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Optical reference geometry for stationary and axially symmetric spacetimes

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Abstract. We discuss the acceleration and Fermi–Walker transport for the circular motion of particles, photons and gyroscopes in stationary, axially symmetric spacetimes in terms of the optical reference geometry.

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1. Optical reference geometry in static spacetimes

Abramowicz, Carter and Lasota [1], hereafter ACL, defined an optical reference geometry on three-dimensional spatial sections in static spacetimes. The optical geometry is obtained from the directly projected geometry of 3-spaces by a proper conformal rescaling. Light trajectories are geodesic lines in the optical geometry. Equations describing dynamics are identical to those describing Newtonian dynamics on a curved two-dimensional surface. The optical reference geometry offers insight into relativistic dynamics by providing a description in accord with Newtonian intuition and explaining effects which otherwise seem to be paradoxical [2, 3]. Therefore, it may be useful to consider the optical geometry in a more general case of stationary and axially symmetric spacetimes which is relevant in many astrophysical applications. We do this here.

2. Optical reference geometry in a general spacetime

In [4] we have introduced the optical geometry and defined the inertial forces for a general spacetime with no symmetries, generalizing the static spacetime definition given by ACL. Our approach employs a vector field n^i, and a scalar function Φ which locally obey the conditions

$$n^k n_k = -1 \qquad \dot{n}_k = n^i \nabla_i n_k = \nabla_k \Phi \qquad n_{[i} \nabla_j n_{k]} = 0 \,. \tag{2.1}$$

It was shown in [4] that at least one solution of (2.1), corresponding to $\Phi =$ constant, exists in every spacetime.

We briefly recall the points of [4] which are relevant here†. The 4-velocity u^i of a test particle (with rest mass m) may be uniquely decomposed into

$$u^i = \gamma(n^i + v\tau^i)\,. \tag{2.2}$$

Here τ^i is a unit spacelike vector orthogonal to n^i, v is the speed, and $\gamma^2 = 1/(1-v^2)$. Let $\tilde{v} = \gamma v$. For photons $v = \pm 1$ and $\tilde{v} = \pm\infty$. In the projected space with the metric $h_{ik} = g_{ik} + n_i n_k$, Frenet's triad (θ_A), $A = 1, 2, 3$, associated with the particle trajectory is given by $(\theta_A{}^i) = (\tau^i, \lambda^i, \Lambda^i)$, where the unit vectors λ^i and Λ^i are the first and second normals. The optical geometry is introduced by conformal rescaling $\tilde{h}_{ik} = e^{2\Phi}h_{ik}$. The covariant derivative in $\tilde{h}_{ik}$ is denoted by $\tilde{\nabla}_i$. For Frenet's triad in optical geometry, $(\tilde{\theta}_A{}^i) = e^{-\Phi}(\theta_A{}^i)$, and $(\tilde{\theta}_{Ai}) = e^{\Phi}(\theta_{Ai})$.

Equation (2.1) defines τ^i only along the world line of the particle. However, in our construction one also needs to know how τ^i changes along trajectories of $\tilde{\eta}^i = e^{\Phi}n^i$. This should be postulated, and here we adopt the ACL gauge

$$\mathcal{L}_{\tilde{\eta}}\tau_k = \tilde{\eta}^i\nabla_i\tau_k - \tau^i\nabla_i\tilde{\eta}_k = 0\,. \tag{2.3}$$

Although different gauges are possible, only this particular one gives physically natural interpretations of different terms in the acceleration formula.

