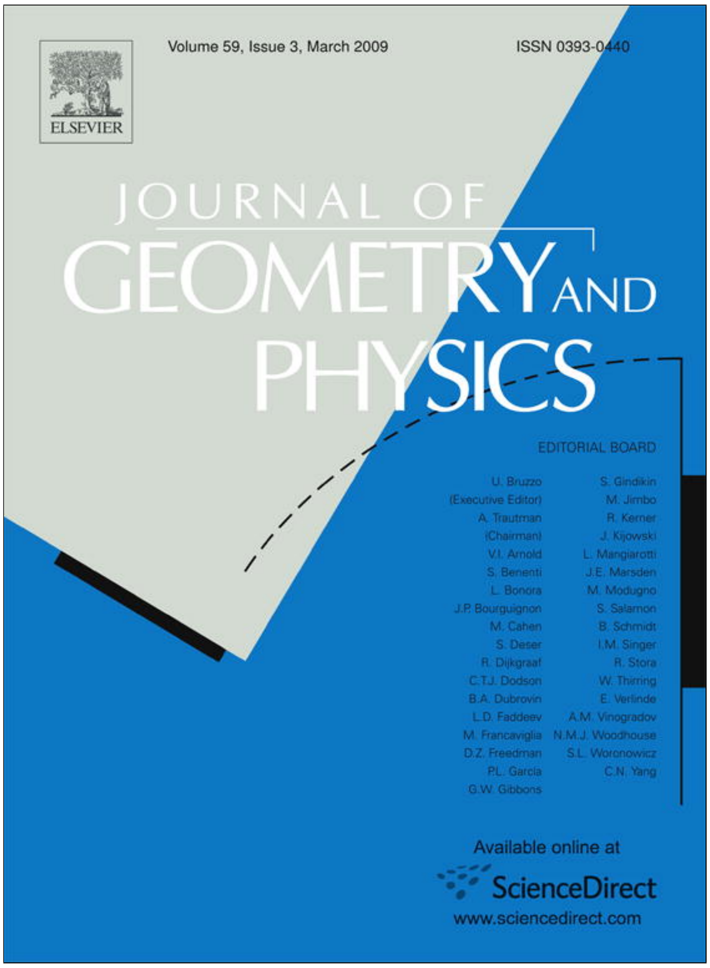
Volume 59, Issue 3, March 2009
ISSN 0393-0440
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GEOMETRY AND
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Journal of Geometry and Physics 59 (2009) 267–278

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

Comment on $\mathbf{GL}(2,\mathbb{R})$ geometry of fourth-order ODEs ☆

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ARTICLE INFO

Article history:
Received 9 October 2007
Accepted 5 November 2008
Available online 24 December 2008

MSC:
53A40
53B05
34C30
34C31

Keyword:
Geometrization of ODEs

ABSTRACT

We describe fourth-order ODEs satisfying two contact invariant conditions of Bryant in terms of the Ricci tensor of a certain $\mathfrak{gl}(2,\mathbb{R})$-valued connection. We also provide nonhomogeneous examples of such ODEs.

1. Introduction

Recently there has been a growing interest in the geometrization program of ODEs [1–6]. Although the program my be traced back to Lie [7] and Tresse [8], and although it was formulated by E. Cartan and S. S. Chern in the 1940s [9–11], it was not very popular until the works of R. Bryant (see e.g. [12]) on the invariants of the fourth-order ODEs. In the present note we restate some of the results of [12] in terms of the invariants of the recently discussed $\mathbf{GL}(2,\mathbb{R})$ geometry of ODEs [4]. In particular we interpret Bryant's results in terms of the Ricci tensor of a certain $\mathfrak{gl}(2,\mathbb{R})$-connection, which characterises the ODEs satisfying contact invariant conditions of Bryant [12].

Our starting point is the following well-known proposition.

Proposition 1.1. *The ordinary differential equation*

$$y^{(4)}=0$$

has $\mathbf{GL}(2,\mathbb{R})\times_\rho\mathbb{R}^4$ *as its group of contact symmetries. Here* $\rho:\mathbf{GL}(2,\mathbb{R})\to\mathbf{GL}(4,\mathbb{R})$ *is the 4-dimensional irreducible representation of* $\mathbf{GL}(2,\mathbb{R})$.

