Class. Quantum Grav. 8 (1991) 493–501. Printed in the UK

Algebraically special solutions of the Einstein equations with pure radiation fields

Jerzy Lewandowski†, Paweł Nurowski‡ and Jacek Tafel†

† Institute of Theoretical Physics, Warsaw University, 00-681 Warsaw, Hoża 69, Poland
‡ International School for Advanced Studies, SISSA/ISAS-Trieste, Strada Costiera 11, 34014 Trieste, Italy

Received 21 June 1990, in final form 17 October 1990

Abstract. The Einstein equations $R_{\mu\nu}=\Phi k_\mu k_\nu$, k^μ being tangent to a twisting shear-free congruence of null geodesics, are formulated as equations in a three-dimensional Cauchy–Riemann space. If the NUT parameter M vanishes and the Cauchy–Riemann space is a hypersurface in C^2 then the equations reduce to a single linear second-order equation. New gravitational solutions are found for the case of the Robinson congruence.

1. Introduction

We study the Einstein equations with pure radiation fields

$$R_{\mu\nu}=\Phi k_\mu k_\nu \tag{1.1}$$

in a four-dimensional spacetime $\mathcal{M}$ with metric g of the signature $(+---)$. All considerations are local. We assume that spacetime admits a twisting shear-free congruence of null geodesics, which are integral lines of $k^\mu\partial_\mu$ (also for $\Phi=0$). Goldberg and Sachs proved [1] that the Weyl tensor of g is algebraically special and k^μ is its multiple null eigenvector. This property of g leads to significant simplifications of the Einstein equations. Many solutions of this type have been found, among them the celebrated Kerr solution (see [2] for a review and references).

Let r, x^i $(i=1,2,3)$ be coordinates such that ∂_r is tangent to the congruence. Locally $\mathcal{M}=\mathbb{R}\times\mathcal{N}$, where $\mathcal{N}$ is a submanifold of $\mathcal{M}$ defined by $r=\text{const}$. The metric tensor of $\mathcal{M}$ can be written in the form [3]

$$g=\kappa\omega-p\alpha\bar{\alpha} \tag{1.2}$$

where κ (real) and α (complex) are 1-forms on $\mathcal{N}$ (no dependence on r) and p, ω are a real function and real 1-form, respectively (r dependent in general). The form $\bar{\alpha}$ is the complex conjugate of α. The non-degeneracy of g requires

$$\kappa\wedge\omega\wedge\alpha\wedge\bar{\alpha}\neq 0.$$

The forms κ, α, $\bar{\alpha}$ constitute a basis of 1-forms on $\mathcal{N}$. The dual vector basis is given by ∂_0, ∂, $\bar{\partial}$. The assumption of a non-vanishing twist of the congruence is equivalent to

$$\kappa\wedge d\kappa\neq 0.$$

The forms κ and α are not uniquely defined by decomposition (1.2). We will restrict ourselves to κ such that

$$\sigma > 0 \tag{1.3}$$

where

$$\kappa \wedge \mathrm{d}\kappa = \mathrm{i}\sigma\kappa \wedge \alpha \wedge \bar{\alpha}.$$

Under condition (1.3) *the forms* κ, α *are defined up to the transformations*

$$\begin{aligned} \kappa' &= A\kappa \qquad A > 0 \\ \alpha' &= B\alpha + C\kappa \qquad B \neq 0 \end{aligned} \tag{1.4}$$

where the function A is real and B, C are complex.

A three-dimensional differential manifold with a pair (κ, α) of 1-forms (real and complex, respectively) satisfying $\omega \wedge \alpha \wedge \bar{\alpha} \neq 0$ and defined up to transformations (1.4) (with $A \neq 0$ rather than $A > 0$) is called the Cauchy–Riemann (CR) space (see [4] and references therein). It is said to be non-degenerate if $\kappa \wedge \mathrm{d}\kappa \neq 0$. Thus every twisting null geodesic shearfree congruence corresponds to a CR space [4]. From now on by a CR space we will always mean a non-degenerate CR space.