The projection of the 4-acceleration $a^{\perp}_k = (\delta^j{}_k + n^j n_k)u^i\nabla_i u_j$ is uniquely decomposed in terms proportional to zeroth, first and second powers of $\tilde{v}$ and its change, $\dot{V} = u^i\nabla_i(\tilde{v}e^{\Phi})$,

$$a^{\perp}_k = G_k(\tilde{v}^0) + C_k(\tilde{v}^1) + Z_k(\tilde{v}^2) + E_k(\dot{V}) = \tilde{\nabla}_k\Phi + 2\tilde{v}\tilde{X}_k + \tilde{v}^2\tilde{\tau}^k\tilde{\nabla}_k\tilde{\tau}_i + \dot{V}\tilde{\tau}_k \tag{2.4}$$

where, $\tilde{X}_k = \gamma e^{-\Phi}\tau^i\nabla_{(i}\tilde{\eta}_{k)}$. From equation (2.4) we have deduced the covariant definitions of inertial forces,

$$\text{Gravitational force: } G_k = -m\tilde{\nabla}_k\Phi \tag{2.5}$$

$$\text{Coriolis (Lense–Thirring) force: } C_k = -2m\gamma^2 v e^{-\Phi}\tau^i\nabla_{(i}\tilde{\eta}_{k)} \tag{2.6}$$

$$\text{Centrifugal force: } Z_k = -m(\gamma v)^2\tilde{\tau}^i\tilde{\nabla}_i\tilde{\tau}_k \tag{2.7}$$

$$\text{Euler force: } E_k = -m\dot{V}\tilde{\tau}_k\,. \tag{2.8}$$

3. Circular motion in stationary and axially symmetric spacetimes

Stationary, axially symmetric spacetimes were discussed by number of authors. We follow here Bardeen's discussion in [5]. In these spacetimes two commuting Killing vector fields exist,

$$\nabla_{(i}\eta_{k)} = 0 \qquad \nabla_{(i}\xi_{k)} = 0 \qquad \eta^i\nabla_i\xi_k - \xi^i\nabla_i\eta_k = 0\,. \tag{3.1}$$

The vector field η^i is (at least asymptotically) timelike and has open trajectories, while the vector field ξ^i is spacelike and has closed trajectories.

† In the paragraph following equation (4) in [4], the formula in the text should read, $\nabla^i\Phi = g^{ij}\nabla_j\Phi = 0$. In the paper, the last $= 0$ is missing.

Let $\alpha_0 > 0$ and ω_0 be two constant numbers. Then, obviously, the vectors

$$\eta^i_* = \alpha_0(\eta^i + \omega_0 \xi^i) \qquad \xi^i_* = \xi^i \tag{3.2}$$

are also commuting Killing vectors, i.e. they obey (3.1). Bardeen [6] has pointed out that any physically meaningful quantity X constructed from the Killing vectors η^i and ξ^i in some covariant way must be invariant under the transformation $X(\eta^i_*, \xi^i_*) = X(\eta^i, \xi^i)$, or the B-invariant as we shall call it for short†.

It is easy to check by a direct substitution that the vector n^i field corresponding to the zero angular momentum observers (ZAMO) introduced by Bardeen [5],

$$n^i = e^{\Phi}(\eta^i + \omega \xi^i) \tag{3.3}$$

$$\omega = -\frac{(\eta\xi)}{(\xi\xi)} \qquad \Phi = -\tfrac{1}{2}\ln\left[(\eta\eta) + 2\omega(\xi\eta) + \omega^2(\xi\xi)\right] \tag{3.4}$$

is B-invariant and that it obeys conditions (2.1), which are themselves B-invariant. Therefore, once the 4-velocity of a particle u^i is specified, the ZAMO vector field n^i can be used for the definition of inertial forces described in the previous section.

In this paper we assume that the motion is *circular*, and therefore that the 4-velocity of a particle on a particular circular trajectory of ξ^i (which we denote by $\mathcal{C}_0$) equals,

$$u^i = A(\eta^i + \Omega\xi^i) \tag{3.5}$$

$$-A^{-2} = (\eta\eta) + 2\Omega(\eta\xi) + \Omega^2(\xi\xi)\,. \tag{3.6}$$

Here Ω is the angular velocity measured by the stationary observer at infinity. In general, $\dot{\Omega} = u^i\nabla_i\Omega \neq 0$. Equating (3.5) with the expression for velocity in terms of ZAMO, given by (2.2), one deduces that

$$v = \tilde{\Omega}\tilde{R} \qquad \tau^i = \xi^i\tilde{r}^{-1} \qquad \tilde{r}^2 = (\xi\xi)\,. \tag{3.7}$$

