LETTER TO THE EDITOR

Pure radiation field solutions of the Einstein equations

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Abstract. Solutions of the Einstein equations with pure radiation fields are obtained in a class of algebraically special metrics related to the Cauchy-Riemann spaces admitting a group of symmetries of Bianchi type VI_h.

In [1] we formulated the Einstein equations

$$R_{\mu\nu} = \Phi k_\mu k_\nu \tag{1}$$

in the case when the vector field $k^\mu \partial_\mu$ was tangent to a shearfree and twisting congruence of null geodesics, as equations in an arbitrary three-dimensional Cauchy-Riemann (CR) space. In particular, we studied these equations for symmetric CR spaces and we found certain solutions of them in the case of the Robinson congruence. In this letter we obtain solutions of (1) for a family of CR spaces with a three-dimensional group of symmetries of Bianchi type VI_h. All considerations are local.

By a CR space we mean a three-dimensional differential manifold $\mathcal{N}$ with a pair (κ, α) of 1-forms, real and complex, respectively, defined up to the transformations

$$\kappa' = A\kappa \qquad A > 0 \tag{2}$$

$$\alpha' = B\alpha + C\kappa \qquad B \neq 0$$

and such that

$$\kappa \wedge \alpha \wedge \bar{\alpha} \neq 0.$$

The vector basis dual to $(\kappa, \alpha, \bar{\alpha})$ will be denoted by $(\partial_0, \partial, \tilde{\partial})$. We assume that

$$\sigma > 0 \tag{3}$$

where σ is defined by

$$\kappa \wedge d\kappa = i\sigma \kappa \wedge \alpha \wedge \bar{\alpha}.$$

Given κ it is convenient to choose α in such a way that

$$d\kappa = i\alpha \wedge \bar{\alpha}. \tag{4}$$

It was shown in [1] that equations (1), k^μ being null, twisting, shearfree and geodesic, reduce to the following system of equations for a complex function ξ and real functions m, M, $\mathscr{P}$ on a CR space $\mathcal{N}$ with forms κ, α satisfying condition (4):

$$\bar{\partial}\xi = 0 \tag{5}$$

$$\partial(m + iM) = -3\bar{q}(m + iM) \tag{6}$$

$$[\partial\bar{\partial} + \bar{\partial}\partial - c\partial - \overline{c\partial} - \tfrac{1}{3}R_F + 3\,\mathrm{Re}(\partial q - \bar{c}q) - \tfrac{3}{2}q\bar{q}]\mathscr{P} = \tfrac{1}{2}M\mathscr{P}^{-3} \tag{7}$$

where

$$R_F = -\tfrac{3}{2}(\partial c + \bar{\partial}\bar{c} - 2c\bar{c} + c_1)$$

$$d\alpha = ic_1\kappa \wedge \alpha + ic_2\kappa \wedge \bar{\alpha} + c\alpha \wedge \bar{\alpha}$$

and q is defined by ξ via

$$\alpha = q_0\,d\xi + iq\kappa. \tag{8}$$

The functions ξ, m, M, $\mathscr{P}$ change in the following way under transformation (2)

$$\xi' = \xi \qquad \mathscr{P}' = A^{-1/2}\mathscr{P} \qquad m' + iM' = A^{-3}(m + iM). \tag{9}$$

Spacetime $\mathcal{M}$ is (locally) a product of a real line by $\mathcal{N}$, $\mathcal{M} = \mathbb{R} \times \mathcal{N}$. To define the metric tensor g of $\mathcal{M}$ in terms of the functions ξ, m, M and $\mathscr{P}$ it is useful to find functions u (real) and L (complex) such that

$$\kappa = du + L\,d\xi + \bar{L}\,d\bar{\xi}.$$

The variables u, $\mathrm{Re}\,\xi$, $\mathrm{Im}\,\xi$ completed by a coordinate r in $\mathbb{R}$ constitute a system of coordinates in $\mathcal{M}$. For suitably chosen r the metric g reads (see e.g. [2])

$$g = 2\kappa(dr + W\,d\xi + \bar{W}\,d\bar{\xi} + H\kappa) - 2P^{-2}(r^2 + \Sigma^2)\,d\xi\,d\bar{\xi} \tag{10}$$

where

$$P = |q_0|^{-1}\mathscr{P} \qquad \Sigma = -\tfrac{1}{2}\mathscr{P}^2$$

$$W = -(r + i\Sigma)\partial_0 L + i\partial\Sigma$$

$$H = -r\partial_0 \ln P - (mr + M\Sigma)(r^2 + \Sigma^2)^{-1} + P^2\,\mathrm{Re}[\partial(\bar{\partial}\ln P - \partial_0\bar{L})]$$

