm{i}r} + \beta\\ H &= -\frac{\bar{m}}{p^4}\mathrm{e}^{2\mathrm{i}r} - \frac{m}{p^4}\mathrm{e}^{-2\mathrm{i}r} + \frac{1}{2}\bar{\phi}\mathrm{e}^{\mathrm{i}r} + \frac{1}{2}\phi\mathrm{e}^{-\mathrm{i}r} + \frac{1}{2}\chi, \end{aligned} \tag{11.6}$$

where the functions p, χ (real) and α, β, m (complex) do not depend on the r coordinate. Thus, using some of the Einstein equation (11.5), one quickly reduces the problem from $\mathcal{M}$ to a system of equations on the CR-manifold with preferred splitting $(M, [\lambda, \mu])$.

Now we introduce a preferred set of vector fields $(\partial_0, \partial, \bar{\partial})$ on M defined as the respective duals of the preferred forms $(\omega, \omega_1, \bar{\omega}_1)$. Note that this notation is in agreement with the notation of CR-structure theory. In particular $\bar{\partial}$ is the tangential CR-operator on M, so that the equation for a CR-function ξ on M is $\bar{\partial}\xi = 0$.

With this notation the remaining Einstein equation (11.5) for g give first:

$$\begin{aligned} \alpha &= 2(\partial \log p - c)\\ \beta &= 2\mathrm{i}(\partial \log p - 2c - A_1)\\ \phi &= (\bar{\partial} + A_1 + \mathrm{i}\bar{B}_1 + \mathrm{i}\bar{\beta})\alpha - 4\frac{m}{p^4}\\ \chi &= 3\alpha\bar{\alpha} + 2\mathrm{i}(\partial + A_1 - \mathrm{i}B_1)\bar{\beta} - 2\mathrm{i}(\bar{\partial} + A_1 + \mathrm{i}\bar{B}_1)\beta \mp 1, \end{aligned} \tag{11.7}$$

where we have introduced a new unknown complex function c on M and used the Cartan invariants $A_1 > 0$, B_1 and ± 1 of the system (7.29) and (7.31).

Finally the differential equations for the unknown functions c, m and p equivalent to the Einstein equation (11.5) are:

$$(\partial - 3A_1 + \mathrm{i}B_1)c - 2c^2 + a_{11} - A_1^2 + \frac{\mathrm{i}}{2}A_1(3B_1 + \bar{B}_1) = 0 \tag{11.8}$$

$$(\bar{\partial} - 6\bar{c})m = 0 \tag{11.9}$$

$$\begin{aligned} &(\partial + 3A_1 - \mathrm{i}B_1)\bar{\partial}p + (\bar{\partial} + 3A_1 + \mathrm{i}\bar{B}_1)\partial p + -3\Big[(\partial + 3A_1 - \mathrm{i}B_1)\bar{c} + (\bar{\partial} + 3A_1 + \mathrm{i}\bar{B}_1)c + 2c\bar{c}\\ &\quad + \frac{8}{3}A_1^2 + \frac{4}{3}a_{11} + \frac{2\mathrm{i}}{3}A_1(\bar{B}_1 - B_1) \pm \frac{1}{6}\Big]p = -\frac{m + \bar{m}}{p^3}. \end{aligned} \tag{11.10}$$

We thus have the following theorem.

Theorem 11.1. *Let $(M,[\lambda,\mu])$ be a structure of an oriented congruence having vanishing shear, nonvanishing twist and the invariants $K_1\neq 0$, $K_2\equiv 0$. Then a Lorentzian metric g associated with $(M,[\lambda,\mu])$ via (11.3) and (11.4) satisfies the Einstein equation (11.5) if and only if the metric functions are given by means of (11.6) and (11.7) with the unknown functions c, m (complex), p (real) on M satisfying the differential equations (11.8)–(11.10).*

Remark 11.2. Note that contrary to the invariants $(\omega,\omega_1,\bar\omega_1)$ the coordinate r, and in turn the differential $\mathrm{d}r$, has no geometric meaning. Actually the coordinate freedom in choosing r is $r\to r+f$, where f is any real function f on M. This induces some gauge transformations on the variables β and χ. Nevertheless the Eqs. (11.8)–(11.10) are not affected by these transformations.