The representation ρ, at the level of the Lie algebra $\mathfrak{gl}(2,\mathbb{R})$, is given in terms of the Lie algebra generators

$$E_+=\begin{pmatrix}0&3&0&0\\0&0&2&0\\0&0&0&1\\0&0&0&0\end{pmatrix},\qquad E_-=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&2&0&0\\0&0&3&0\end{pmatrix},\qquad E_0=\begin{pmatrix}-3&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&3\end{pmatrix},$$

$$E=-\begin{pmatrix}3&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3\end{pmatrix}. \qquad (1.1)$$

☆ This research was supported by the KBN grant 1 P03B 07529.

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0393-0440/$ – see front matter © 2009 Published by Elsevier B.V.
doi:10.1016/j.geomphys.2008.11.015

These matrices satisfy the $\mathfrak{gl}(2,\mathbb{R})$ commutation relations

$$[E_0,E_+]=-2E_+,\qquad [E_0,E_-]=2E_-,\qquad [E_+,E_-]=-E_0,$$

where the commutator in the $\mathfrak{gl}(2,\mathbb{R})=\mathrm{Span}_{\mathbb{R}}(E_-,E_+,E_0,E)\subset\mathrm{End}(\mathbb{R}^4)$ is the usual commutator of matrices.

Now, we consider a general fourth-order ODE

$$y^{(4)}=F\left(x,y,y',y'',y^{(3)}\right). \tag{1.2}$$

To simplify the notation, we introduce the coordinates $x, y, y_1=y', y_2=y'', y_3=y^{(3)}$ on the 5-dimensional *jet space* J. Introducing the four *contact forms*

$$\begin{aligned}\omega^0&=\mathrm{d}y-y_1\mathrm{d}x\\ \omega^1&=\mathrm{d}y_1-y_2\mathrm{d}x\\ \omega^2&=\mathrm{d}y_2-y_3\mathrm{d}x\\ \omega^3&=\mathrm{d}y_3-F(x,y,y_1,y_2,y_3)\mathrm{d}x,\end{aligned} \tag{1.3}$$

and an additional 1-form

$$w_+=\mathrm{d}x,$$

we define a *contact transformation* to be a diffeomorphism $\phi: J\to J$ which transforms the above five 1-forms via:

$$\begin{aligned}\phi^*\omega^0&=\alpha^0{}_0\omega^0\\ \phi^*\omega^1&=\alpha^1{}_0\omega^0+\alpha^1{}_1\omega^1\\ \phi^*\omega^2&=\alpha^2{}_0\omega^0+\alpha^2{}_1\omega^1+\alpha^2{}_2\omega^2\\ \phi^*\omega^3&=\alpha^3{}_0\omega^0+\alpha^3{}_1\omega^1+\alpha^3{}_2\omega^2+\alpha^3{}_3\omega^3\\ \phi^*w_+&=\alpha^4{}_0\omega^0+\alpha^4{}_1\omega^1+\alpha^4{}_4w_+.\end{aligned} \tag{1.4}$$

Here $\alpha^i{}_j$, $i,j=0,1,2,3,4,5$, are real functions on J such that

$$\alpha^0{}_0\alpha^1{}_1\alpha^2{}_2\alpha^3{}_3\alpha^4{}_4\neq 0.$$

The contact equivalence problem for the fourth-order ODEs (1.2) can be studied in terms of the invariant forms $(\theta^0,\theta^1,\theta^2,\theta^3,\Omega_+)$ defined by

$$\begin{pmatrix}\theta^0\\ \theta^1\\ \theta^2\\ \theta^3\\ \Omega_+\end{pmatrix}=\begin{pmatrix}\alpha^0{}_0&&&&\\ \alpha^1{}_0&\alpha^1{}_1&&&\\ \alpha^2{}_0&\alpha^2{}_1&\alpha^2{}_2&&\\ \alpha^3{}_0&\alpha^3{}_1&\alpha^3{}_2&\alpha^3{}_3&\\ \alpha^4{}_0&\alpha^4{}_1&&&\alpha^4{}_4\end{pmatrix}\begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega^3\\ w_+\end{pmatrix}. \tag{1.5}$$

Among all ODEs (1.2) considered modulo contact transformations (1.4) there is a remarkable class for which the invariant forms satisfy

$$\begin{aligned}\mathrm{d}\theta^0&=3(\Omega+\Omega_0)\wedge\theta^0-3\Omega_+\wedge\theta^1\\ \mathrm{d}\theta^1&=-\Omega_-\wedge\theta^0+(3\Omega+\Omega_0)\wedge\theta^1-2\Omega_+\wedge\theta^2\\ \mathrm{d}\theta^2&=-2\Omega_-\wedge\theta^1+(3\Omega-\Omega_0)\wedge\theta^2-\Omega_+\wedge\theta^3\\ \mathrm{d}\theta^3&=-3\Omega_-\wedge\theta^2+3(\Omega-\Omega_0)\wedge\theta^3.\end{aligned} \tag{1.6}$$

This system is defined on an 8-dimensional $\mathbf{GL}(2,\mathbb{R})$ principal fibre bundle P over the solution space M^4 for the corresponding ODE (1.2). The invariant forms $(\theta^0,\theta^1,\theta^2,\theta^3,\Omega_+)$ together with the additional three 1-forms $(\Omega_-,\Omega_0,\Omega)$ constitute a well-defined coframe on P.

