On a certain formulation of the Einstein equations

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We define a certain differential system on an open set of $\mathbf{R}^6$. The system locally defines a Lorentzian four-manifold satisfying the Einstein equations. The converse statement is indicated and its details are postponed to the forthcoming paper. © *1998 American Institute of Physics.* [S0022-2488(98)01310-3]

Let $\mathscr{U}$ be an open subset of $\mathbf{R}^6$. Suppose that on $\mathscr{U}$ we have six one-forms $(F,\bar{F},T,\Lambda,E,\bar{E})$ which satisfy the following conditions:

(i) T, Λ are real- and F, E are complex-valued one-forms;

(ii) $F \wedge \bar{F} \wedge T \wedge \Lambda \wedge E \wedge \bar{E} \neq 0$ at each point p of $\mathscr{U}$;

(iii) there exist complex-valued one-forms Ω and Γ on $\mathscr{U}$, and a certain complex function α on $\mathscr{U}$ such that

$$dF = (\Omega - \bar{\Omega}) \wedge F + E \wedge T + \bar{\Gamma} \wedge \Lambda,$$

$$dT = \Gamma \wedge F + \bar{\Gamma} \wedge \bar{F} - (\Omega + \bar{\Omega}) \wedge T,$$

$$dΛ = \bar{E} \wedge F + E \wedge \bar{F} + (\Omega + \bar{\Omega}) \wedge \Lambda, \tag{1}$$

$$dE = 2\Omega \wedge E + \bar{F} \wedge T + \alpha \Lambda \wedge F.$$

The aim of this paper is to prove that we can associate to such $\mathscr{U}$ a Lorentzian four-manifold that satisfies the Einstein equations. The proof will be a direct consequence of the following sequence of Lemmas.

Lemma 1: If $(F,\bar{F},T,\Lambda,E,\bar{E})$ *satisfy* (1) *then there exist complex functions a, h on* $\mathscr{U}$ *and a real constant* λ *such that*

$$d\Gamma = 2\Gamma \wedge \Omega + \alpha T \wedge \bar{F} + a(T \wedge \Lambda + F \wedge \bar{F}) + h\Lambda \wedge F,$$

$$d\Omega = E \wedge \Gamma - \left(\alpha + \frac{\lambda}{2} \right)(T \wedge \Lambda + F \wedge \bar{F}) + a\Lambda \wedge F. \tag{2}$$

Sketch of the proof: Condition (iii) can be understood as follows. We consider the forms $(F,\bar{F},T,\Lambda,E,\bar{E})$ as the basic objects. We assume that their differentials have a special form (iii) with certain auxiliary forms $(\Omega,\bar{\Omega},\Gamma,\bar{\Gamma})$. Suppose that we have the system of ten forms $(F,\bar{F},T,\Lambda,E,\bar{E},\Omega,\bar{\Omega},\Gamma,\bar{\Gamma})$ satisfying (1). Then

$$d^2\Lambda = d^2T = d^2F = d^2E = 0 \tag{3}$$

imply that the differentials $d\Gamma$ and $d\Omega$ have a special form. The proof that $d\Gamma$ and $d\Omega$ have the form (2) is a direct (but lengthy) calculation which uses (3) and the independence condition (ii). To obtain information about the differentials of the functions appearing in (2) one has to use

$$d^2\Gamma = d^2\Omega = 0. \tag{4}$$

These equations show, in particular, that λ must be a real constant.   □

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There are also other equations that are implied by (3) and (4). They carry information about the differentials of α, a, h and about the decompositions of Γ and Ω onto the basis one-forms $(F,\bar{F},T,\Lambda,E,\bar{E})$. Explicitly we have

$$d\alpha = \alpha_1 F + \gamma_4 \bar{F} + \gamma_1 T + \alpha_4 \Lambda - 2aE,$$

$$da = a_1 F + \alpha_4 \bar{F} + \alpha_1 T + a_4 \Lambda + hE - (3\alpha+\lambda)\Gamma - 2a\Omega, \tag{5}$$

$$dh = h_1 F - a_4 \bar{F} - a_1 T + h_4 \Lambda + 4a\Gamma - 4h\Omega,$$

where the possible forms of Ω and Γ are

$$\Omega = \omega_1 F + \omega_2 \bar{F} + \omega_3 T + \omega_4 \Lambda,$$

$$\Gamma = \gamma_1 F - 4\omega_4 \bar{F} - 4\omega_1 T + \gamma_4 \Lambda - (3\alpha+\lambda)E \tag{6}$$

and $\gamma_1,\gamma_4,\alpha_1,\alpha_4,a_1,a_4,h_1,h_4,\omega_1,\omega_2,\omega_3,\omega_4$ are complex functions on $\mathscr{U}$. The differentials of these functions can be further analyzed by looking at equations such as $d^2\alpha=0$, etc.

Reformulating the above statements in the language of the theory of differential systems we say that the differential system (1), (2), (5), (6) is still not closed. For example, the equations of the sort $d^2\alpha=0$ should still be added.