Remark 11.3. Eqs. (11.8)–(11.10) should be understood in the following way. Start with a structure of an oriented congruence $(M,[\lambda,\mu])$ having vanishing shear, nonvanishing twist and the invariants $K_1\neq 0$, $K_2\equiv 0$. Calculate its invariants $(\omega,\omega_1,\bar\omega_1)$, $(\partial_0,\partial,\bar\partial)$, A_1, B_1, a_{11} of (7.29) and (7.31). Using this data write down Eqs. (11.8)–(11.10) for the unknowns c,m,p. As a hint for solving these equations observe that the Eq. (11.8) involves only the unknown c. Thus, solve it first. Once you have the general solution for c insert it into the Eq. (11.9). Then this equation becomes an equation for the unknown m. In particular $m=0$ is always a solution of (11.9). Once this equation for m is solved, insert c and m to the Eq. (11.10), which becomes a real, second order equation for the real unknown p. In particular, if it happens that you are only interested in solutions for which $m+\bar m=0$, this equation is a linear second order PDE on M. For particular choices of $(M,[\lambda,\mu])$ it can be reduced to well known equations of mathematical physics, such as, for example, the hypergeometric equation [14].

Remark 11.4. The unknown variable m is related to a notion known to physicists as *complex mass*. For physically interesting solutions, such as for example the Kerr black hole, the imaginary part of m is related to the mass of the gravitational source. The real part of m is related to the so called NUT parameter. Moreover m is responsible for algebraical specialization of the Weyl tensor of the metric. If $m\equiv 0$ the metric is of type III, or its specializations, in the Cartan–Petrov–Penrose algebraic classification of gravitational fields.

11.2. Examples of solutions

Here we give examples of metrics (11.4) satisfying the Einstein equation (11.5). In all these examples the structures of oriented congruences $(M,[\lambda,\mu])$ will be isomorphic to the structures with a 3-dimensional group of symmetries described by Proposition 7.16. The invariant forms $(\omega,\omega_1,\bar\omega_1)$ for these structures are:

$$\begin{aligned}\omega&=\frac{2\tau^2}{1\mp 4\tau^2}(y^{-2(1\mp 2\tau^2)}\mathrm{d}u-y^{-1}\mathrm{d}x),\\ \omega_1&=\pm\mathrm{i}\tau y^{-1}(\mathrm{d}x+\mathrm{i}\mathrm{d}y),\\ \bar\omega_1&=\mp\mathrm{i}\tau y^{-1}(\mathrm{d}x-\mathrm{i}\mathrm{d}y).\end{aligned}\tag{11.11}$$

We recall that the real parameter τ is related to the invariants A_1,B_1 of the structures (11.11) via:

$$A_1=-\frac{\mp 1+2\tau^2}{2\tau},\qquad B_1=\mathrm{i}\tau.$$

Since these invariants are *constant*, all the higher order invariants for these structures, such as for example the a_{11} in (7.31), are *identically vanishing*. Although Proposition 7.16 excludes the values $\tau^2=\frac12$ in the upper sign case, we include it in the discussion below. This value corresponds to $A_1=0$ and therefore must describe one of the two nonequivalent structures $(M,[\lambda,\mu])$ of Example 7.7. From the two structures of this example, the one corresponding to $\tau^2=\frac12$ is defined by $(\epsilon_1,\epsilon_2)=(0,1)$. In particular, it has a strictly 4-dimensional symmetry group.

First we assume that the metric (11.4) has the same *conformal symmetries* as the structures (11.11). This assumption, together with Einstein's equation (11.5), which are equivalent to the Eqs. (11.6) and (11.7), (11.8)–(11.10), implies that *all the metric functions p, m, c must be constant*. Then the system (11.8)–(11.10) reduces to the following algebraic equations for m,p,c:

$$(-3A_1+\mathrm{i}B_1)c-2c^2-A_1^2+\frac{\mathrm{i}}{2}A_1(3B_1+\bar B_1)=0\tag{11.12}$$

$$\bar c m=0\tag{11.13}$$

$$3\left[(3A_1-\mathrm{i}B_1)\bar c+(3A_1+\mathrm{i}\bar B_1)c+2c\bar c+\frac83A_1^2+\frac{2\mathrm{i}}{3}A_1(\bar B_1-B_1)\pm\frac16\right]p=\frac{m+\bar m}{p^3}.\tag{11.14}$$

Thus we have two cases.