As noted by Bryant [12], the class of ODEs having forms $(\theta^0,\theta^1,\theta^2,\theta^3,\Omega_+,\Omega_-,\Omega_0,\Omega)$ of system (1.6), is distinguished by the demand that their defining functions $F=F(x,y,y_1,y_2,y_3)$ satisfy the following two conditions:

$$\begin{aligned}&4D^2F_3-8DF_2+8F_1-6DF_3F_3+4F_2F_3+F_3^3=0,\\ &160D^2F_2-640DF_1+144(DF_3)^2-352DF_3F_2+144F_2^2-\\ &80DF_2F_3+160F_1F_3-72DF_3F_3^2+88F_2F_3^2+9F_3^4+16000F_y=0.\end{aligned} \tag{1.7}$$

Here $F_i=\frac{\partial F}{\partial y_i}$ and $D=\partial_x+y_1\partial_y+y_2\partial_{y_1}+y_3\partial_{y_2}+F\partial_{y_3}$. Bryant's conditions (1.7), considered simultaneously, are *contact invariant*; if the ODE undergoes contact transformation of its variables, conditions (1.7) are preserved. Examples are known of ODEs satisfying these conditions [12], the simplest being

$$y^{(4)}=(y^{(3)})^{(4/3)}. \tag{1.8}$$

The purpose of this note is to establish a theorem on speciality of a $\mathfrak{gl}(2,\mathbb{R})$-valued connection defined by such ODEs on their solution spaces.

2. The closed system

Let us make the following choice

$$\begin{aligned}
\alpha^0{}_0&=-3\alpha^1{}_1\alpha^4{}_4\\
\alpha^2{}_0&=-\frac{(\alpha^1{}_0)^2}{3\alpha^1{}_1\alpha^4{}_4}+\frac{\alpha^1{}_1}{240\alpha^4{}_4}(-24DF_3+36F_2+11F_3^2)\\
\alpha^2{}_1&=-\frac{2\alpha^1{}_0}{3\alpha^4{}_4}+\frac{\alpha^1{}_1}{12\alpha^4{}_4}F_3\\
\alpha^2{}_2&=-\frac{\alpha^1{}_1}{2\alpha^4{}_4}\\
\alpha^3{}_0&=\frac{(\alpha^1{}_0)^3}{9(\alpha^1{}_1\alpha^4{}_4)^2}+\frac{\alpha^1{}_0}{240(\alpha^4{}_4)^2}(24DF_3-36F_2-11F_3^2)+\frac{\alpha^1{}_1}{720(\alpha^4{}_4)^2}(36(DF_2-4F_1)+18(DF_3-2F_2)F_3-7F_3^3)\\
\alpha^3{}_1&=\frac{(\alpha^1{}_0)^2}{3\alpha^1{}_1(\alpha^4{}_4)^2}-\frac{\alpha^1{}_0}{12(\alpha^4{}_4)^2}F_3+\frac{\alpha^1{}_1}{240(\alpha^4{}_4)^2}(36DF_3-84F_2-19F_3^2)\\
\alpha^3{}_2&=\frac{\alpha^1{}_0}{2(\alpha^4{}_4)^2}-\frac{\alpha^1{}_1}{4(\alpha^4{}_4)^2}F_3\\
\alpha^3{}_3&=\frac{\alpha^1{}_1}{2(\alpha^4{}_4)^2}\\
\alpha^4{}_0&=-\frac{\alpha^4{}_4}{60}(12DF_{33}-6F_{23}+F_3F_{33})\\
\alpha^4{}_1&=\frac{\alpha^4{}_4}{6}F_{33}
\end{aligned} \tag{2.1}$$

for the group parameters defining forms $(\theta^0,\theta^1,\theta^2,\theta^3,\Omega_+)$ of (1.5). Then we have the following theorem.