Here $\tilde{\Omega}$ is the angular velocity measured by ZAMO, and $\tilde{R}$ is the radius of gyration

$$\tilde{\Omega} = \Omega - \omega \qquad \tilde{R} = \tilde{r}e^{\Phi}\,. \tag{3.8}$$

Although (3.7) determines τ^i only on $\mathcal{C}_0$, it obviously makes sense everywhere in the spacetime, and therefore it is natural to postulate that (3.7) indeed gives τ^i everywhere. This is equivalent to adopting the ACL gauge (2.4).

We call $\tilde{R}$ the radius of gyration, because $\tilde{R}^2 = \tilde{\ell}/\tilde{\Omega}$, where $\tilde{\ell} = \tilde{\mathcal{L}}/\tilde{\mathcal{E}}$ is the specific angular momentum, $\tilde{\mathcal{L}} = (u\xi)$ is the angular momentum, and $\tilde{\mathcal{E}} = -(u\tilde{\eta}) = \gamma e^{\Phi}$ is the energy of the particle. *Nota bene*, this proves that the von Zeipel cylinders in stationary spacetimes should be defined by the B-invariant condition $\tilde{R}$ = constant‡. See [7] for a discussion of these concepts in static spacetimes.

† In the coordinate frame in which $\eta^i = \delta^i{}_t$, and $\xi^i = \delta^i{}_\phi$, with t being time and ϕ being the azimuthal angle around the symmetry axis, one has $g_{tt} = (\eta\eta) = -e^{-2\Phi}$, $g_{t\phi} = (\eta\xi)$, $g_{\phi\phi} = (\xi\xi)$. In this frame (3.2) becomes, $t_* = \alpha_0^{-1}t$, $\phi_* = \phi - \omega_0 t$.

‡ Some authors wrongly define the von Zeipel cylinders in non-static spacetimes by a non-B-invariant equation R = constant with $R^2 = \ell/\Omega$ and $\ell = -(u\xi)/(u\eta)$.

One derives, after a short piece of algebra

$$G_k = -m\nabla_k\Phi \tag{3.9}$$

$$C_k = m\gamma^2 v\tilde{R}\nabla_k\omega \tag{3.10}$$

$$Z_k = -m(\gamma v)^2\tilde{R}^{-1}\nabla_k\tilde{R} \tag{3.11}$$

$$E_k = -me^{\Phi}\gamma^3\tilde{R}\dot{\Omega}\tilde{\tau}_k\,. \tag{3.12}$$

It is easy to check that the four forces G_k, C_k, Z_k, E_k, and the velocity v are B-invariant. Indeed, it is instructive to see how different quantities which are *not* B-invariant combine to form the B-invariant ones in (3.9)–(3.12). For example, the gravitational potential is not B-invariant, $\Phi_* = \Phi - \ln\alpha_0$, and for this reason it is determined only up to an additive constant (as should be expected). However, its gradient $\nabla_k\Phi$ is B-invariant and therefore physically meaningful. Similarly, $\tilde{R}_* = \alpha_0^{-1}\tilde{R}$, $\omega_* = \alpha_0(\omega-\omega_0)$ and $\tilde{\Omega}_* = \alpha_0\tilde{\Omega}$.

Note that

$$\tilde{\tau}^i\tilde{\nabla}_i\tilde{\tau}_k = \kappa\tilde{\lambda}_k = -\tilde{R}^{-1}\nabla_k\tilde{R}\,. \tag{3.13}$$

Here $\kappa = 1/\mathcal{R}$ is the curvature, and $\mathcal{R}$ the curvature radius of the circle C_0, as measured in the optical geometry. It is, $\mathcal{R}_* = \alpha_0^{-1}\mathcal{R}$. For geodesic lines $\kappa = 0$.