- Either $c=0$
- or $m=0$.

Strangely enough in both cases Eqs. (11.12)–(11.14) admit solutions *only* for the *upper* sign in (11.14).

If $c = 0$ then we have only one solution corresponding to $\tau = \pm\frac{1}{\sqrt{2}}$ with *arbitrary* constant $p \neq 0$ and $m = \frac{p^4}{4} + \mathrm{i}M$, where M is *real* constant. The corresponding metric

$$\mathrm{d}s^2 = \frac{p^2}{\cos^2\frac{r}{2}}\left[\frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2} + 2\left(\frac{\mathrm{d}x}{y}-\mathrm{d}u\right)\left(\mathrm{d}r - 2\cos^2\frac{r}{2}(\cos r + 4M\sin r)\left(\frac{\mathrm{d}x}{y}-\mathrm{d}u\right)\right)\right]$$

is *vacuum* i.e. it satisfies Eq. (11.5) with $\Phi \equiv 0$.

If $m = 0$ then $p \neq 0$ is an arbitrary constant, and we have the following solutions:

- $\tau = \frac{\epsilon_1}{4}\sqrt{5+\epsilon_2\sqrt{17}}$, $c = -\frac{\epsilon_1}{\sqrt{5+\epsilon_2\sqrt{17}}}$,
- $\tau = \frac{\epsilon_1}{2}\sqrt{\frac{1}{2}(7+\epsilon_2\sqrt{17})}$, $c = \frac{\epsilon_1}{4}\sqrt{\frac{1}{2}(7+\epsilon_2\sqrt{17})}(3+\epsilon_2\sqrt{17})$.

Here $\epsilon_1^2 = \epsilon_2^2 = 1$. Sadly, irrespectively of the signs of ϵ_1, ϵ_2, all these solutions have $\Phi = \mathrm{const} < 0$, and as such do not correspond to physically meaningful sources.

In the next example we still consider structures $(M, [\lambda, \mu])$ with the invariants (11.11), and assume that the metrics have only *two* conformal symmetries ∂_u and ∂_x. For simplicity we consider only solutions with $m = 0$ in (11.9). Under these assumptions we find that the general solution of (11.8)–(11.10) includes a free real parameter t and is given by

$$c = \frac{-2+4\tau^2}{4\tau} + \frac{1-4\tau^2}{4\tau}\frac{1}{1-ty^{(4\tau^2-1)}}, \tag{11.15}$$

with the real function $p = p(y)$ satisfying a linear 2nd order ODE:

$$\begin{aligned}&4y(y-ty^{4\tau^2})^2[yp'' + (4\tau^2-2)p'] + [(-32\tau^4+20\tau^2-1)y^2 + 4t^2(4\tau^4-7\tau^2+2)y^{8\tau^2}\\ &\quad - 16t(8\tau^4-5\tau^2+1)y^{(4\tau^2+1)}]p = 0.\end{aligned} \tag{11.16}$$

If this equation is satisfied, the only *a'priori* nonvanishing component of the Ricci tensor is

$$\begin{aligned}R_{33} = {}& -\frac{1}{8}\left(\frac{\cos(\frac{r}{2})}{\tau(y-ty^{4\tau^2})p}\right)^4(((8\tau^2-3)(128\tau^6-160\tau^4+92\tau^2-21)y^4\\ &+8t^4\tau^2(32\tau^6+8\tau^4-28\tau^2+9)y^{16\tau^2} + 4t(8\tau^2-3)(256\tau^6-248\tau^4+58\tau^2+3)y^{3+4\tau^2}\\ &+36t^2(4\tau^4+\tau^2-1)(32\tau^4-12\tau^2-1)y^{2+8\tau^2} + 16t^3\tau^2(128\tau^6-184\tau^4+122\tau^2-27)y^{1+12\tau^2})p^2\\ &-4y(y-ty^{4\tau^2})((8\tau^2-3)(16\tau^4-3)y^3 + 4t^3\tau^2(16\tau^4-3)y^{12\tau^2}\\ &+6t(8\tau^2-3)y^{2+4\tau^2} + 96t^2\tau^2(1-2\tau^2)^2y^{1+8\tau^2})pp' + 4y^2(y-ty^{4\tau^2})^2((8\tau^2-3)y + 4t\tau^2y^{4\tau^2})^2p'^2).\end{aligned}$$