Theorem 2.1. *If a fourth-order ODE*

$$y^{(4)}=F\left(x,y,y',y'',y^{(3)}\right) \tag{2.2}$$

satisfies contact invariant conditions

$$\begin{aligned}
&4D^2F_3-8DF_2+8F_1-6DF_3F_3+4F_2F_3+F_3^3=0,\\
&160D^2F_2-640DF_1+144(DF_3)^2-352DF_3F_2+144F_2^2\\
&\quad-80DF_2F_3+160F_1F_3-72DF_3F_3^2+88F_2F_3^2+9F_3^4+16000F_y=0
\end{aligned} \tag{2.3}$$

then the manifold P parametrised by $(x,y,y_1,y_2,y_3,\alpha^1{}_0,\alpha^1{}_1,\alpha^4{}_4)$ is a principal $\mathbf{GL}(2,\mathbb{R})$ *bundle $P\to M^4$ over the solution space M^4 of (2.2) and forms $(\theta^0,\theta^1,\theta^2,\theta^3,\Omega_+)$, together with additional three 1-forms $(\Omega_-,\Omega_0,\Omega)$, constitute an invariant coframe on P satisfying*

$$\begin{aligned}
d\theta^0&=3(\Omega+\Omega_0)\wedge\theta^0-3\Omega_+\wedge\theta^1\\
d\theta^1&=-\Omega_-\wedge\theta^0+(3\Omega+\Omega_0)\wedge\theta^1-2\Omega_+\wedge\theta^2\\
d\theta^2&=-2\Omega_-\wedge\theta^1+(3\Omega-\Omega_0)\wedge\theta^2-\Omega_+\wedge\theta^3\\
d\theta^3&=-3\Omega_-\wedge\theta^2+3(\Omega-\Omega_0)\wedge\theta^3
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
d\Omega_+ &= 2\Omega_0\wedge\Omega_+ + \frac{1}{12}(-3a_0+4b_1)\theta^0\wedge\theta^1 + \frac{1}{4}(a_1+2b_2)\theta^0\wedge\theta^2\\
&\quad + \frac{1}{24}(3a_2+4b_3)\theta^0\wedge\theta^3 + \frac{1}{8}(-5a_2+4b_3)\theta^1\wedge\theta^2 + \frac{1}{6}b_4\theta^1\wedge\theta^3\\
d\Omega_- &= -2\Omega_0\wedge\Omega_- + \frac{1}{6}b_0\theta^0\wedge\theta^2 + \frac{1}{24}(-3a_0+4b_1)\theta^0\wedge\theta^3\\
&\quad + \frac{1}{8}(5a_0+4b_1)\theta^1\wedge\theta^2 + \frac{1}{4}(-a_1+2b_2)\theta^1\wedge\theta^3 + \frac{1}{12}(3a_2+4b_3)\theta^2\wedge\theta^3 \qquad (2.5)\\
d\Omega_0 &= \Omega_+\wedge\Omega_- - \frac{1}{6}b_0\theta^0\wedge\theta^1 + \frac{1}{24}(-3a_0-4b_1)\theta^0\wedge\theta^2 + \frac{1}{4}a_1\theta^0\wedge\theta^3 - \frac{1}{4}a_1\theta^1\wedge\theta^2\\
&\quad + \frac{1}{24}(-3a_2+4b_3)\theta^1\wedge\theta^3 + \frac{1}{6}b_4\theta^2\wedge\theta^3\\
d\Omega &= -\frac{1}{6}b_0\theta^0\wedge\theta^1 - \frac{1}{3}b_1\theta^0\wedge\theta^2 - \frac{1}{6}b_2\theta^0\wedge\theta^3 - \frac{1}{2}b_2\theta^1\wedge\theta^2 - \frac{1}{3}b_3\theta^1\wedge\theta^3 - \frac{1}{6}b_4\theta^2\wedge\theta^3.
\end{aligned}$$

The coefficients $a_0, a_1, a_2, b_0, b_1, b_2, b_3, b_4$ are totally determined by (2.2) and are expressible in terms of the derivatives of function F and the coordinates. The simplest of these coefficients are:

$$\begin{aligned}
b_4 &= -2\frac{(\alpha^4{}_4)^3}{(\alpha^1{}_1)^2}F_{333}\\
b_3 &= -\frac{(\alpha^4{}_4)^2}{12(\alpha^1{}_1)^2}(6DF_{333}+5F_3F_{333}) - \frac{2\alpha^1{}_0(\alpha^4{}_4)^2}{3(\alpha^1{}_1)^3}F_{333}\\
b_2 &= -\frac{2\alpha^4{}_4(\alpha^1{}_0)^2}{9(\alpha^1{}_1)^4}F_{333} - \frac{\alpha^4{}_4\alpha^1{}_0}{18(\alpha^1{}_1)^3}(6DF_{333}+5F_3F_{333})\\
&\quad + \frac{\alpha^4{}_4}{360(\alpha^1{}_1)^2}[60(2DF_{233}+4F_{133}-2F_{223}+DF_{333}F_3) + (-36DF_3+204F_2+79F_3^2)F_{333}]\\
a_2 &= -\frac{(\alpha^4{}_4)^2}{45(\alpha^1{}_1)^2}(18DF_{333}+24F_{233}+4F_{33}^2+27F_3F_{333}).
\end{aligned}$$