One says that a CR space is realizable (as a hypersurface in C^2) if the equation

$$\mathrm{d}\xi \wedge \kappa \wedge \alpha = 0 \tag{1.5}$$

or, equivalently,

$$\bar{\partial}\xi = 0 \tag{1.6}$$

for a complex function ξ has two independent solutions ξ_1, ξ_2. The functions ξ_1, ξ_2 exist, e.g. *when the* CR *structure is real analytic*. They define a three-dimensional surface $\tilde{\mathcal{N}}$ in C^2. And conversely, given a surface in C^2 defined by one real equation

$$R(\xi_1, \xi_2, \bar{\xi}_1, \bar{\xi}_2) = 0$$

one can impose a CR structure on it by taking [5]

$$\kappa = \mathrm{i}(\partial_{\xi_1} R \, \mathrm{d}\xi_1 + \partial_{\xi_2} R \, \mathrm{d}\xi_2) \qquad \alpha = \mathrm{d}\xi_1 \text{ or } \alpha = \mathrm{d}\xi_2.$$

In terms of ξ_1, ξ_2 the general solution of (1.5) is given by

$$\xi = f(\xi_1, \xi_2) \tag{1.7}$$

where f is a holomorphic function of two complex variables defined on one side (at least) of the surface $\tilde{N}$ [3].

With every representative (κ, α) of the CR structure one can connect the so-called Fefferman metric g_{F} (of the Lorentzian signature) on $\mathbb{R} \times \mathcal{N}$ [6, 7]. This metric transforms conformally

$$g'_{\mathrm{F}} = A g_{\mathrm{F}} \tag{1.8}$$

under transformation (1.4). It does not depend on a particular choice of α. To define g_{F} it is convenient to assume

$$\mathrm{d}\kappa = \mathrm{i}\alpha \wedge \bar{\alpha} \tag{1.9}$$

a condition which can be achieved by means of transformation (1.4) with $A = 1$. Under assumption (1.9) the Fefferman metric is given by

$$g_{\mathrm{F}} = 2\kappa[\mathrm{d}r + \tfrac{2}{3}\mathrm{i}\bar{c}\alpha - \tfrac{2}{3}\mathrm{i}c\bar{\alpha} - \tfrac{2}{3}(c_1 + \tfrac{1}{6}R_{\mathrm{F}})\kappa] - 2\alpha\bar{\alpha} \tag{1.10}$$

where

$$R_F = -\tfrac{3}{2}(\partial c + \overline{\partial c} - 2c\bar{c} + c_1) \tag{1.11}$$

and c_1 (real) and c (complex) are defined by

$$d\alpha = ic_1\kappa \wedge \alpha + ic_2\kappa \wedge \bar{\alpha} + c\alpha \wedge \bar{\alpha}. \tag{1.12}$$

The function R_F is the curvature scalar of g_F.

If ξ is a solution of (1.5) then $d\xi = B\alpha + C\kappa$ and the CR structure can be represented by κ and $d\xi$. Re ξ, Im ξ can be completed by u to form a coordinate system on $\mathcal{N}$ such that

$$\kappa = du + L\,d\xi + \bar{L}\,d\bar{\xi}. \tag{1.13}$$

In terms of these coordinates $\partial_0 = \partial_u$, $\partial = \partial_\xi - L\partial_u$ and

$$d\kappa = \partial_u L\kappa \wedge d\xi + \partial_u \bar{L}\kappa \wedge d\bar{\xi} + i\sigma\, d\xi \wedge d\bar{\xi} \tag{1.14}$$

where

$$\sigma = i(\bar{\partial} L - \partial \bar{L}). \tag{1.15}$$

Condition (1.9) is satisfied, in particular, by

$$\alpha = \sqrt{\sigma}\, d\xi - \frac{i}{\sqrt{\sigma}} \partial_u L\kappa. \tag{1.16}$$