4. Steady circular motion

In this section we assume that the circular motion is steady, $\dot{\Omega} = 0$. A unit spacelike vector e_k parallel to $\nabla_k\tilde{r}$ defines the direction outwards of the symmetry axis $\tilde{r} = 0$. We denote by $G(\tilde{r}) = -e^k\nabla_k\Phi$, $C(\tilde{r}) = e^k\tilde{R}\nabla_k\omega$, $Z(\tilde{r}) = -e^k\tilde{R}^{-1}\nabla_k\tilde{R}$ the *velocity independent* parts of the gravitational, Coriolis and centrifugal accelerations in the direction of e_k, and for the total acceleration in this direction we write, $a = e^k a_k$.

The Kerr metric, and several other stationary and axially symmetric ones, display a discrete mirror symmetry $g_{ik}(z) = g_{ik}(-z)$, which invariantly defines the equatorial plane $z = 0$. Let Q^i denotes the acceleration vector a^i, or the gradient of an axially symmetric, stationary, and mirror-symmetric function. On the equatorial plane

$$Q^i\Lambda_i = 0 = Q^i\tau_i\,. \tag{4.1}$$

This means that on the equatorial plane Q^i points in the direction of the vector $\lambda^i = \varepsilon e^i$, where $\varepsilon = (\lambda e)$ is equal either $+1$ or -1. On this plane we define $E^i = \varepsilon\Lambda^i$. The vectors (τ^i, e^i, E^i) form an orthonormal triad whose orientation in space relates to the axis of symmetry independent of the particle's trajectory.

4.1. Acceleration and geodesic motion on the equatorial plane

On a particular circle C_0 defined by $\tilde{r} =$ constant one has

$$a(\tilde{v},\tilde{r}) = -G(\tilde{r}) - \tilde{v}^2 Z(\tilde{r}) + (1+\tilde{v}^2)^{1/2}\tilde{v}C(\tilde{r})\,. \tag{4.2}$$

Orbital velocities of free particles (circular geodesic motion) are given by solutions of the equation $a(\tilde{v},\tilde{r}) = 0$, or

$$\tilde{v}_\pm = \pm\left[\frac{\frac{1}{2}C^2 - ZG \mp \frac{1}{2}C\left(C^2 - 4ZG + 4G^2\right)^{1/2}}{Z^2 - C^2}\right]^{1/2}. \tag{4.3a}$$

If $C(\tilde{r}) \neq 0$, free orbits could exist at a particular circle $\tilde{r}$ = constant if and only if $C^2(\tilde{r}) - 4Z(\tilde{r})G(\tilde{r}) + 4G^2(\tilde{r}) > 0$. Additional (obvious) inequalities must be obeyed for *existence of either prograde or retrograde orbits. If $C(\tilde{r}) = 0$, i.e. in a static spacetime*, the free orbit exists if and only if $GZ < 0$, i.e. if and only if gravitational and centrifugal forces points in opposite directions. The orbital velocity is given by

$$\tilde{v}_{\pm} = \pm\left(-\frac{G}{Z}\right)^{1/2}. \tag{4.3b}$$

The condition $Z(\tilde{r})G(\tilde{r}) < 0$ is *not* fulfilled in the Schwarzschild spacetime for circles with $\tilde{r} < 3\sqrt{3}M$. (This corresponds to $r < 3M$ in Schwarzschild coordinates, where M is the central mass.)

For ultra-relativistic particles $\tilde{v}^2 \gg 1$ and (4.2) becomes

$$\frac{a(\tilde{v},\tilde{r})}{\tilde{v}^2} = -Z(\tilde{r}) \pm C(\tilde{r}) + [-G(\tilde{r}) \pm \tfrac{1}{2}C(\tilde{r})]\tilde{v}^{-2} + \mathcal{O}(\tilde{v}^{-4}). \tag{4.4}$$