and ∂_0, ∂, $\bar{\partial}$ correspond to the representation $(\kappa, d\xi)$ of the CR structure (hence $\partial_0 = \partial_u$, $\partial = \partial_\xi - L\partial_u$). Metric (10) satisfies equations (1) with indefinite sign of Φ. The energy density is non-negative if

$$\partial_u[P^{-3}(m + iM)] + P(\partial - 2G)\partial(\bar{\partial}\bar{G} - \bar{G}^2) \leq 0 \tag{11}$$

where $G = \partial_u L - \partial \ln P$. This inequality can always be satisfied (for certain modified m) if

$$M = 0 \tag{12}$$

and

$$\partial_u(mP^{-3}) \neq 0 \tag{13}$$

since then one can change a sign of the LHS of (11) by multiplying m by an appropriate constant. Assumption (12) is also very convenient from the point of view of solving equations (5)-(7), since then all these equations are linear. Conditions (6), (12), (13) are compatible only when the CR structure admits a Lie group of symmetries [1]. In

this paper we will assume that it admits a three-dimensional symmetry group of Bianchi type VI_h (including type VI_0 and III). Such a CR structure can be represented by $\alpha' = d\xi'$ and [3]

$$\kappa' = du' - 2(n-1)y'^{n-1}\,dx' \qquad n \neq 1 \tag{14}$$

where $\xi' = x' + iy'$, $y' > 0$ and n is a real constant (we introduce the factor $(n-1)$ in (14) in order to satisfy (3)). Let $M = 0$ and

$$\xi = \xi' \tag{15}$$

or

$$\xi = \begin{cases} \dfrac{n}{2(n-1)}u' + iy'^n & \text{for } n \neq 0 \\[2mm] -\tfrac{1}{2}u' + i\ln y' & \text{for } n = 0. \end{cases} \tag{16}$$

(This choice of ξ guarantees substantial simplifications of equation (7). See the discussion preceding equation (3.12) in [1].) It follows from (6) that $m = \text{const}$ or $m = \text{const } y'^{3-3n}$, respectively. Equation (7) reads

$$\partial^2_{y'}\mathscr{P}' + [\partial_{x'} + 2(n-1)y'^{n-1}\partial_{u'}]^2\mathscr{P}' + ay'^{-2}\mathscr{P}' = 0 \tag{17}$$

where

$$a = \tfrac{1}{4}(n-2) - \tfrac{3}{4}\varepsilon(n^2-1)$$

and $\varepsilon = 0$ or $\varepsilon = 1$ in the case (15) or (16), respectively. Given a solution of (17) for $\varepsilon = 0$ we can easily calculate the metric tensor (10). In the case (16) it is convenient to make first the transformation

$$(\kappa', d\xi') \rightarrow (\kappa, d\xi) \tag{18}$$

where

$$\kappa = y'^{1-n}\kappa' = du + 2L\,dx \tag{19}$$
$$L = \begin{cases} (1-\hat{n})y^{\hat{n}-1} & \hat{n} = 1/n & \text{for } n \neq 0 \\ -e^y & & \text{for } n = 0 \end{cases}$$

and

$$u = 2(1-n)x' \qquad \xi = x + iy.$$

For $n \neq 0$ this transformation has a peculiar property of merely changing n into $1/n$ in the expression for κ'. Form (19) for $n = 0$ coincides with (14) with $n = 2$ under the subsitution

$$\xi = 2i\ln\xi' \qquad u = u' - x'y'.$$

Thus we could restrict ourselves to the case (15) from the beginning. However, considering both cases has some advantages; they are described by the following lemma.

Lemma. If $n \neq 0$ then the transformation

$$\mathscr{P}'(u', x', y') \rightarrow y'^{(1-\hat{n})/2}\mathscr{P}'\left(2(1-\hat{n})x', \frac{\hat{n}}{2(\hat{n}-1)}u', y'^{\hat{n}}\right) \tag{20}$$

transforms solutions of equation (17) into solutions of (17) with n, ε replaced by $\hat{n} = 1/n$ and $1 - \varepsilon$, respectively. Equation (17) for $n = 0$ is equivalent to the equation

$$\partial_y^2 \mathscr{P} + (\partial_x + 2\,e^y\partial_u)^2\mathscr{P} + \tfrac{3}{4}(\varepsilon - 1)\mathscr{P} = 0 \tag{21}$$

under the substitution

$$\mathscr{P}' = \sqrt{y'}\,\mathscr{P} \qquad u = 2x' \qquad x = -\tfrac{1}{2}u' \qquad y = \ln y'.$$

