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New algebraically special solutions of the Einstein–Maxwell equations

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Abstract. New algebraically special solutions of the Einstein-Maxwell equations are constructed. Among them are the first examples of solutions with twisting rays and a purely radiative Maxwell field.

1. Introduction

In recent papers with Lewandowski [1-3] we studied Einstein's equations for pure radiation $R_{\mu\nu} = k_\mu k_\nu$, where k_μ is a vector field tangent to a twisting shear-free congruence of null geodesics in spacetime. We showed that these equations reduce to a single linear second-order equation when the NUT parameter M vanishes. Solving this linear equation led us to new solutions of the Einstein equations. In this paper we consider algebraically special solutions of the Einstein–Maxwell equations. We construct the first twisting solutions of Petrov type II which are not just charged vacuums. Also the first examples of twisting Petrov type II and III solutions with pure radiation Maxwell fields are presented. Thus the 'no go' theorem of Debever *et al* [4] (see also Wils [5]) cannot be extended to the above Petrov types.

We assume that spacetime admits a twisting shear-free congruence of null geodesics, which is aligned with one of the eigenvectors of the Maxwell tensor $F_{\mu\nu}$. It follows from the Goldberg–Sachs theorem [6] that the Weyl tensor of the metric is algebraically special. The Einstein–Maxwell equations reduce to the following equations [7] (we follow the notation used in [8]) for functions m, M, P (all real) and L, ϕ_1^0, ϕ_2^0 (all complex) of coordinates u, Re ξ, Im ξ (the fourth coordinate is denoted by r):

$$(\partial - 2L_u)\phi_1^0 = 0 \tag{1.1}$$

$$(\partial - L_u)(P^{-1}\phi_2^0) + \partial_u(P^{-2}\phi_1^0) = 0 \tag{1.2}$$

$$P(\partial - 3L_u)(m + \mathrm{i}M) = -2\phi_1^0\bar{\phi}_2^0 \tag{1.3}$$

$$P^4(\partial - 2G)\partial(\bar{G}^2 - \bar{\partial}\bar{G}) - P^3[P^{-3}(m + \mathrm{i}M)]_u = \phi_2^0\bar{\phi}_2^0 \tag{1.4}$$

$$M = -2P^2\Sigma\,\mathrm{Re}(\partial\bar{G}) + P^2\,\mathrm{Re}(\partial\bar{\partial}\Sigma - 2\bar{L}_u\partial\Sigma - \Sigma\partial_u\partial\bar{L}) \tag{1.5}$$

where

$$\partial = \partial_\xi - L\partial_u \tag{1.6}$$

$$G = L_u - \partial \ln P \tag{1.7}$$

$$2\mathrm{i}\Sigma = P^2(\bar{\partial}L - \partial\bar{L}). \tag{1.8}$$

Given a solution of equations (1.1)-(1.5), the corresponding metric tensor g and the electromagnetic field 2-form $F=\frac{1}{2}F_{\mu\nu}\vartheta^\mu \wedge \vartheta^\nu$ are defined by

$$g=2(\vartheta^1\vartheta^2-\vartheta^3\vartheta^4) \tag{1.9}$$

$$F=(\bar{\phi}_1-\phi_1)\vartheta^1\wedge\vartheta^2-(\bar{\phi}_1+\phi_1)\vartheta^3\wedge\vartheta^4+\bar{\phi}_2\vartheta^1\wedge\vartheta^3+\phi_2\vartheta^2\wedge\vartheta^3 \tag{1.10}$$

where the null tetrad ϑ^μ is given by

$$\begin{gathered}\vartheta^1=-\mathrm{d}\xi/P\bar{\rho}=\bar{\vartheta}^2 \qquad \vartheta^3=\mathrm{d}u+L\,\mathrm{d}\xi+\bar{L}\,\mathrm{d}\bar{\xi}\\ \vartheta^4=\mathrm{d}r+W\,\mathrm{d}\xi+\bar{W}\,\mathrm{d}\bar{\xi}+H\vartheta^3\end{gathered} \tag{1.11}$$