It follows that this R_{33}, with p satisfying (11.16), *may* identically vanish for some values of parameter τ. This happens *only* when the parameter $t = 0$. If

$$t = 0$$

the values of τ for which R_{33} may be identically zero and for which the function $p = p(y)$ satisfies (11.16) are:

$$\tau = \pm\frac{1}{2}\sqrt{2}, \qquad \tau = \pm\frac{1}{2}\sqrt{\frac{3}{2}}, \qquad \tau = \pm\frac{1}{2}\sqrt{\frac{5}{3}}, \qquad \tau = \pm\frac{1}{2}\sqrt{3},$$
$$\tau_- = \pm\frac{1}{2}\sqrt{\frac{1}{6}(11-\sqrt{13})}, \qquad \tau_+ = \pm\frac{1}{2}\sqrt{\frac{1}{6}(11+\sqrt{13})}.$$

Of these *distinguished* values the most interesting (modulo sign) are the last two, τ_- and τ_+, since for them the corresponding metrics (11.4) may be *vacuum* and *not* conformally flat. Actually, restricting our attention to the plus signs above and assuming $t = 0$, we have the following possibilities:

- $\tau_\varepsilon = \frac{1}{2}\sqrt{\frac{1}{6}(11+\varepsilon\sqrt{13})}$, $\varepsilon = \pm 1$; for these two values of τ the general solution of (11.16) is

  $$p_\varepsilon = y^{\frac{1}{12}(1-\varepsilon\sqrt{13})}(s_2 + s_1 y),$$

  and the only potentially nonvanishing component of the Ricci tensor is

  $$R_{33} = -\frac{4}{9}(7+\varepsilon\sqrt{13})s_2^2 y^{-\frac{1}{6}(1-\varepsilon\sqrt{13})}\left(\frac{\cos\frac{r}{2}}{s_2+s_1y}\right)^4.$$

  This *vanishes* when $s_2 = 0$. If $s_2 = 0$ the corresponding metrics g_ε, as defined in (11.4), read

  $$g_\varepsilon = 2P^2\left(\omega_1\bar{\omega}_1 + \omega\left(\mathrm{d}r + W\omega_1 + \bar{W}\bar{\omega}_1 + \frac{3+(9-20\tau_\varepsilon^2)\cos r}{12\tau_\varepsilon^2}\omega\right)\right),$$

with

$$P=\frac{s_1 y^{2(1-\tau_\varepsilon^2)}}{\cos\frac{r}{2}},\qquad W=\mathrm{i}\frac{2(20\tau_\varepsilon^2-9)+(8\tau_\varepsilon^2-9)\mathrm{e}^{-\mathrm{i}r}}{24\tau_\varepsilon^3},$$

and ω, ω_1, $\bar{\omega}_1$ given by (11.11). For both values of $\varepsilon=\pm1$ the metric is *Ricci flat* and of Petrov type *III*. In particular it is *neither* flat, *nor* of type *N*.

In all other cases of the *distinguished* τs the corresponding *vacuum* metrics are the *flat* Minkowski metrics. In fact,

- if $\tau=\frac{1}{2}\sqrt{\frac{3}{2}}$, the general solution to (11.16) is

$$p=s_1\sqrt{y}+s_2 y,$$

  and the corresponding metric (11.4) is flat.
- if $\tau=\frac{1}{2}\sqrt{\frac{5}{3}}$, the general solution to (11.16) is

$$p=y^{\frac{2}{3}}(s_1+s_2\log y),$$

  and the potentially nonvanishing Ricci component R_{33} is

$$R_{33}=-\frac{8}{25}s_2(2s_1+s_2+2s_2\log y)\left(\frac{\cos\frac{r}{2}}{(s_1+s_2\log y)y^{\frac{1}{3}}}\right)^4.$$