Substituting (1.16) into (1.12) yields the functions c_i, c in terms of L and its derivatives. In consequence the Fefferman metric and its curvature scalar take the form [8]

$$g_F = 2\kappa(dr - \tfrac{1}{3}ib\, d\xi + \tfrac{1}{3}i\bar{b}\, d\bar{\xi} - \tfrac{2}{3}V\kappa) - 2\sigma\, d\xi\, d\bar{\xi} \tag{1.17}$$

$$R_F = \frac{3}{2\sigma}\,\mathrm{Re}(\partial\bar{\partial} \ln \sigma - 3\partial\partial_u \bar{L} + 2\partial_u L \partial_u \bar{L}) \tag{1.18}$$

where

$$b = \partial_u L + \partial \ln \sigma \qquad V = \frac{1}{4\sigma}\partial\bar{b}. \tag{1.19}$$

The functions σ and b can be equivalently defined by

$$d(\sigma^{-1}\kappa) = b\sigma^{-1}\kappa \wedge d\xi + \bar{b}\sigma^{-1}\kappa \wedge d\bar{\xi} + i\, d\xi \wedge d\bar{\xi}. \tag{1.20}$$

It follows from (1.20) that $b' = b$, $\sigma' = A\sigma$, $V' = A^{-1}V$ under the transformation $\kappa' = A\kappa$, $\xi' = \xi$.

2. Equations

The main purpose of this work is to reduce equations (1.1) to a form which is invariant under transformations (1.4). This problem was partially solved by Lewandowski and Nurowski [9]. In this paper we present another approach. To simplify calculations and to keep contact with classical work on the subject we start with partially solved equations (1.1) that can be found in textbooks (we follow [2]). In suitably chosen coordinates r, u, ξ, $\bar{\xi}$ these equations read

$$\partial(m + iM) = 3(m + iM)\partial_u L \tag{2.1}$$

$$P^2\,\mathrm{Re}[\partial\bar{\partial}\Sigma - 2\partial\Sigma\partial_u \bar{L} - \Sigma\partial_u\partial\bar{L} + 2\Sigma(\partial\bar{\partial} \ln P - \partial\partial_u \bar{L})] = M \tag{2.2}$$

where

$$\Sigma = -\tfrac{1}{2}\sigma P^2 \tag{2.3}$$

and all the unknown functions, m, M, P (all real, $P \neq 0$) and L, are independent of r. There is also a condition of the non-negativity of the energy density of matter ($\Phi \geqslant 0$ in (1.1))

$$\partial_u[P^{-3}(m+\mathrm{i}M)] + P(\partial - 2G)\partial(\bar{\partial}\bar{G} - \bar{G}^2) \leqslant 0 \tag{2.4}$$

where

$$G = \partial_u L - \partial \ln P.$$

(Note that the LHS of (2.4) is real due to (2.2).) Given L, m, M and P the metric tensor is defined by

$$g = 2\kappa(\mathrm{d}r + W\,\mathrm{d}\xi + \bar{W}\,\mathrm{d}\bar{\xi} + H\kappa) - 2P^{-2}(r^2+\Sigma^2)\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi} \tag{2.5}$$

where

$$W = -(r + \mathrm{i}\Sigma)\partial_u L + \mathrm{i}\partial\Sigma \tag{2.6}$$

$$H = -r\partial_u \ln P - (mr + M\Sigma)(r^2+\Sigma^2)^{-1} + P^2\,\mathrm{Re}[\partial(\bar{\partial}\ln P - \partial_u \bar{L})]. \tag{2.7}$$

Equations (2.1), (2.2) can be regarded as equations for m, M and P in a given CR geometry represented by L. They are invariant under the transformation

$$u' = F(u, \xi, \bar{\xi}) \qquad \xi' = h(\xi)$$

followed by an appropriate transformation of other variables. They are not invariant under the transformation $\xi \to \xi'$, where ξ' is an arbitrary solution of (1.5). This means that, given a CR structure, a class of solutions $h(\xi)$ of (1.5) is distinguished by the Einstein equations.