Lemma 2: $\mathscr{U}$ is locally foliated by two-dimensional manifolds $\mathscr{S}_x$, which are tangent to the real distribution $\mathscr{V}$ defined by

$$\Lambda(\mathscr{V}) = F(\mathscr{V}) = T(\mathscr{V}) = 0.$$

Here the proof is an immediate application of the Froebenius theorem, since the forms $(F,\bar{F},T,\Lambda)$ form a closed differential ideal due to the relations (1). □

Let us define

$$G = F\bar{F} - T\Lambda \tag{7}$$

on $\mathscr{U}$. G constitutes a degenerate metric on $\mathscr{U}$. It has the signature $(+,+,+,-,0,0)$.

Lemma 3: G is constant along any leaf $\mathscr{S}_x$ of the foliation $\{\mathscr{S}_x\}$.

To prove this we define a basis of vector fields $(f,\bar{f},X,Y,e,\bar{e})$, which is the respective dual of $(F,\bar{F},T,\Lambda,E,\bar{E})$. Then we notice that the distribution $\mathscr{V}$ is spanned by vector fields of the form

$$V = Ue + \bar{U}\bar{e},$$

where U is any complex function on $\mathscr{U}$. Using an arbitrary V and the explicit form of G one easily finds that $\mathscr{L}_V G = 0$ due to Eq. (1) and the properties of the Lie derivative $\mathscr{L}_V$. □

Now we introduce an equivalence relation $\sim$ on $\mathscr{U}$ which identifies points lying on the same leave of $\{\mathscr{S}_x\}$. We assume that the quotient space $\mathscr{M} = \mathscr{U}/\sim$ is a four-manifold. According to Lemma 3, G projects down to a well-defined nondegenerate Lorentzian metric g on $\mathscr{M}$.

Theorem 1: *The Lorentzian metric g on $\mathscr{M}=\mathscr{U}/\sim$ satisfies the Einstein equations $R_{ij}=\lambda g_{ij}$ and is not conformally flat.*

Proof: To prove the theorem, we first consider a four-dimensional submanifold $\mathscr{M}'$ of $\mathscr{U}$ that is transversal to the leaves of $\{\mathscr{S}_x\}$. We have a natural inclusion $\iota\colon \mathscr{M}'\hookrightarrow\mathscr{U}$. $\mathscr{M}'$ may be equipped with a Lorentzian metric $g'=\iota^*G$. It is clear that $(\mathscr{M},g)$ and $(\mathscr{M}',g')$ are locally isometrically equivalent. This statement does not depend on the choice of $\mathscr{M}'$. Thus, we may represent $(\mathscr{M},g)$ by $(\mathscr{M}',g')$. In this way, it is enough to show that g' satisfies the Einstein equations. To do this, we first observe that $g'=\iota^*(F)\iota^*(\bar{F})-\iota^*(T)\iota^*(\Lambda)$. Thus, a set of forms $\theta^i=(\iota^*(F),\iota^*(\bar{F}),\iota^*(T),\iota^*(\Lambda))$, $(i=1,2,3,4)$ constitutes a null cotetrad for g'. To calculate the curvature we need to know $d\theta^i$. But these, due to the fact that $\iota^*d=d\iota^*$, are given by the relations (1). To simplify the notation, we will omit the signs of the pullback ι^* in all of the formulas. On doing that we find that the connection one-forms for g' determined by

$$dF = -\Gamma^1{}_1 \wedge F - \Gamma^1{}_3 \wedge T - \Gamma^1{}_4 \wedge \Lambda,$$

$$d\bar{F} = -\Gamma^2{}_2 \wedge \bar{F} - \Gamma^2{}_3 \wedge T - \Gamma^2{}_4 \wedge \Lambda,$$

$$dT = -\Gamma^3{}_1 \wedge F - \Gamma^3{}_2 \wedge \bar{F} - \Gamma^3{}_3 \wedge T,$$

$$d\Lambda = -\Gamma^4{}_1 \wedge F - \Gamma^4{}_2 \wedge \bar{F} - \Gamma^4{}_4 \wedge \Lambda,$$

$$g_{ik}\Gamma^k{}_j + g_{jk}\Gamma^k{}_i = 0, \quad g_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

read

$$\Gamma^i{}_j = \begin{pmatrix} \bar{\Omega} - \Omega & 0 & -E & -\bar{\Gamma} \\ 0 & \Omega - \bar{\Omega} & -\bar{E} & -\Gamma \\ -\Gamma & -\bar{\Gamma} & \Omega + \bar{\Omega} & 0 \\ -\bar{E} & -E & 0 & -\Omega - \bar{\Omega} \end{pmatrix}.$$

The curvature $\mathscr{R}^i{}_j = d\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j$ of this connection can be calculated using the relations (1) and their consequences (2). Modulo the obvious reality conditions its components read as follows (we lower indices by means of the metric g_{ij}).

$$\mathscr{R}_{23} = -\bar{F} \wedge T - \alpha \Lambda \wedge F,$$

$$\mathscr{R}_{14} = \bar{\alpha} \bar{F} \wedge T + a(\Lambda \wedge T - F \wedge \bar{F}) - h \Lambda \wedge F,$$

$$\tfrac{1}{2}(\mathscr{R}_{43} - \mathscr{R}_{12}) = -(\alpha + \tfrac{1}{2}\lambda)(\Lambda \wedge T - F \wedge \bar{F}) - a \Lambda \wedge F.$$