Other coefficients are given in the next two sections.

The proof of this theorem is a lengthy calculation based on a variant of Cartan's equivalence method. In the next section we outline the main points of the proof.

3. Proof of the main theorem

The basic idea in the proof of Theorem 2.1 is to force 1-forms (1.5) to satisfy system (1.6). This requirement makes restrictions on the free parameters $\alpha^i{}_j$ and, more importantly, on the possible functions $F = F(x, y, y', y'', y^{(3)})$ defining the ODE.

The main steps when imposing (1.6) on (1.5) are:

(1) equation $d\theta^0\wedge\theta^0\wedge\theta^2 = 3\Omega_+\wedge\theta^0\wedge\theta^1\wedge\theta^2$ requires $\alpha^0{}_0 = -3\alpha^1{}_1\alpha^4{}_4$,
(2) the first equation (1.6) gives a relation between Ω, Ω_0, $d\alpha^4{}_4$ and $d\alpha^1{}_1$,
(3) similarly, equation $d\theta^1\wedge\theta^1\wedge\theta^2 = -\Omega_-\wedge\theta^0\wedge\theta^1\wedge\theta^2$ gives a relation between Ω_-, $d\alpha^1{}_0$, $d\alpha^1{}_1$,
(4) equation $d\theta^1\wedge\theta^0\wedge\theta^1 = -2\Omega_+\wedge\theta^0\wedge\theta^1\wedge\theta^2$ gives $\alpha^2{}_2 = -\frac{\alpha^1{}_1}{2\alpha^4{}_4}$,
(5) equation $d\theta^1\wedge\theta^0\wedge\theta^2 = -(3\Omega+\Omega_0)\wedge\theta^0\wedge\theta^1\wedge\theta^2$ gives a relation between Ω, Ω_0 and $d\alpha^1{}_1$,
(6) now, the expressions for $d\theta^2\wedge\theta^0\wedge\theta^1\wedge\theta^3$, $d\theta^2\wedge\theta^0\wedge\theta^1\wedge\theta^3$, $d\theta^3\wedge\theta^0\wedge\theta^1\wedge\theta^2$ enable us to fix $\alpha^3{}_2$, $\alpha^3{}_3$, and $\alpha^2{}_1$ respectively,
(7) considering successively $d\theta^3\wedge\theta^0\wedge\theta^1$, $d\theta^2\wedge\theta^0$, $d\theta^2\wedge\theta^1\wedge\theta^2\wedge\theta^3$ we fix $\alpha^2{}_0$, $\alpha^3{}_1$, $\alpha^3{}_0$,
(8) now the requirement $d\theta^3\wedge\theta^0\wedge\theta^2\wedge\theta^3 = 0$ gives the first Bryant condition $4D^2F_3-8DF_2+8F_1-6DF_3F_3+4F_2F_3-F_3^3 = 0$,
(9) the second of Bryant's conditions (1.7) is equivalent to the requirement that $d\theta^3\wedge\theta^1\wedge\theta^2\wedge\theta^3 = 0$,
(10) now, having Bryant's conditions determined, it is straightforward to obtain the required system (1.6) and express all the $\alpha^i{}_j$'s in terms of $\alpha^1{}_0$, $\alpha^1{}_1$ and $\alpha^4{}_4$ only,
(11) the expressions for $\alpha^i{}_j$'s are given by (2.1); inserting them to (1.5) we get the invariant forms $(\theta^0, \theta^1, \theta^2, \theta^3, \Omega_+)$
(12) forms Ω_0, Ω_-, Ω are determined by the linear relations from points (2), (3) and (5).