Equation (2.1) can be replaced by equation (1.5) for a complex function η related to m and M by the formula

$$m + \mathrm{i}M = (\partial_u \bar{\eta})^3. \tag{2.8}$$

Indeed, substituting (2.8) into (2.1) yields $\partial\partial_u\bar{\eta} = \partial_u L \partial_u \bar{\eta}$, hence $\partial_u \partial\bar{\eta} = 0$ in virtue of the identity $[\partial, \partial_u] = \partial_u L \partial_u$. We can modify η, without violating (2.8), to obtain

$$\bar{\partial}\eta = 0. \tag{2.9}$$

If the CR space is realized as a hypersurface in C^2 then the general solution for η is known (see (1.7)). In this case the only equation to be solved is equation (2.2) for P. The latter is not as non-linear as it seems. In terms of the variable $\mathscr{P}$ related to P by

$$\mathscr{P} = \sqrt{\sigma}\, P \tag{2.10}$$

(2.2) can be written in the form

$$\partial\bar{\partial}\mathscr{P} + \bar{\partial}\partial\mathscr{P} - \partial_u L \bar{\partial}\mathscr{P} - \partial_u \bar{L}\partial\mathscr{P} + \sigma V_1 \mathscr{P} = \tfrac{1}{2}\sigma M \mathscr{P}^{-3} \tag{2.11}$$

where

$$\sigma V_1 = -\tfrac{1}{4}(\partial\bar{\partial} + \bar{\partial}\partial)\ln\sigma + \tfrac{1}{2}\partial_u L \partial_u \bar{L} - \tfrac{3}{4}(\partial\partial_u \bar{L} + \bar{\partial}\partial_u L). \tag{2.12}$$

Equation (2.11) becomes linear with respect to $\mathscr{P}$ when $M = 0$. We will use this fact in the next section, where examples of solutions will be constructed.

Equation (2.11) resembles the Laplace equation in the complex plane. For instance, if $\partial_u \mathcal{P} = 0$, then the part of (2.11) with highest order derivatives is $2\partial_\xi \partial_{\bar\xi} \mathcal{P}$. For general $\partial_u \mathcal{P}$ equation (2.11) is equivalent to

$$\sigma_1 * \mathrm{d} * \mathrm{d}\mathcal{P} + \sigma V_1 \mathcal{P} = \tfrac{1}{2}\sigma M \mathcal{P}^{-3} \tag{2.13}$$

where the Hodge star corresponds to any metric g_1 on $\mathcal{M}$ of the form

$$g_1 = 2\kappa(\mathrm{d}r + W_1\,\mathrm{d}\xi + \bar W_1\,\mathrm{d}\bar\xi + H_1\kappa) - 2\sigma_1\,\mathrm{d}\xi\,\mathrm{d}\bar\xi \qquad \partial_r W_1 = 0 = \partial_r \sigma_1 .$$

It is convenient to use the Fefferman metric (1.17) in place of g_1. It follows from (1.18), (1.19) and (2.12) that

$$V_1 = \tfrac{1}{6}R_{\mathrm{F}} - 3V \tag{2.14}$$

hence equation (2.13) reads

$$\underset{\mathrm{F}}{*}\,\mathrm{d}\underset{\mathrm{F}}{*}\,\mathrm{d}\mathcal{P} + (\tfrac{1}{6}R_{\mathrm{F}} - 3V)\mathcal{P} = \tfrac{1}{2}M\mathcal{P}^{-3} \tag{2.15}$$

where $\underset{\mathrm{F}}{*}$ denotes the Hodge star with respect to g_{F}. Unlike (2.2) equation (2.15) is invariant under transformation (1.4). This is due to (1.8) and the following transformation properties of the other variables

$$\mathcal{P}' = (\sqrt{A})^{-1}\mathcal{P} \qquad V' = A^{-1}V \qquad M' = A^{-3}M. \tag{2.16}$$

A price paid for this invariance is the change of the character of ξ. In (2.2) it was an *independent* variable and in (2.15) it is a *dependent* variable (constrained by equation (1.5)). The potential V depends essentially on ξ.