$$\begin{gathered}\rho^{-1}=-(r+\mathrm{i}\Sigma) \qquad W=\rho^{-1}L_u+\mathrm{i}\partial\Sigma\\ H=-r(\ln P)_u-(mr+M\Sigma-\phi_1^0\bar{\phi}_1^0)\rho\bar{\rho}-P^2\,\mathrm{Re}(\partial\bar{G})\end{gathered} \tag{1.12}$$

and

$$\phi_1=\rho^2\phi_1^0 \qquad \phi_2=\rho\phi_2^0+\rho^2P(2\bar{L}_u-\bar{\partial})\phi_1^0+2\mathrm{i}\rho^3P(\Sigma\bar{L}_u-\bar{\partial}\Sigma)\phi_1^0. \tag{1.13}$$

The energy-momentum tensor of the electromagnetic field is given by

$$\begin{gathered}T_{12}=T_{34}=2\bar{\phi}_1\phi_1 \qquad T_{13}=\bar{T}_{23}=2\bar{\phi}_2\phi_1\\ T_{33}=2\bar{\phi}_2\phi_2 \qquad T_{11}=\bar{T}_{22}=T_{14}=\bar{T}_{24}=T_{44}=0.\end{gathered} \tag{1.14}$$

Equations (1.3)-(1.5) with $\phi_1^0=0$ are equivalent to the (reduced) Einstein equations with a pure radiation field. If, in addition, $\phi_2^0=0$, they reduce to the vacuum equations [9].

Equations (1.1)-(1.5) are invariant under a coordinate transformation

$$u'=U(\xi,\bar{\xi},u) \qquad \xi'=h(\xi) \qquad r'=U_u^{-1}r \tag{1.15}$$

which induces the following transformation laws of dependent variables

$$L'=-h_\xi^{-1}\partial U \qquad (m+\mathrm{i}M)'=U_u^{-3}(m+\mathrm{i}M) \qquad P'=U_u^{-1}|h_\xi|P \tag{1.16}$$

$$\phi_1^{0\prime}=U_u^{-2}\phi_1^0 \qquad \phi_2^{0\prime}=U_u^{-2}(\bar{h}_{\bar{\xi}}^{-1}h_\xi)^{1/2}\phi_2^0 \tag{1.17}$$

$$\Sigma=U_u\Sigma' \qquad h_\xi G'=G-\tfrac{1}{2}h_{\xi\xi}h_\xi^{-1}. \tag{1.18}$$

2. Equations

Equations (1.1)-(1.5) can be written in the form

$$(\partial-2G)q=0 \tag{2.1}$$

$$(\partial-G)f+P^{-1}\partial_u q=0 \tag{2.2}$$

$$(\partial-3G)(\tilde{m}+\mathrm{i}\tilde{M})=-2q\bar{f} \tag{2.3}$$

$$(\partial-2G)\partial(\overline{\partial G}-\bar{G}^2)+P^{-1}\partial_u(\tilde{m}+\mathrm{i}\tilde{M})=-f\bar{f} \tag{2.4}$$

$$\mathrm{Re}[\partial\bar{\partial}S-2G\bar{\partial}S+(-3\bar{\partial}G+G\bar{G})S]=-2\tilde{M} \tag{2.5}$$

where

$$q=P^{-2}\phi_1^0 \qquad f=P^{-2}\phi_2^0 \qquad S=-2P^{-1}\Sigma \qquad \tilde{m}+\mathrm{i}\tilde{M}=P^{-3}(m+\mathrm{i}M). \tag{2.6}$$

The functions G, q, f, $\tilde{m}$, $\tilde{M}$ and S are invariant under transformation (1.15) with $\xi'=\xi$.