In this way one finds the explicit expressions for the invariant coframe satisfying system (1.6). Instead of giving these formulae we present formulae for $(\theta^0, \theta^1, \theta^2, \theta^3, \Omega_+)$ evaluated at $(\alpha^1{}_0, \alpha^1{}_1, \alpha^4{}_4) = (0, 1, 1)$. Denoting these forms by $(\theta^0_0, \theta^1_0, \theta^2_0, \theta^3_0, \Omega^0_+)$, we have

$$\theta^0_0 = -3\omega^0$$
$$\theta^1_0 = \omega^1$$
$$\theta^2_0 = \frac{1}{240}(-24DF_3 + 36F_2 + 11F_3^2)\omega^0 + \frac{1}{12}F_3\omega^1 - \frac{1}{2}\omega^2$$
$$\theta^3_0 = \frac{1}{720}\left(36(DF_2 - 4F_1) + 18(DF_3 - 2F_2)F_3 - 7F_3^2\right)\omega^0 + \frac{1}{240}(36DF_3 - 84F_2 - 19F_3^2)\omega^1 - \frac{1}{4}F_3\omega^2 + \frac{1}{2}\omega^3$$
$$\Omega^0_+ = -\frac{1}{60}(12DF_{33} - 6F_{23} + F_3F_{33})\omega^0 + \frac{1}{6}F_{33}\omega^1 + w_+.$$

The remaining three 1-forms $(\Omega_0, \Omega_-, \Omega)$, when written in the gauge $({\alpha^1}_0, {\alpha^1}_1, {\alpha^4}_4) = (0, 1, 1)$, read:

$$\begin{aligned}
\Omega^0_0 \;=\; & \frac{1}{4320}(72DF_{23} + 432F_{13} - 288F_{22} + 60DF_{33}F_3 - 216F_{23}F_3 - 108DF_3F_{33} + 324F_2F_{33} + 47F_3^2F_{33})\theta^0_0 \\
& + \frac{1}{180}(3DF_{33} - 9F_{23} - F_3F_{33})\theta^1_0 + \frac{1}{6}F_{33}\theta^2_0 - \frac{1}{12}F_3\theta^4_0 \\
\Omega^0_- \;=\; & \frac{1}{64800}(720DF_{22} + 288DF_3DF_{33} - 2160F_{12} - 432DF_{33}F_2 + 216DF_3F_{23} \\
& + 216F_2F_{23} + 720DF_{23}F_3 - 1080F_{13}F_3 - 360F_{22}F_3 + 48DF_{33}F_3^2 \\
& - 174F_{23}F_3^2 - 360DF_2F_{33} + 1440F_1F_{33} + 24DF_3F_3F_{33} + 324F_2F_3F_{33} + 29F_3^3F_{33} + 3600F_{3y})\theta^0_0 \\
& + \frac{1}{1080}(-108DF_{23} - 288F_{13} + 252F_{22} - 54DF_{33}F_3 + 186F_{23}F_3 + 66DF_3F_{33} - 252F_2F_{33} - 31F_3^2F_{33})\theta^1_0 \\
& + \frac{1}{90}(12DF_{33} - 6F_{23} + F_3F_{33})\theta^2_0 + \frac{1}{360}(-24DF_3 + 36F_2 + 11F_3^2)\theta^4_0 \\
\Omega^0 \;=\; & \frac{1}{4320}(120DF_{23} + 240F_{13} - 240F_{22} + 36DF_{33}F_3 - 168F_{23}F_3 - 36DF_3F_{33} + 204F_2F_{33} + 17F_3^2F_{33})\theta^0_0 \\
& + \frac{1}{12}(-DF_{33} + F^{23})\theta^1_0 - \frac{1}{6}F_{33}\theta^2_0 + \frac{1}{12}F_3\theta^4_0.
\end{aligned}$$

All the eight forms $(\theta^0, \theta^1, \theta^2, \theta^3, \Omega^0_+, \Omega^0_0, \Omega^0_-, \Omega^0)$ satisfy system (2.4) and (2.5), with corresponding coefficients $(a^0_0, a^0_1, a^0_2, b^0_0, b^0_1, b^0_2, b^0_3, b^0_4)$ given by:

$$b^0_4 = -2F_{333}$$
$$b^0_3 = \frac{1}{12}(-6DF_{333} - 5F_3F_{333})$$
$$b^0_2 = \frac{1}{360}(120DF_{233} + 240F_{133} - 120F_{223} + 60DF_{333}F_3 - 36DF_3F_{333} + 204F_2F_{333} + 79F_3^2F_{333})$$
$$\begin{aligned}
b^0_1 \;=\; & \frac{1}{1080}(-180DF_{223} - 540F_{123} + 360F_{222} - 90DF_{33}F_{23} + 270F_{23}^2 + 90DF_3F_{233} - 540F_2F_{233} \\
& - 180DF_{233}F_3 - 270F_{133}F_3 + 360F_{223}F_3 - 45DF_{333}F_3^2 - 45F_{233}F_3^2 + 90DF_{23}F_{33} + 90F_{13}F_{33} - 360F_{22}F_{33} \\
& - 90F_{23}F_3F_{33} + 90F_2F_{33}^2 + 18DF_2F_{333} - 72F_1F_{333} + 54DF_3F_3F_{333} - 288F_2F_3F_{333} - 71F_3^3F_{333} - 180F_{33y})
\end{aligned}$$
$$\begin{aligned}
b^0_0 \;=\; & \frac{1}{129600}(-8640DF_{233}DF_3 - 12960DF_{23}DF_{33} - 4320DF_2DF_{333} + 43200DF_{33y} \\
& + 17280DF_{333}F_1 + 129600F_{113} - 64800F_{122} - 34560DF_{33}F_{13} - 86400DF_3F_{133} \\
& + 12960DF_{233}F_2 + 194400F_{133}F_2 + 30240DF_{33}F_{22} + 32400DF_3F_{223} - 64800F_2F_{223} \\
& + 6480DF_{23}F_{23} - 25920F_{13}F_{23} - 15120F_{22}F_{23} + 2160DF_2F_{233} - 8640F_1F_{233} \\
& - 64800F_{23y} - 6480DF_{33}^2F_3 - 6480DF_3DF_{333}F_3 - 21600F_{123}F_3 + 10800DF_{333}F_2F_3 \\
& - 10800F_{222}F_3 + 25560DF_{33}F_{23}F_3 - 18360F_{23}^2F_3 + 13320DF_3F_{233}F_3 - 25920F_2F_{233}F_3 \\
& + 3960DF_{233}F_3^2 + 50400F_{133}F_3^2 - 28800F_{223}F_3^2 + 2820DF_{333}F_3^3 - 10980F_{233}F_3^3 \\
& - 18000DF_{22}F_{33} + 6480DF_3DF_{33}F_{33} + 86400F_{12}F_{33} - 28080DF_{33}F_2F_{33} \\
& - 11880DF_3F_{23}F_{33} + 10800F_2F_{23}F_{33} - 19080DF_{23}F_3F_{33} + 18720F_{13}F_3F_{33} \\
& + 16920F_{22}F_3F_{33} - 8100DF_{33}F_3^2F_{33} + 7200F_{23}F_3^2F_{33} + 7560DF_2F_{33}^2 - 30240F_1F_{33}^2 \\
& - 11520F_2F_3F_{33}^2 - 1620F_3^3F_{33}^2 + 11664DF_3^2F_{333} - 63072DF_3F_2F_{333} + 76464F_2^2F_{333} \\
& - 2520DF_2F_3F_{333} + 10080F_1F_3F_{333} - 17712DF_3F_3^2F_{333} + 42768F_2F_3^2F_{333} + 5299F_3^4F_{333} \\
& - 18000F_3F_{33y} - 75600F_{33}F_{3y} + 43200F_{333}F_y)
\end{aligned}$$
$$a^0_2 = \frac{1}{45}(-18DF_{333} - 24F_{233} - 4F_{33}^2 - 27F_3F_{333})$$

$$a^0_1 = \frac{1}{540}(-72DF_{233} - 432F_{133} + 216F_{223} - 36DF_{333}F_3 + 96F_{233}F_3 + 48DF_{33}F_{33} + 16F_3F^2_{33} + 108DF_3F_{333} - 324F_2F_{333} - 81F^2_3F_{333})$$

$$a^0_0 = \frac{1}{4050}(-180DF_{223} + 288DF^2_{33} - 4860F_{123} + 2520F_{222} - 378DF_{33}F_{23} + 1782F^2_{23} + 810DF_3F_{233} - 2700F_2F_{233} - 180DF_{233}F_3 - 2430F_{133}F_3 + 2880F_{223}F_3 - 45DF_{333}F^2_3 + 435F_{233}F^2_3 + 810DF_{23}F_{33} + 810F_{13}F_{33} - 2520F_{22}F_{33} + 408DF_{33}F_3F_{33} - 594F_{23}F_3F_{33} + 810F_2F^2_{33} + 122F^2_3F^2_{33} - 270DF_2F_{333} + 1080F_1F_{333} + 270DF_3F_3F_{333} - 1080F_2F_3F_{333} - 135F^3_3F_{333} + 2700F_{33y}).$$