These should be compared with the definitions of the spinorial coefficients of the Weyl tensor Ψ_μ, $\mu = 0,1,2,3,4$, the traceless Ricci tensor S_{ij}, and the Ricci scalar R given by

$$\mathscr{R}_{23} = \Psi_4 \bar{F} \wedge T + \Psi_3 (\Lambda \wedge T - F \wedge \bar{F}) + (\Psi_2 + \tfrac{1}{12}R)\Lambda \wedge F$$
$$+ \tfrac{1}{2}S_{33}F \wedge T + \tfrac{1}{2}S_{32}(\Lambda \wedge T + F \wedge \bar{F}) + \tfrac{1}{2}S_{22}\Lambda \wedge \bar{F},$$

$$\mathscr{R}_{14} = (-\Psi_2 - \tfrac{1}{12}R)\bar{F} \wedge T - \Psi_1 (\Lambda \wedge T - F \wedge \bar{F}) - \Psi_0 \Lambda \wedge F$$
$$- \tfrac{1}{2}S_{11}F \wedge T - \tfrac{1}{2}S_{41}(\Lambda \wedge T + F \wedge \bar{F}) - \tfrac{1}{2}S_{44}\Lambda \wedge \bar{F},$$

$$\tfrac{1}{2}(\mathscr{R}_{43} - \mathscr{R}_{12}) = \Psi_3 \bar{F} \wedge T + (\Psi_2 - \tfrac{1}{24}R)(\Lambda \wedge T - F \wedge \bar{F}) + \Psi_1 \Lambda \wedge F$$
$$+ \tfrac{1}{2}S_{31}F \wedge T + \tfrac{1}{4}(S_{12} + S_{34})(\Lambda \wedge T + F \wedge \bar{F}) + \tfrac{1}{2}S_{42}\Lambda \wedge \bar{F}.$$

Looking at these equations we easily get $S_{ij} = R_{ij} - \tfrac{1}{4}Rg_{ij} \equiv 0$. This proves the Einstein property of the metric.

The spinorial coefficients for the Weyl tensor are also easy to obtain. They read

$$\Psi_0 = h, \quad \Psi_1 = -a, \quad \Psi_2 = -\alpha - \frac{\lambda}{3}, \quad \Psi_3 = 0, \quad \Psi_4 = -1.$$

Due to the nonvanishing of Ψ_4 the metric is never conformally flat.

We can also interpret the constant λ which for the first time appeared in Eq. (2). One easily reads from the above that it is proportional to the Ricci scalar $R = 4\lambda$. This concludes the proof of the Theorem.   □

We, additionally, note that the metric is of the Cartan–Petrov–Penrose type D iff

$$a = 0; \quad h = -(3\alpha + \lambda)^2 \tag{8}$$

and of the type N iff

$$h = a = 0, \quad \lambda = -3\alpha. \tag{9}$$

The metric is algebraically special iff $I^3 = 27 J^2$, where

$$I = -h + \tfrac{1}{3}(3\alpha + \lambda)^2, \quad J = a^2 + \tfrac{1}{27}(3\alpha + \lambda)^3 + \tfrac{1}{3}h(3\alpha + \lambda).$$

The converse of the present paper can be stated in the following theorem (see Ref. 1 for details).

Theorem 2: *Let $\mathscr{M}$ be a four-dimensional manifold with a Lorentzian conformally nonflat metric g satisfying the Einstein equations $R_{ij} = \lambda g_{ij}$. Then, there exists a double branched cover $\widetilde{\mathscr{P}}$ of the bundle of null directions $\mathscr{P}$ over $\mathscr{M}$*[2,3]

    (i) which is a fibration over $\mathscr{M}$ with fibers being two-dimensional tori (or, in algebraically special cases, their degenerate counterparts[1]*)*

and

    (ii) on which there exists a unique system of one-forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ which satisfies the following three conditions:

(i) T, Λ are real- and F, E are complex-valued one-forms,
(ii) $F \wedge \bar{F} \wedge T \wedge \Lambda \wedge E \wedge \bar{E} \neq 0$ at each point p of $\widetilde{\mathscr{P}}$;
(iii)
$$dF = (\Omega - \bar{\Omega}) \wedge F + E \wedge T + \bar{\Gamma} \wedge \Lambda,$$
$$dT = \Gamma \wedge F + \bar{\Gamma} \wedge \bar{F} - (\Omega + \bar{\Omega}) \wedge T,$$
$$d\Lambda = \bar{E} \wedge F + E \wedge \bar{F} + (\Omega + \bar{\Omega}) \wedge \Lambda,$$
$$dE = 2\Omega \wedge E + \bar{F} \wedge T + \alpha \Lambda \wedge F$$

with certain complex-valued one-forms Ω and Γ, and a certain complex function α on $\widetilde{\mathscr{P}}$.

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