Following the method of Robinson and Robinson [10] for vacuum fields, we will assume that

$$G_u = q_u = f_u = \tilde{m}_u = \tilde{M}_u = 0. \tag{2.7}$$

Conditions (2.7) are invariant under transformation (1.15), which can be used to obtain

$$P_u = 0. \tag{2.8}$$

Condition (2.8) still admits the following changes of the coordinates u, ξ, r

$$u' = U_1(\xi, \bar{\xi})u + U_2(\bar{\xi}, \xi) \qquad \xi' = h(\xi) \qquad r' = r/U_1. \tag{2.9}$$

Under assumptions (2.7) and (2.8), integrating (1.7) with respect to L yields

$$L = u(G + \partial \ln P) + w/P \tag{2.10}$$

where $w \equiv w(\xi, \bar{\xi})$ is an arbitrary function. It follows from (1.8), (2.6) and (2.8) that

$$S = u\gamma P + \hat{S} \tag{2.11}$$

where

$$\gamma = \mathrm{i}(\bar{\partial} G - \partial \bar{G}) \tag{2.12}$$

and

$$\hat{S} = -2\,\mathrm{Im}[(\bar{\partial} + \bar{G})w]. \tag{2.13}$$

Equations (2.1)–(2.5) take the form

$$(\partial - 2G)q = 0 \tag{2.14}$$

$$(\partial - G)f = 0 \tag{2.15}$$

$$(\partial - 3G)(\tilde{m} + \mathrm{i}\tilde{M}) = -2q\bar{f} \tag{2.16}$$

$$(\partial - 2G)\partial(\overline{\partial G} - \bar{G}^2) = -f\bar{f} \tag{2.17}$$

$$\mathrm{Re}[\partial\bar{\partial}\hat{S} - 2G\bar{\partial}\hat{S} + \hat{S}(-3\bar{\partial}G + G\bar{G}) - \gamma\bar{\partial}w + w(3\bar{G}\gamma - 2\bar{\partial}\gamma)] = -2\tilde{M}. \tag{2.18}$$

We will consider w, G, q, f, $\tilde{m}$, $\tilde{M}$ as basic unknown variables. They are subject to equations (2.14)–(2.18), where γ and $\hat{S}$ are given by (2.12) and (2.13), respectively. The function $P(\xi, \bar{\xi})$ is arbitrary. Once a solution of (2.14)–(2.18) is known and a function P is chosen, the functions m, M, L, ϕ_1^0, ϕ_2^0 can be obtained from (2.6) and (2.10). The metric tensor and the electromagnetic field can be computed according to (1.9)–(1.13).

A transformation (2.9) can be used to simplify equation (2.18). For $\xi' = \xi$ it induces the following change of w and $\hat{S}$:

$$w' = w - (\partial + G)(PU_2U_1^{-1}) \qquad \hat{S}' = \hat{S} - \gamma PU_2U_1^{-1}. \tag{2.19}$$

Since $PU_2U_1^{-1}$ can be any real function of ξ and $\bar{\xi}$ we can impose one real condition on w, e.g.

$$w = \bar{w} \tag{2.20}$$

or (only if $\gamma \neq 0$)

$$\hat{S} = -2\,\mathrm{Im}[(\bar{\partial} + \bar{G})w] = 0. \tag{2.21}$$

Equations (2.14)–(2.18) can be partially integrated as follows. Let $F(\xi, \bar{\xi})$ be a function (complex, in general) related to G by

$$G = \partial \ln F. \tag{2.22}$$

By virtue of a freedom we have in defining F, solutions to equations (2.14), (2.15), without loss of generality, can be written as

$$f = \varepsilon F \qquad \varepsilon = 0, 1 \tag{2.23}$$

$$q = \bar{A}F^2 \tag{2.24}$$

where $A \equiv A(\xi)$ is a holomorphic function. Equations (2.16) and (2.17) are equivalent to the equations

$$\partial[F^{-3}(\tilde{m} + \mathrm{i}\tilde{M})] = -2\varepsilon\bar{A}\bar{F}/F \tag{2.25}$$

and

$$\partial[F^{-2}\partial(\overline{\partial G} - \bar{G}^2)] = -\varepsilon\bar{F}/F \tag{2.26}$$

respectively. It follows from them that

$$\tilde{m} + \mathrm{i}\tilde{M} = -2\varepsilon\bar{A}F\partial I + \bar{B}F^3 \tag{2.27}$$

where B is another holomorphic function of ξ and I is defined by

$$I = \bar{G}^2 - \overline{\partial G}. \tag{2.28}$$

Thus all the functions $G, q, f, \tilde{m}, \tilde{M}$ can be expressed in terms of F and w (and arbitrary holomorphic functions A, B), which are subject to equations (2.26) and (2.18). Equation (2.18) can be written in the form

$$\mathrm{Im}[(\bar{\partial} - \bar{G})(\bar{\partial} + \bar{G})(\partial - G)w + 2w\partial(\overline{\partial G} - \bar{G}^2)] = MP^{-3}. \tag{2.29}$$

Equation (2.26) does not involve w. Given a solution F of (2.26) equation (2.29) is a linear equation for w.