Under assumption

$$\partial_{u'}\mathscr{P}' = 0 \tag{22}$$

equation (17) (with primes omitted) becomes

$$\Delta\mathscr{P} + ay^{-2}\mathscr{P} = 0 \tag{23}$$

where Δ is the two-dimensional Laplace operator, $\Delta = 4\partial_\xi\partial_{\bar\xi}$. Equation (23) is the Helmholtz equation

$$*\,d*d\mathscr{P} + a\mathscr{P} = 0$$

on a pseudosphere with the metric $y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$. This metric is invariant under the following action of the group $SL(2,\mathbb{R})$

$$L_h : \xi \to \frac{h_{11}\xi + h_{12}}{h_{21}\xi + h_{22}} \qquad h = (h_{ij}) \in SL(2,\mathbb{R}). \tag{24}$$

Hence the transformation

$$\mathscr{P} \to L_h^*\mathscr{P} \tag{25}$$

can be used to generate new solutions of (23) from a given one.

For $a = 0$ a general solution of (23) is given by

$$\mathrm{Re}\,f(x + iy) \tag{26}$$

where f is a holomorphic function of ξ. This is the case when $\varepsilon = 0$, $n = 2$ or $\varepsilon = 1$, $3n^2 - n - 1 = 0$. For $a \neq 0$ equation (23) can be separated in various systems of coordinates. Separation of x and y leads to complex solutions of the form

$$\sqrt{y}\,e^{px}(A_1 J_\nu(py) + A_2 Y_\nu(py)) \qquad (A_3 x + A_4)y^{1/2 \pm \nu} \tag{27}$$

where p and A_i are complex constants

$$\nu = (\tfrac{1}{4} - a)^{1/2}$$

(one of the roots) and $J_\nu(z)$, $Y_\nu(z)$ are two independent solutions of the Bessel equation [4]

$$z^2\partial_z^2 J + z\partial_z J + (z^2 - \nu^2)J = 0.$$

All the functions given by (27) become elementary when ν equals half an integer.

To get real solutions of (23) one can take e.g.

$$\mathscr{P} = \sqrt{y}\,\mathrm{Re}\int \mathrm{d}p_1 \int \mathrm{d}p_2\,e^{px}[A(p_1,p_2)J_\nu(py) + B(p_1,p_2)Y_\nu(py)] \tag{28}$$

where $p = p_1 + ip_2$ and A, B are functions or distributions such that the RHS of (28) is well defined. The arbitrariness of functions of two variables in (28) is superficial. For instance if $\mathcal{P}$ admits the Fourier transform with respect to x it is necessarily of the form

$$\mathcal{P} = \sqrt{y}\,\operatorname{Re}\int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}tx}[\hat{A}(t)I_\nu(ty) + \hat{B}(t)K_\nu(ty)] \tag{29}$$

where $I_\nu(z)$, $K_\nu(z)$ are independent solutions of the modified Bessel equation

$$z^2\partial_z^2 J + z\partial_z J - (z^2 + \nu^2)J = 0.$$

Formula (29) can be obtained from (28) by taking A and B proportional to the Dirac distribution $\delta(p_1)$.

The following solutions of (17), for $n \neq 0$, are generated from (27) and (26) in virtue of the lemma:

$$\sqrt{y}\,\mathrm{e}^{p\hat{u}}[A_1 J_{\nu/n}(py^n) + A_2 Y_{\nu/n}(py^n)] \qquad (A_3 u + A_4)y^{1/2 \pm \nu} \tag{30}$$

$$y^{(1-n)/2}f(\hat{u} + \mathrm{i}y^n) \qquad (\text{only for } n = \tfrac{1}{2}(-1 \pm \sqrt{13})) \tag{31}$$

where

$$\hat{u} = \frac{n}{2(n-1)}\,u.$$

Also for $n = \varepsilon = 0$ the lemma can be applied to obtain u-dependent solutions of (17). For $\varepsilon = 0$ and assumption (22) satisfied equation (21) becomes the Helmholtz equation in plane. Hence the function $\mathcal{P}$, for $n = \varepsilon = 0$, can take the form

$$\sqrt{y}\,f(u, \tau) \tag{32}$$

where $\tau = 2\ln y$ and

$$\partial_u^2 f + \partial_\tau^2 f = \tfrac{3}{16}f. \tag{33}$$

It is possible to separate variables in equation (33) in many different ways [5]. For instance separation of u and τ shows that $\mathcal{P}$ can be a linear superposition of the functions

$$uy^{(1\pm\sqrt{3})/2} \qquad \sqrt{y}\,\ln y\,\mathrm{e}^{\pm\sqrt{3}\,u/4} \qquad y^{2k+1/2}\,\mathrm{e}^{pu} \tag{34}$$

where $k^2 + p^2 = \tfrac{3}{16}$.