  This vanishes when $s_2=0$. In such case the metric is flat.
- if $\tau=\frac{1}{2}\sqrt{2}$, the general solution of (11.16) is

$$p=\sqrt{y}(s_1+s_2\log y),$$

  and

$$R_{33}=-\frac{2s_2^2}{y}\left(\frac{\cos\frac{r}{2}}{s_1+s_2\log y}\right)^4;$$

  this vanishes when $s_2=0$; in such cases the metric is flat.
- if $\tau=\frac{1}{2}\sqrt{3}$, the general solution of (11.16) is

$$p=s_1 y+s_2 y^{-1},$$

  and

$$R_{33}=-32s_2^2y^2\left(\frac{\cos\frac{r}{2}}{s_2+s_1y^2}\right)^4;$$

  this vanishes when $s_2=0$; in such case the metric is the flat Minkowski metric.

We close this section with an example of a metric that goes a bit beyond the formulation of the Einstein equations presented here. Remaining with the structures of an oriented congruence with the upper sign in (11.11), we take c as in (11.15) with $t=0$, and consider the metric (11.4), (11.6) and (11.7) with a *constant* function p given by

$$p=\frac{\sqrt{3}}{4s\tau}\sqrt{\varepsilon(-1+20\tau^2-32\tau^4)}.$$

Here the ε is ±1, and is chosen to be such that the value $\varepsilon(-1+20\tau^2-32\tau^4)$ is positive; s is a nonzero constant. A short calculation shows that the Ricci tensor for this metric has the following form

$$Ric=(\tau^2-1)(8\tau^2-5)\frac{16\Lambda(4\tau^2+1)\cos^4\frac{r}{2}}{3\tau^2(1-20\tau^2+32\tau^4)}k\odot k+\Lambda g.$$

Thus, this metric is *Einstein*, with cosmological constant equal to $\Lambda=\varepsilon s^2$, provided that

$$\tau=\pm1,\quad\text{or}\quad\tau=\pm\frac{1}{2}\sqrt{\frac{5}{2}}.$$

It is remarkable that the Einstein metric

$$g=-\frac{3}{5\Lambda\cos^2\frac{r}{2}}\left(\omega_1\bar{\omega}_1+\omega\left(\mathrm{d}r+\frac{\mathrm{i}(2\mathrm{e}^{-\mathrm{i}r}+5)}{\sqrt{10}}\omega_1-\frac{\mathrm{i}(2\mathrm{e}^{\mathrm{i}r}+5)}{\sqrt{10}}\bar{\omega}_1+\frac{7}{10}(3+2\cos r)\omega\right)\right),$$

corresponding to $\tau = \pm\frac{1}{2}\sqrt{\frac{5}{2}}$, is of *Petrov type N* with the quadruple principal null direction of the Weyl tensor being *twisting*. It was first obtained by Leroy [11] and recently discussed in [17]. The Einstein metric

$$g = -\frac{39}{8\Lambda\cos^2\frac{r}{2}}\left(\omega_1\bar{\omega}_1 + \omega\left(\mathrm{d}r + \frac{\mathrm{i}(\mathrm{e}^{-\mathrm{i}r}+4)}{2}\omega_1 - \frac{\mathrm{i}(\mathrm{e}^{\mathrm{i}r}+4)}{2}\bar{\omega}_1 + \frac{5}{8}(3+2\cos r)\omega\right)\right),$$

corresponding to $\tau = \pm 1$ is of Petrov type *III*.

11.3. Discussion of the reduced equations

Here we discuss the integration procedures for Eqs. (11.8)–(11.10) along the lines indicated in Remark 11.3. We start with Eq. (11.8). This is an equation for the unknown c. Remarkably, the existence of a function c satisfying this equation is equivalent to an existence of a certain CR function η on M. To see this we proceed as follows. We consider a 1-form Π on M given by

$$\Pi = \omega_1 + 2\mathrm{i}(A_1 + \bar{c})\omega, \tag{11.17}$$

where c is an arbitrary complex function on M. Of course

$$\Pi \wedge \bar{\Pi} \neq 0, \tag{11.18}$$

since otherwise the forms ω_1 and $\bar{\omega}_1$ would not be independent. Now using the differentials $\mathrm{d}\omega$, $\mathrm{d}\omega_1$, $\mathrm{d}A_1$ given in (7.29) and (7.31), we easily find that

$$\mathrm{d}\Pi \wedge \Pi = 2\mathrm{i}\left[(\bar{\partial} - 3A_1 - \mathrm{i}\bar{B}_1)\bar{c} - 2\bar{c}^2 + a_{11} - A_1^2 - \frac{\mathrm{i}}{2}A_1(3\bar{B}_1 + B_1)\right]\omega_1 \wedge \bar{\omega}_1 \wedge \omega.$$