Equations (2.1), (2.15) and the metric tensor (2.5) can be written in any representation (κ, α) of the CR structure. We will do it for (κ, α) satisfying condition (1.9). Let ξ be a distinguished solution of (1.5). We introduce variables q, q_0 related to ξ by

$$\alpha = q_0\,\mathrm{d}\xi + \mathrm{i}q\kappa. \tag{2.17}$$

It follows from (2.17) and (1.14) that

$$\partial_u L = -\bar q q_0 \qquad \sigma = q_0 \bar q_0 . \tag{2.18}$$

Substituting these relations into (1.19) and using the exterior derivative of (2.17) yields

$$V = \tfrac{1}{6}R_{\mathrm{F}} + V_2 \qquad V_2 = \mathrm{Re}(\bar c q - \partial q) + \tfrac{1}{2}q\bar q \tag{2.19}$$

where now ∂ corresponds to (κ, α) and c is defined by (1.12). With V_2 defined in this way equation (2.15) reads

$$[\partial\bar\partial + \bar\partial\partial - c\partial - \overline{c\partial} - \tfrac{1}{3}R_{\mathrm{F}} - 3V_2]\mathcal{P} = \tfrac{1}{2}M\mathcal{P}^{-3}. \tag{2.20}$$

In a similar way one shows that equation (2.1) is equivalent to

$$\partial(m + \mathrm{i}M) = -3\bar q(m + \mathrm{i}M) \tag{2.21}$$

or (in virtue of (2.8) and (2.9))

$$m - \mathrm{i}M = (\partial_0\eta + \mathrm{i}q\partial\eta)^3 \qquad \bar\partial\eta = 0 \tag{2.22}$$

and that the metric tensor takes the form

$$g = 2\kappa(\mathrm{d}r + \hat W\alpha + \bar{\hat W}\bar\alpha + \hat H\kappa) - 2\mathcal{P}^{-2}(r^2 + \tfrac{1}{4}\mathcal{P}^4)(\alpha - \mathrm{i}q\kappa)(\bar\alpha + \mathrm{i}\bar q\kappa) \tag{2.23}$$

where

$$\hat{W}=\bar{q}(r-\tfrac{1}{2}\mathrm{i}\mathscr{P}^2)-\mathrm{i}\mathscr{P}\partial\mathscr{P} \tag{2.24}$$

$$\hat{H}=-r\partial_u \ln P+\frac{\frac{1}{2}M\mathscr{P}^2-mr}{r^2+\frac{1}{4}\mathscr{P}^4}+\frac{M}{4\mathscr{P}^2}-\mathscr{P}^2(|\partial\ln\mathscr{P}+\tfrac{3}{2}\bar{q}|^2+V-q\bar{q}) \tag{2.25}$$

$$\partial_u \ln P=\partial_0 \ln\mathscr{P}+\mathrm{Im}(-2q\partial\ln\mathscr{P}+\partial q-\bar{c}q). \tag{2.26}$$

Equations (2.20)-(2.26) depend on ξ only via the function q. It follows from (2.17) that q satisfies

$$\bar{\partial}q+q^2+cq=-c_2. \tag{2.27}$$

And conversely, given a solution q of (2.27) one can find ξ and q_0 such that (2.17) and (1.5) are satisfied. Thus one can use either ξ or q to describe the metric and the field equations.

Theorem. The Einstein equations (1.1) with k^μ being a twisting shear-free null geodesic vector field reduce to equations (2.20), (2.21), (2.27) for functions q, m, M, $\mathscr{P}$ in a CR space represented by (κ, α) satisfying condition (1.9). The metric tensor is given by (2.23)-(2.26).

An advantage of equations (2.20), (2.21), (2.27) is that we can write them in the simplest possible coordinates describing the CR structure whereas dependence of this structure on u, ξ, $\bar{\xi}$, where ξ is a fixed solution of (1.5), can be complicated even for highly symmetric congruences. Given a solution to these equations condition (2.4) should be verified. To do this one should first calculate P and $\partial_u L$ from (2.10), (2.18) and to find, using (2.17), a transformation between the derivatives ∂_0, ∂ related to κ, $\mathrm{d}\xi$ on one hand and κ, α on the other hand. We have not managed to write condition (2.4) in terms of q, m, M, $\mathscr{P}$ in a digestible form. Fortunately, if $M=0$ and $\partial_u(mP^{-3})\neq 0$ (hence $m\neq 0$) then (2.4) can always be satisfied by a suitable choice of m (note that for $M=0$ equations (2.20), (2.21) are invariant under the multiplication of m by a constant). We will use this fact in the next section.