Theorem 2.1. Let $P(\xi, \bar{\xi})$ be an arbitrary real function ($P \neq 0$) and let $m, M, L, \phi_1^0, \phi_2^0$ be given by

$$m + \mathrm{i}M = 2\varepsilon\bar{A}P^3F\partial(\overline{\partial G} - \bar{G}^2) + \bar{B}P^3F^3$$

$$L = u(G + \partial \ln P) + w/P$$

$$\phi_1^0 = \bar{A}P^2F^2 \qquad \phi_2^0 = \varepsilon P^2 F$$

where $\varepsilon = 0$ or $\varepsilon = 1$, A and B are holomorphic functions of ξ and $G = \partial \ln F$. If functions $F(\xi, \bar{\xi})$ and $w(\xi, \bar{\xi})$ satisfy equations (2.26) and (2.29), then relations (1.9)–(1.13) define an algebraically special solution of the Einstein–Maxwell equations.

The case $\varepsilon = 0$ corresponds to vacuum or charged vacuum metrics. In this case, integrating equation (2.26) yields

$$\partial I = \bar{C}F^2 \tag{2.30}$$

where $C \equiv C(\xi)$. If $C = 0$ then equations (2.30), (2.29) can be explicitly solved [10]. If $C \neq 0$ then

$$w = w_0 - \bar{B}F/2\bar{C} \tag{2.31}$$

where w_0 satisfies equation (2.29) with $M_0 = 0$.

The case $\varepsilon = 1$ corresponds to radiative Maxwell fields. Substituting

$$w = w_0 + \bar{A}F \tag{2.32}$$

into equation (2.29) transforms its RHS to $\tilde{M}_0 = \mathrm{Im}(\bar{B}F^3)$.

Thus, if $\partial I \neq 0$, then equation (2.29) can be reduced to

$$\mathrm{Im}[(\bar{\partial} - \bar{G})(\bar{\partial} + \bar{G})(\partial - G)w_0 - 2w_0\partial I] = \varepsilon\ \mathrm{Im}(\bar{B}F^3). \tag{2.33}$$

This means, in particular, that one can easily generate charged radiative Einstein–Maxwell fields from purely radiative ($\phi_1 = 0$) ones.

If $A \neq 0$ (or $B \neq 0$) then it can be transformed to $A' = 1$ (or $B' = 1$) by a transformation (2.9). Function w_0 undergoes the same transformation law as w under (2.9). Due to this, w_0 can be assumed to be real. Then (2.33) is a third-order equation for w_0. If

$$\gamma \equiv \mathrm{i}\partial\bar{\partial} \ln(F/\bar{F}) \neq 0 \tag{2.34}$$

then we can assume

$$\mathrm{Im}[(\bar{\partial} + \bar{G})w_0] = 0. \tag{2.35}$$

In this case equation (2.33) is equivalent to

$$\mathrm{Re}[\gamma\bar{\partial}w_0 + (2\bar{\partial}\gamma - 3\bar{G}\gamma)w_0] = 2\varepsilon\ \mathrm{Im}(\bar{B}F^3). \tag{2.36}$$

Equations (2.35) and (2.36) form a first-order system of equations for the complex function w_0. Another form of equation (2.33) is obtained if, following Robinson and Robinson [10], one introduces functions ϕ and ψ according to

$$w_0 = (\partial + G)(\phi + \mathrm{i}\psi). \tag{2.37}$$

Then equation (2.33) takes the form

$$\bar{\partial}^2\partial^2\psi - \bar{\partial}^2(\bar{I}\psi) - \partial^2(I\psi) + (I\bar{I} + \varepsilon F\bar{F})\psi = \varepsilon\ \mathrm{Im}(\bar{B}F^3) \tag{2.38}$$

where I is given by (2.28). The function ϕ is disposable. It can be transformed to $\phi' = 0$ by (2.9).