Thus our Eq. (11.8) is satisfied for c if and only if $\mathrm{d}\Pi \wedge \Pi = 0$. Due to our Lemma 5.1, Π satisfying $\mathrm{d}\Pi \wedge \Pi = 0$ defines a complex valued function η on M such that $\Pi = h\mathrm{d}\eta$. Because of (11.18) we have $h\bar{h}\mathrm{d}\eta \wedge \mathrm{d}\bar{\eta} \neq 0$. Furthermore, since Π is given by (11.17) then $\Pi \wedge \omega \wedge \omega_1 = 0$, which after factoring out by h gives $\mathrm{d}\eta \wedge \omega \wedge \omega_1 = 0$. Thus η is a CR-function on M.

Conversely, suppose that we have a CR-function η on M such that

$$\mathrm{d}\eta \wedge \mathrm{d}\bar{\eta} \neq 0. \tag{11.19}$$

Then the three one-forms ω_1, ω and $\mathrm{d}\eta$ are linearly dependent at each point. Thus there exist complex functions x, y on M such that

$$\mathrm{d}\eta = x\omega_1 + y\omega. \tag{11.20}$$

Due to the nondegeneracy condition (11.19) we must have $x\bar{x}\omega_1 \wedge \bar{\omega}_1 + x\bar{y}\omega_1 \wedge \omega - \bar{x}y\bar{\omega}_1 \wedge \omega \neq 0$, so that the complex function x must be nonvanishing. In such case we may rewrite (11.20) in the more convenient form $h\mathrm{d}\eta = \omega_1 + \bar{z}\omega$, where $h = 1/x$ and $\bar{z} = y/x$. Now, defining c to be $c = \frac{\mathrm{i}z}{2} - A_1$, we see that the trivially satisfied equation $(h\mathrm{d}\eta) \wedge \mathrm{d}(h\mathrm{d}\eta) = 0$ implies that the function c must satisfy Eq. (11.8). Summarizing we have the following proposition.

Proposition 11.5. *Every solution η of the tangential CR equation $\bar{\partial}\eta = 0$ satisfying $\mathrm{d}\eta \wedge \mathrm{d}\bar{\eta} \neq 0$ defines a solution c of Eq. (11.8). Given η, the function c satisfying Eq. (11.8) is defined by*

$$c = \frac{\mathrm{i}}{2}\frac{\bar{y}}{\bar{x}} - A_1, \tag{11.21}$$

where $\mathrm{d}\eta = x\omega_1 + y\omega$. Also the converse is true: every solution c of Eq. (11.8) defines a CR function η such that $\mathrm{d}\eta \wedge \mathrm{d}\bar{\eta} \neq 0$.

Remark 11.6. Recall that the structures $(M, [\lambda, \mu])$ satisfying the system (7.29) and (7.31) admit at least one CR-function ζ, since they have zero shear $s \equiv 0$. Associated to ζ, by the above Proposition, there should be a solution c of the Einstein equation (11.8). One checks by direct calculation that

$$c = -A_1$$

automatically satisfies (11.8). And this is the solution c associated with ζ. This is consistent with formula (11.21), since $y \equiv 0$ means that $\mathrm{d}\eta \wedge \mathrm{d}\zeta \equiv 0$ (compare with (11.20)).