3. Symmetric congruences and examples of solutions

There is no problem in finding general forms of ξ (or q), m, M when the CR space is defined by a hypersurface in C^2. In order to simplify equation (2.20) for $\mathscr{P}$ we assume in this section that

$$M=0. \tag{3.1}$$

It follows from (3.1) and (2.1) that $m=0$ (type III or N) or the CR structure has a continuous symmetry preserving ξ. Indeed, if $M=0$ and $m\neq 0$ then we can use transformation (1.4) to obtain $m=\text{const}$. Equation (2.1) shows that $\partial_u L=0$ in that gauge, i.e. the transformed form κ is invariant under translations in u.

Let the CR structure admit a non-trivial Lie group of symmetries [5, 10]. Then it can be represented by $\mathrm{d}\xi'$ and $\kappa'=\mathrm{d}u'+L'\,\mathrm{d}\xi'+\bar{L}'\,\mathrm{d}\bar{\xi}'$, where

$$L'=-\mathrm{i}\partial_{\xi'}s \qquad \partial_{u'}s=0 \tag{3.2}$$

and s is a real function such that

$$\sigma' = 2\partial_{\bar{\xi}'}\partial_{\xi'} s > 0.$$

A general solution of equation (1.5) is given by $\xi = f(\xi_1, \xi_2)$, where

$$\xi_1 = \xi' \qquad \xi_2 = u' + \mathrm{i}s. \tag{3.3}$$

It follows from (2.17) with $\kappa = \kappa'$ and $\alpha = \sqrt{\sigma'}\,\mathrm{d}\xi'$ that

$$q = \mathrm{i}\sqrt{\sigma'}\,(\partial_{\xi_2} f)(\partial_{\xi_1} f - 2L'\partial_{\xi_2} f)^{-1}. \tag{3.4}$$

Equation (2.20) is invariant under translations of u', hence easier to solve, when q is independent of u'. This is the case when

$$\xi = \xi' \tag{3.5}$$

or

$$\xi = u' + \mathrm{i}s + h(\xi') \tag{3.6}$$

where h is a holomorphic function of ξ'. In the first case we can have $m \neq 0$, in the latter this is possible only when the symmetry group is at least two-dimensional.

If a CR space admits an Abelian two-dimensional symmetry group [10] then we can assume that

$$L' = -\tfrac{1}{2}\partial_{y'} s \qquad s = s(y') \qquad \sigma' = \tfrac{1}{2}\partial^2_{y'} s > 0 \tag{3.7}$$

where $\xi' = x' + \mathrm{i}y'$. There are two cases for which $m \neq 0$ and q is independent of u', namely

$$\xi = \xi' \qquad q = 0 \qquad m = m_0 \tag{3.8}$$

and

$$\xi = u' + \mathrm{i}s + 2a\xi' \qquad q = \frac{\mathrm{i}\sqrt{\sigma'}}{2(a - L')} \qquad m = \frac{m_0}{(L' - a)^3} \tag{3.9}$$

where m_0 and a are real constants.

In the case (3.9)

$$\kappa' = \tfrac{1}{2}|L' - a|\kappa \qquad \kappa = \mathrm{d}u + 2L(y)\,\mathrm{d}x \tag{3.10}$$

where

$$u = 4x'\,\mathrm{sgn}(L' - a) \qquad L = \frac{1}{|L' - a|}. \tag{3.11}$$

In principle one can consider equation (2.20) in the representation $(\kappa, \sqrt{\sigma}\,\mathrm{d}\xi)$ (then $q = 0$) but to find an explicit dependence of L on y one has to solve the equation

$$y = s(y') + 2ay' \tag{3.12}$$

with respect to y'. This is often not possible in practice. In these cases it is more convenient to use the original coordinates u', ξ', $\bar{\xi}'$.