The upper signs are for the prograde motion, ($\tilde{v} > 0$), and the lower signs for the retrograde ($\tilde{v} < 0$) motion. It follows from (4.4) that, with accuracy $\mathcal{O}(\tilde{v}^{-2})$, *motion of ultra-relativistic particles is not influenced by the gravitational force.* With the same accuracy, the ultra-relativistic particles move along the circles given by

$$Z(\tilde{r}) - C(\tilde{r}) = 0 \qquad \text{(for prograde motion)} \tag{4.5a}$$

$$Z(\tilde{r}) + C(\tilde{r}) = 0 \qquad \text{(for retrograde motion)}. \tag{4.5b}$$

Photons move *exactly* along these circles. Ultra-relativistic particles moving progradely along the prograde free-photon orbit (4.5a) have acceleration $a = -G + C/2$ *independent* of the orbital speed (with the above-mentioned accuracy). The same surprising effect occurs *for ultra-relativistic particles moving retrogradely along the retrograde free-photon orbit* (4.5b), with accuracy $\mathcal{O}(\tilde{v}^{-2})$, acceleration $a = -G - C/2$ is *independent* of the orbital speed. For static spacetimes $C = 0$. Abramowicz and Lasota [8] noticed that in this case the acceleration *exactly* equals $G(\tilde{r})$ and is *exactly* independent of the speed for *all* particles, not only the ultra-relativistic ones, which move either progradely or retrogradely along the unique (prograde = retrograde) circular free-photon orbit given by $Z(\tilde{r}) = 0$. From equation (3.13) one deduces that this circle is a geodesic line in the optical geometry $\tilde{h}_{ik}$. For static spacetimes *all* the geodesic lines in optical geometry always coincide with trajectories of free photons (ACL), and this explains the name 'optical geometry'.

4.2. Fermi–Walker transport and gyroscope precession

The Fermi–Walker derivative of τ^i with respect to u^i equals $\delta\tau_k/\delta s = u^i\nabla_i\tau_k - (a_k u_i - a_i u_k)\tau^i$, and the vector Ω^j_G,

$$\Omega^j_G = \left(\frac{\delta\tau^k}{\delta s}\Lambda_k\right)\lambda^j - \left(\frac{\delta\tau^k}{\delta s}\lambda_k\right)\Lambda^j \tag{4.6}$$

gives the rate of precession (with respect to τ^i) of a gyroscope which moves with the 4-velocity u^i, see, for example, [9].

For a steady circular motion,

$$-mv\gamma^{-1}\frac{\delta\tau_k}{\delta s} = \tfrac{1}{2}(1+v^2)C_k + Z_k. \tag{4.7}$$

One sees that the *precession of the gyroscope is not influenced by the gravitational force.* On the equatorial plane,

$$\Omega^j_G = \gamma^3 \left[\tfrac{1}{2}(1 + v^2)C(\tilde{r}) + \varepsilon e^{\Phi} \tilde{\Omega} \frac{\tilde{R}}{\mathcal{R}} \right] E^j \,. \tag{4.8}$$

In this formula one easily recognizes different types of precession (Thomas, geodesic, Lense–Thirring). Its generalization will be discussed in [10].

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Appendix

The main steps in the derivation of (2.4) are

$$u^k \nabla_k u_i = \dot{\gamma} n_i + (\gamma v)\dot{} \tau_i + \gamma^2 \dot{n}_i + \gamma^2 v^2 \tau^k \nabla_k \tau_i + \gamma^2 v (n^k \nabla_k \tau_i + \tau^k \nabla_k n_i)$$
$$\gamma^2 \dot{n}_i + \gamma^2 v^2 \tau^k \nabla_k \tau_i = (\gamma^2 - 1 + 1)\dot{n}_i + \gamma^2 v^2 \tau^k \nabla_k \tau_i = \dot{n}_i + v^2 \gamma^2 [(\tau^k \nabla_k \tau_i) + (n^k \nabla_k n_i)]$$
$$\tilde{\tau}^k \tilde{\nabla}_k \tilde{\tau}_i = \tau^k \nabla_k \tau_i + \nabla_i \Phi - \tau_i \tau^k \nabla_k \Phi \,.$$

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