We now pass to the discussion of the second Einstein equation (11.9). Eq. (11.9), the equation for the function m, has a principal part resembling the tangential CR-equation. Remarkably its solutions m are also expressible in terms of CR-functions. To see this consider an arbitrary complex valued function ξ and define m to be

$$m = [\partial_0\xi - 2\mathrm{i}(A_1 + \bar{c})\partial\xi + 2\mathrm{i}(A_1 + c)\bar{\partial}\xi]^3. \tag{11.22}$$

Here c is supposed to be a solution to the first Einstein equation (11.8). Observe, that since the vector field $\partial_0 - 2\mathrm{i}(A_1 + \bar{c})\partial + 2\mathrm{i}(A_1 + c)\bar{\partial}$ is real, then given m one can always locally solve for ξ. Our goal now is to show that if ξ is a CR-function on M, then m given by (11.22) satisfies Eq. (11.9). To prove this one inserts (11.22) into Eq. (11.9) and commutes the operators $\bar{\partial}\partial_0$ and $\bar{\partial}\partial$. After this is performed the Eq. (11.9) for m becomes the following equation for ξ:

$$(\partial_0 + 2i\bar{\partial}(A_1 + c) + 2i(A_1 + c)\bar{\partial} - 2i(A_1 + \bar{c})\partial - 4i\bar{c}(A_1 + c) + A_1 - iB_1)\bar{\partial}\xi = 0.$$

This, in particular, means that if ξ is a CR-function then this equation is satisfied automatically. Thus given a CR-function ξ, via (11.22), we constructed m which satisfies Eq. (11.9). To see that all solutions m of (11.9) can be constructed in this way is a bit more subtle (see [6]).

References

[1] L. Bianchi, Sugli spazii a tre dimensioni che ammettono un gruppo continuo di movimenti, Soc. Ital. Sci. Mem. di Mat. 11 (1898) 267–352.
[2] R.J. Baston, L.J. Mason, Conformal gravity, the Einstein equations and spaces of complex null geodesics, Classical Quantum Gravity 4 (1987) 815–826.
[3] E. Cartan, Sur la geometrie pseudo-conforme des hypersurfaces de deux variables complexes, Ann. Math. Pure Appl. 11 (1932) 17–90 (part I); Ann. Sc. Norm. Super. Pisa 1 333–354 (part II).
[4] A.R. Gover, P. Nurowski, Obstructions to conformally Einstein metrics in n dimensions, J. Geom. Phys. 56 (2006) 450–484.
[5] C.L. Fefferman, Monge-Ampere equations, the Bergman kernel, and geometry of pseudoconvex domains, Ann. of Math. 103 (1976) 395–416; erratum Ann. of Math. 104 393–394.
[6] C.D. Hill, J. Lewandowski, P. Nurowski, Einstein equations and the embedding of 3-dimensional CR manifolds, math.DG, Indiana Univ. Math. J. (2009) (in press) arXiv: 0709.3660.
[7] S. Kobayashi, Transformation Groups in Differential Geometry, Springer, 1995.
[8] C. Kozameh, E.T. Newman, E.T. Nurowski, Conformal Einstein equations and Cartan conformal connection, Classical Quantum Gravity 20 (2003) 3029–3035.
[9] C. Kozameh, E.T. Newman, K.P. Tod, Conformal Einstein spaces, GRG 17 (1985) 343–352.
[10] D. Kramer, H. Stephani, M. MacCallum, E. Herlt, Exact Solutions of Einstein's Field Equations, Cambridge University Press, 1980. See also the second edition: H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, Exact Solutions of Einstein's Field Equations Cambridge University Press, 2003.
[11] J. Leroy, Un espace d'Einstein de type N a rayons non integrables, C. R. Acad. Sci. (Paris) A270 (1970) 1078–1080.
[12] J. Lewandowski, P. Nurowski, J. Tafel, Einstein's equations and realizibility of CR manifolds, Classical Quantum Gravity 7 (1990) L241–L246.
[13] C.W. Misner, Taub-NUT space as a counterexample to almost anything, in: J. Ehlers (Ed.), Relativity Theory and Astrophysics. Vol. 1 Relativity and Cosmology, in: Lectures in Applied Mathematics, vol. 8, AMS, 1967, p. 160.
[14] P. Nurowski, Einstein equations and Cauchy–Riemann geometry, Ph.D. Thesis, SISSA, 1993.
[15] P. Nurowski, J. Tafel, Symmetries of Cauchy–Riemann Spaces, Lett. Math. Phys. 15 (1988) 31–38.
[16] P. Nurowski, J.F. Plebanski, Nonvacuum twisting type N metrics, Classical Quantum Gravity 18 (2001) 341–351.
[17] P. Nurowski, Twisting type N vacuums with cosmological constant, J. Geom. Phys. 58 (2008) 615–618.