Let us summarize the obtained results so far in the following proposition.

Proposition. The Einstein equations (1) are satisfied by the metric tensor (10), where $L = (1-n)y^{n-1}$ $(n \neq 1)$, $M = 0$, $m = \text{const}$, $q_0 = (n-1)y^{n/2-1}$ and $\mathcal{P}$ is a real linear superposition of the functions given by (27) ((26) if $n = 2$) and (30) ((34) if $n = 0$, (31) if $n = \tfrac{1}{2}(-1 \pm \sqrt{13})$) with $\nu = \tfrac{1}{2}\sqrt{3-n}$. If $\partial_u\mathcal{P} \neq 0$ then m can be chosen in such a way that the energy density of matter is non-negative $(\phi \geq 0)$.

The space of admissible $\mathcal{P}$ can be easily extended by means of symmetries (24) possibly followed by transformation (20). Moreover a separation of variables in (23) can be done in systems of coordinates different from (x, y) and coordinates related to the latter via (24). For instance equation (17) has solutions of the form $Q(\phi)\rho^\mu$ and $Q_0(\phi)\ln\rho$, where

$$x = \rho\cos\phi \qquad y = \rho\sin\phi$$

Q satisfies

$$\partial_\phi^2 Q + (\mu^2 + a \sin^{-2} \phi)Q = 0 \tag{35}$$

and Q_0 satisfies equation (35) with $\mu = 0$. Equation (35) is equivalent to the Legendre equation

$$(1-z^2)\partial_z^2 Q - 2z\partial_z Q - \left(a + \frac{\mu^2}{1-z^2}\right)Q = 0$$

under the substitution $z = \mathrm{i} \cot \phi$. Hence we obtain the following solutions of (17) in terms of the Legendre functions P_β^μ, Q_β^μ [4]:

$$\begin{aligned}
&|\xi|^\mu (A_1 P_{\nu-1/2}^\mu(\mathrm{i}x/y) + A_2 Q_{\nu-1/2}^\mu(\mathrm{i}x/y)) \\
&\ln |\xi|(A_3 P_{\nu-1/2}^0(\mathrm{i}x/y) + A_4 Q_{\nu-1/2}^0(\mathrm{i}x/y)).
\end{aligned} \tag{36}$$

Transformations (25) and (20) can be applied to (36) in order to obtain further solutions of (17). In this way one can obtain, e.g.

$$\mathscr{P} = y^{(1-n)/2}(\hat{u}^2 + y^{2n})^{\mu/2}(A_1 P_{\nu/n-1/2}^\mu(\mathrm{i}\hat{u}y^{-n}) + A_2 Q_{\nu/n-1/2}^\mu(\mathrm{i}\hat{u}y^{-n})) \tag{37}$$

a solution which is expressible in terms of elementary functions for some values of n and μ. For $n = 2$ (the Robinson congruence) the function $\mathscr{P}$ (not elementary) given by (37) with $\mu = 1$ corresponds to the Hauser vacuum metric [2].

We do not know whether equation (17), for general n, admits solutions corresponding to vacuum metrics. It may happen that congruences given by (14) are not admitted by the vacuum Einstein equations. Thus our approach to (1) cannot be considered as a practical realization of the technique of Stephani (see [2], p 283) of generating pure radiation solutions from vacuum ones.

References

[1] Lewandowski J, Nurowski P and Tafel J 1990 Algebraically special solutions of the Einstein equations with pure radiation fields 1991 *Class. Quantum Grav.* **8** 493–501
[2] Kramer D, Stephani H, MacCallum M and Herlt E 1980 *Exact Solutions of Einstein's Field Equations* (Berlin: Deutscher)
[3] Nurowski P and Tafel J 1988 Symmetries of Cauchy–Riemann Spaces *Lett. Math. Phys.* **15** 31–8
[4] Abramowitz M and Stegun I A (ed) 1964 *Handbook of Mathematical Functions* (National Bureau of Standards, Applied Mathematics Series 55) (Washington, DC: US Govt Printing Office)
[5] Miller W Jr 1977 *Symmetry and Separation of Variables* (London: Addison-Wesley)