For ξ given by (3.8) or (3.9) equation (2.20) is independent of u' and x':

$$\partial^2_{y'}\mathscr{P}' + (\partial_{x'} - 2L'\partial_{u'})^2\mathscr{P}' + [-\tfrac{1}{4}\partial^2_{y'}\ln(\partial_{y'}L') + \tfrac{3}{2}\varepsilon\partial^2_{y'}\ln(L' - a) - \tfrac{3}{4}\varepsilon(\partial_{y'}L')^2(L' - a)^{-2}]\mathscr{P}' = 0 \tag{3.13}$$

where ε equals 0 or 1 for the case (3.8) or (3.9), respectively. In some cases equation (3.13) can be reduced to certain well known equations of mathematical physics. For instance let us consider the CR structure related to the Robinson congruence [11]. This can be represented by $d\xi'$ and

$$\kappa' = du' + (2/y')\,dx'. \tag{3.14}$$

(The standard representation of this structure is given by $d\xi''$ and $\kappa'' = du'' + 2y''\,dx''$. The primed and double primed coordinates are related by the transformation $\xi'' = \sqrt{y'}\exp(\frac{1}{4}iu'), 2u'' = -x' + y'\sin(\frac{1}{2}u')$.) If $\varepsilon = 0$ or $\varepsilon = 1$, $a = 0$ then we can easily transform to the coordinates u, ξ, $\bar{\xi}$. Here we focus on the case (3.9) with $a \neq 0$. In this case we can assume $a = 1$ without loss of generality due to the transformation $\xi' \to a^{-1}\xi'$ preserving κ'. Then

$$\xi = u' - 2i\ln y' + 2\xi' \qquad q = \frac{i}{2(y'-1)} \qquad m = \frac{-m_0 y'^3}{(y'-1)^3}. \tag{3.15}$$

Let us assume that $\mathcal{P}'$ is independent of x' and that it admits the Fourier transform with respect to u',

$$\mathcal{P}' = \int dk\,\exp(iku')\,Y(k, y'). \tag{3.16}$$

Substituting (3.16) into (3.13) with $L' = y'^{-1}$ and $a = 1$ yields the following equation for Y

$$\ddot{Y} + \left(\frac{3}{2y'(y'-1)} + \frac{1-16k^2}{4y'^2} - \frac{9}{4(y'-1)^2}\right) Y = 0 \tag{3.17}$$

where a dot denotes the differentiation with respect to y'. Equation (3.17) becomes the hypergeometric equation

$$y'(1-y')\ddot{F} + [4k+1-(4k+1+2k_0)y']\dot{F} - [(4k+1)k_0 + \tfrac{3}{2}]F = 0 \tag{3.18}$$

under the substitution

$$Y = |y'|^{2k+1/2}|y'-1|^{k_0}F \qquad k_0 = \tfrac{1}{2}(1 \pm \sqrt{10}). \tag{3.19}$$

It follows from (3.16)-(3.19) that

$$\mathcal{P}' = |y'|^{1/2}|y'-1|^{k_0}\,\mathrm{Re}\int dk\,e^{iku'}|y'|^{2k}A(k)F(k, y') \tag{3.20}$$

where $A(k)$ is a free complex function and $F(k, y')$ is any solution of (3.18) such that $y'^{2k}F(k, y')$ and $y'^{-2k}\bar{F}(-k, y')$ are linearly independent. For instance, when $y' \in (0, 1)$ one can take for $F(k, y')$ the hypergeometric function $F(a, b, c; y')$ with

$$a = 2k + k_0 + (4k^2 + \tfrac{3}{4})^{1/2} \qquad b = 2k + k_0 - (4k^2 + \tfrac{3}{4})^{1/2} \qquad c = 4k + 1. \tag{3.21}$$

Formulae (3.1), (3.15), (3.20) define all the functions needed to obtain the metric tensor. For sufficiently large m_0 the metric satisfies the Einstein equations with a non-negative energy density.