3. Radiative solutions admitting the Killing vector ∂_u

In this section we will assume that $\varepsilon = 1$ (i.e. the Maxwell field is radiative, $\phi_2^0 \neq 0$) and

$$\gamma \equiv \mathrm{i}(\bar{\partial}G - \partial\bar{G}) = 0 \tag{3.1}$$

in addition to assumptions (2.7), (2.8). In this case ∂_u is the Killing vector since transformation (2.9) can be used to obtain

$$\partial_u L = 0. \tag{3.2}$$

Instead of considering equations (2.26), (2.33), (3.1) for F and w_0 it is now more natural to reduce equations (1.1)–(1.5) to equations for P and Σ. Equations (1.1)–(1.3) yield

$$\phi_1^0 = \bar{a} \qquad \phi_2^0 = \overline{\partial b}P \qquad m + \mathrm{i}M = -2\bar{a}b - \bar{c} \tag{3.3}$$

where a, b, c are holomorphic functions of ξ and $b \neq$ constant. Equations (1.4), (1.5) reduce to

$$\partial\bar{\partial}(P^2\partial\bar{\partial} \ln P) = |\partial b|^2 \tag{3.4}$$

$$\partial\bar{\partial}\Sigma + 2\Sigma\partial\bar{\partial} \ln P = P^{-2}\ \mathrm{Im}(2a\bar{b} + c). \tag{3.5}$$

Given P and Σ, the function L can be found by integrating equation (1.8). Up to a transformation (2.9) (with $U_1 = 1$) L is given by

$$L = \mathrm{i}\int \Sigma P^{-2}\,\mathrm{d}\bar{\xi}. \tag{3.6}$$

Double integration of equation (3.4) yields

$$P^2\partial\bar{\partial}\ln P = b\bar{b} + e + \bar{e} \tag{3.7}$$

where $e \equiv e(\xi)$. Transformation (2.32) corresponds to the following decomposition of Σ

$$\Sigma = \Sigma_0 + \mathrm{Im}(\partial a_0 - 2a_0\partial\ln P) \tag{3.8}$$

where

$$a_0 = a/\partial b. \tag{3.9}$$

The function Σ_0 is subject to the equation

$$\partial\bar{\partial}\Sigma_0 + 2\Sigma_0\partial\bar{\partial}\ln P = P^{-2}\,\mathrm{Im}(c_0) \tag{3.10}$$

where

$$c_0 = c + 2a_0\partial e. \tag{3.11}$$

The splitting (3.9) of Σ corresponds to the following decomposition of L:

$$L = L_0 - \bar{a}_0 P^{-2} \tag{3.12}$$

where

$$2\mathrm{i}\Sigma_0 = P^2(\bar{\partial}L_0 - \partial\bar{L}_0). \tag{3.13}$$

Given P and Σ_0 function L_0 can be defined by a formula analogous to (3.7).

Theorem 3.1. Let a, b ($b \neq$ constant), c_0 and e be holomorphic functions of ξ. If real functions P, Σ_0 of ξ and $\bar{\xi}$ satisfy equations (3.8) and (3.10) then functions

$$L = \mathrm{i}\int \Sigma_0 P^{-2}\,\mathrm{d}\bar{\xi} - \bar{a}/(\overline{\partial b}P^2)$$

$$\phi_1^0 = \bar{a} \qquad \phi_2^0 = \overline{\partial b}P \qquad m + \mathrm{i}M = -2\bar{a}b - \bar{c}_0 + 2\overline{a\partial e}/\overline{\partial b}$$

define an algebraically special Einstein–Maxwell field via relations (1.9)–(1.13), (3.9).

For $e = 0$ equation (3.8) is equivalent to the Liouville equation, which is exactly soluble. Due to the freedom (restricted by (3.2) and (3.6)) of transformations (2.9) it is sufficient to take only one solution of the Liouville equation, say

$$P = |b|(1 + \xi\bar{\xi}). \tag{3.14}$$

Given this P, equation (3.10) takes the form

$$(1 + \xi\bar{\xi})^2\partial\bar{\partial}\Sigma_0 + 2\Sigma_0 = \mathrm{Im}\, c/b\bar{b}. \tag{3.15}$$

Solutions of (3.15) have the form

$$\Sigma_0 = Y + S \tag{3.16}$$

where S is a particular solution of (3.15) and Y satisfies the homogeneous equation

$$(1 + \xi\bar{\xi})^2\partial\bar{\partial}Y = -2Y. \tag{3.17}$$

Equation (3.17) is the eigenfunction equation (with eigenvalue equal to −2) for the Laplace operator on the two-dimensional sphere. The variable ξ can be interpreted as the complex projective coordinate on the sphere

$$\xi=\tan\tfrac{1}{2}\vartheta\, \mathrm{e}^{\mathrm{i}\varphi} \tag{3.18}$$

where ϑ, φ are the usual spherical angles. Separation of ϑ and φ in (3.17) leads to Y being a linear combination of functions

$$(\xi/|\xi|)^{\mu}P_1^{\mu}(z) \qquad (\xi/|\xi|)^{\mu}Q_1^{\mu}(z) \tag{3.19}$$

where P_1^{μ}, Q_1^{μ} are the associated Legendre functions [11] and

$$z=(1-\xi\bar{\xi})/(1+\xi\bar{\xi}). \tag{3.20}$$

Solutions which are regular on the whole sphere are given by

$$Y=(1+\xi\bar{\xi})^{-1}(\alpha-\alpha\xi\bar{\xi}+\beta\xi+\overline{\beta\xi}) \tag{3.21}$$

where α and β are constants, real and complex, respectively. Whether solutions (3.21) are more important than others is an open question.

It is quite easy to find a particular solution of (3.15) when

$$c=c_1b+c_2 \qquad c_2\in R \tag{3.22}$$

where c_1, c_2 are constants. In this case

$$S=\mathrm{Im}(c_1/2\bar{b}). \tag{3.23}$$

Theorem 3.2. Let a and b, $b\neq$ constant, be arbitrary holomorphic functions of ξ and let $P, m, M, L, \phi_1^0, \phi_2^0$ be given by

$$P=|b|(1+\xi\bar{\xi}) \tag{3.24}$$

$$m+\mathrm{i}M=c_1b+c_2 \tag{3.25}$$

$$L=\mathrm{i}\int\Sigma_0P^{-2}\,\mathrm{d}\bar{\xi}-\bar{a}/(\overline{\partial b}P^2) \tag{3.26}$$

$$\phi_1^0=\bar{a} \qquad \phi_2^0=\overline{\partial b}P \tag{3.27}$$

where

$$\Sigma_0=Y+\mathrm{Im}(c_1/2\bar{b}) \tag{3.28}$$

Y satisfies (3.17) (e.g. it is given by (3.21)) and c_2, α (both real), c_1, β (both complex) are constants. The above relations, together with (1.9)–(1.13), (3.9), define an algebraically special solution of the Einstein–Maxwell equations. The Weyl tensor of the solution is of Petrov type II if $m+\mathrm{i}M\neq0$ or $a\neq0$, and it is of Petrov type III if $m+\mathrm{i}M=a=0$.

Metric functions $P, m, M, L, \phi_1^0, \phi_2^0$ given by (3.24)–(3.28) with $a=0\neq\Sigma_0$ describe purely radiative Einstein–Maxwell fields with twisting rays.

Below we give two examples of solutions covered by theorem 3.2.