As an example we will calculate constituents of the metric tensor in a relatively simple case when

$$\mathcal{P}' = y'^{-3/4k_0}(y'-1)^{k_0}\sin(k_1 u') \qquad k_1 = \tfrac{1}{12}(2k_0 + 1). \tag{3.22}$$

Then the metric is defined by (2.23), where $q=\frac{1}{2}\mathrm{i}(y'-1)^{-1}$, $\kappa=\kappa'$, $\alpha=y'^{-1}\,\mathrm{d}\xi'$ and

$$\hat{W}=-\frac{\mathrm{i}r}{2(y'-1)}+k_1\mathcal{P}'^2\left(\mathrm{i}\cot(k_1u')-\frac{3}{(y'-1)}-2\right) \tag{3.23}$$

$$\hat{H}=-\frac{m_0ry'^3}{(r^2+\frac{1}{4}\mathcal{P}^4)(y'-1)^3}-\frac{k_1ry'}{y'-1}\cot(k_1u')$$
$$-\mathcal{P}'\left(k_1^2\cot^2(k_1u')+\frac{k_0+\frac{5}{4}}{(y'-1)^2}+\frac{k_0+\frac{3}{2}}{y'-1}+4k_1^2+\frac{1}{8}\right). \tag{3.24}$$

4. Conclusions

We have shown that the Einstein equations $R_{\mu\nu}=\Phi k_\mu k_\nu$ with k^μ being a twisting shear-free geodesic null vector field can be considered as equations in an arbitrarily chosen non-degenerate Cauchy–Riemann geometry. The equations have been written in various forms. For instance, a complete set of equations consists of equations (1.5), (2.9), (2.15) for ξ, η and $\mathcal{P}$. Another set is given by equations (2.20), (2.21), (2.27) for q, m, M and $\mathcal{P}$. All the equations of the first system become linear when $M=0$. If the CR space is realized as a hypersurface in C^2 then the general solution of equations (1.5), (2.9) (or (2.21), (2.27)) is known. The remaining equation (2.15) (in version (2.20)) was considered in more detail for symmetric congruences and $M=0$. Under some assumptions on ξ and $\mathcal{P}$ it was reduced to a linear second-order ordinary differential equation (section 3). Using this formulation of the Einstein equations we have found new gravitational solutions related to the Robinson congruence.

Acknowledgments

We are grateful to Professor I Robinson and Professor A Trautman for stimulating discussions. This work was partially supported by the Polish Research Program CPBP 01.03.

References

[1] Goldberg J N and Sachs R K 1962 *Acta Phys. Polon. Suppl.* **22** 13–8
[2] Kramer D, Stephani H, MacCallum M and Herlt E 1980 *Exact Solutions of Einstein's Field Equations* (Berlin: Deutscher)
[3] Tafel J 1985 *Lett. Math. Phys.* **10** 33–9
[4] Robinson I and Trautman A 1985 Cauchy–Riemann structures in optical geometry *Proc. 4th Marcel Grossmann Meeting on General Relativity* ed R Ruffini (Amsterdam: Elsevier); 1989 Optical geometry *New Theories in Physics* ed Z Ajduk, S Pokorski and A Trautman (Singapore: World Scientific) pp 454–97
[5] Cartan E 1932 *Ann. Math. Pura Appl.* **11** 17–90
[6] Fefferman C L 1976 *Ann. Math.* **103** 395–416
[7] Burns D Jr, Diederich K and Shnider S 1977 *Duke Math. J.* **44** 407–31
[8] Lewandowski J 1988 *Lett. Math. Phys.* **15** 129–35
[9] Lewandowski J and Nurowski P 1990 *Class. Quantum Grav.* **7** 309–28
[10] Nurowski P and Tafel J 1988 *Lett. Math. Phys.* **15** 31–8
[11] Penrose R 1987 On the origins of twistor theory *Gravitation and Geometry* ed W Rindler and A Trautman (Naples: Bibliopolis) pp 341–61