If $Y=c_1=0$ then the metric corresponding to (3.24)–(3.28) reads:

$$g=2\frac{(r^2+\Sigma^2)\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}}{b\bar{b}(1+\xi\bar{\xi})^2}-2\vartheta^3\left\{\mathrm{d}r+\mathrm{i}(\partial\Sigma\,\mathrm{d}\xi-\bar{\partial}\Sigma\,\mathrm{d}\bar{\xi})+\left(b\bar{b}+\frac{a\bar{a}-c_2r}{r^2+\Sigma^2}\right)\vartheta^3\right\} \tag{3.29}$$

where

$$\vartheta^3 = \mathrm{d}u - \frac{\bar{a}_0\,\mathrm{d}\xi + a_0\,\mathrm{d}\bar{\xi}}{b\bar{b}(1+\xi\bar{\xi})^2} \qquad a_0 = \frac{a}{\partial b} \tag{3.30}$$

$$\Sigma = \mathrm{Im}\{\partial a_0 - a_0 \partial \ln[b(1+\xi\bar{\xi})^2]\}. \tag{3.31}$$

The electromagnetic field is given by (1.10) with

$$\phi_1 = \bar{a}\rho^2 \tag{3.32}$$

$$\phi_2 = \rho|b|(1+\xi\bar{\xi})\{\overline{\partial b} - \rho\overline{\partial a} - 2\mathrm{i}\rho^2\overline{a\partial}\Sigma\} \tag{3.33}$$

where

$$\rho = -(r+\mathrm{i}\Sigma)^{-1}. \tag{3.34}$$

Solution (3.29)–(3.34) with $a \neq 0$ is twisting, charged and radiative. In the limit $a \to 0$ it yields a non-twisting pure radiation (no charge) Einstein–Maxwell field. The latter solution belongs to the class of solutions obtained by Bartrum [12]. In the limit $b \to \sqrt{2}^{-1}$ it reduces to the Schwarzschild solution with mass $m = c_2$.

An example of a solution with a purely radiative ($\phi_1 = 0$) Maxwell field follows from theorem 3.2 if, e.g. $a = c_1 = 0$ and

$$Y = \alpha(1-\xi\bar{\xi})/(1+\xi\bar{\xi}). \tag{3.35}$$

In this case the metric and the electromagnetic field read:

$$g = 2\frac{(r^2+\Sigma^2)\,\mathrm{d}\xi\,\mathrm{d}\bar{\xi}}{b\bar{b}(1+\xi\bar{\xi})^2} - 2\vartheta^3\left\{\mathrm{d}r + 2\alpha\mathrm{i}\frac{\xi\,\mathrm{d}\bar{\xi} - \bar{\xi}\,\mathrm{d}\xi}{(1+\xi\bar{\xi})^2} + \left(b\bar{b} - \frac{c_2 r}{r^2+\Sigma^2}\right)\vartheta^3\right\} \tag{3.36}$$

$$F = \vartheta^3 \wedge (\partial b\,\mathrm{d}\xi + \overline{\partial b}\,\mathrm{d}\bar{\xi}) \tag{3.37}$$

where

$$\Sigma = \alpha\frac{1-\xi\bar{\xi}}{1+\xi\bar{\xi}} \qquad \vartheta^3 = \mathrm{d}u + L\,\mathrm{d}\xi + \bar{L}\,\mathrm{d}\bar{\xi} \tag{3.38}$$

$$L = \mathrm{i}\alpha\frac{\bar{\xi}}{b\bar{b}(1+\xi\bar{\xi})^2} + \mathrm{i}\alpha\int\frac{\overline{\xi\partial}\ln\bar{b}}{b\bar{b}(1+\xi\bar{\xi})^2}\,\mathrm{d}\bar{\xi}. \tag{3.39}$$

This solution is of Petrov type II if $c_2 \neq 0$ and it is of Petrov type III if $c_2 = 0$. In the limit $b \to \sqrt{2}^{-1}$ it reduces to the Kerr metric with mass parameter $m = c_2$ and angular momentum parameter equal to α. In the limit $\alpha \to 0$ it coincides with solution (3.29)–(3.34) with $a \to 0$.

Solution (3.36)–(3.39) provides the first example of a pure radiation Einstein–Maxwell field with twisting light rays. Recently, it has been proven by Debever *et al* [4] that aligned twisting pure radiation Einstein–Maxwell fields of Petrov type D do not exist. The above example shows that this theorem cannot be generalized to Petrov types II and III.

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