One can use these, relatively simple, formulae to generate expressions for the invariant forms on P. This may be achieved by means of a matrix

$$m = \begin{pmatrix} \alpha^1{}_1\alpha^4{}_4 & 0 & 0 & 0 \\ -\frac{\alpha^1{}_0}{3} & \alpha^1{}_1 & 0 & 0 \\ \frac{(\alpha^1{}_0)^2}{9\alpha^1{}_1\alpha^4{}_4} & -\frac{2\alpha^1{}_0}{3\alpha^4{}_4} & \frac{\alpha^1{}_1}{\alpha^4{}_4} & 0 \\ \frac{-(\alpha^1{}_0)^3}{27(\alpha^1{}_1\alpha^4{}_4)^2} & \frac{(\alpha^1{}_0)^2}{3\alpha^1{}_1(\alpha^4{}_4)^2} & -\frac{\alpha^1_0}{(\alpha^4{}_4)^2} & \frac{\alpha^1{}_1}{(\alpha^4{}_4)^2} \end{pmatrix}. \tag{3.1}$$

Then the expression for the invariant 1-forms $(\theta^i) = (\theta^0, \theta^1, \theta^2, \theta^3)$ can be written as

$$\theta^i = m^i{}_j\theta^j_0, \quad i, j = 0, 1, 2, 3. \tag{3.2}$$

The residual group $G = \{m \mid \alpha^1{}_1, \alpha^4{}_4 \neq 0, \alpha^1{}_0 \in \mathbb{R}\}$ has the Lie algebra $\mathfrak{g} = \mathfrak{h}_2 \oplus \mathfrak{h}_1$ isomorphic to the direct sum of the 2-dimensional noncommuting Lie algebra $\mathfrak{h}_2$ and a 1-dimensional Lie algebra $\mathfrak{h}_1$. Algebra $\mathfrak{h}_2$ is related to the parameters $(\alpha^1{}_0, \alpha^4{}_4)$ and algebra $\mathfrak{h}_1$ is associated with $\alpha^1{}_1$.

The action of G on θ^i_0, induces its action on $(\Omega^0_+, \Omega^0_0, \Omega^0_-, \Omega^0)$. Indeed, defining

$$\overset{0}{\Gamma} = \Omega^0_-E_- + \Omega^0_+E_+ + \Omega^0_0E_0 + \Omega^0E,$$

and

$$\Gamma = \Omega_-E_- + \Omega_+E_+ + \Omega_0E_0 + \Omega E,$$

where 4×4 matrices (E_-, E_+, E_0, E) are the generators of the Lie algebra $\mathfrak{gl}(2, \mathbb{R})$ given in (1.1), we find that

$$\Gamma = m\,\overset{0}{\Gamma}\,m^{-1} + m\mathrm{d}m^{-1}.$$

This enables us to find the explicit expressions for the invariant forms $(\Omega_+, \Omega_0, \Omega_-, \Omega)$.

The transformation rule for Γ resembles the transformation rule for a connection. Since Γ is $\mathfrak{gl}(2, \mathbb{R})$-valued, it is reasonable to look for a $\mathbf{GL}(2, \mathbb{R})$ principal fibre bundle associated with the corresponding ODE (2.2).

Due to properties of system (2.4) and (2.5) the desired bundle is just P of Theorem 2.1. To see this, note that Eqs. (2.4) ensure that $(\theta^1, \theta^2, \theta^3, \theta^4)$ form a closed differential ideal. Thus a 4-dimensional distribution $\mathcal{V}$ on P such that $\mathcal{V} \lrcorner \theta^i = 0$, $\forall i = 0, 1, 2, 3$, is integrable. As a consequence, the manifold P is foliated by 4-dimensional integral leaves of this distribution. Looking at Eq. (2.5) we see that *on each leaf* of $\mathcal{V}$ the forms $(\Omega_+, \Omega_0, \Omega_-, \Omega)$ satisfy the Maurer–Cartan equations for the $\mathbf{GL}(2, \mathbb{R})$ group. This means that P is a principal $\mathbf{GL}(2, \mathbb{R})$ bundle over the leaf space $M^4 = P/\mathcal{V}$. This 4-dimensional space may be identified with a solution space of ODE (2.2).

Remark 3.1. For local calculations, it may be convenient to pass from coordinates $(x, y, y_1, y_2, y_3, \alpha^1{}_0, \alpha^1{}_1, \alpha^4{}_4)$ on P to coordinates $(c_0, c_1, c_2, c_3, s, \alpha^1{}_0, \alpha^1{}_1, \alpha^4{}_4)$ on P, where (c_0, c_1, c_2, c_3) are the integration constants of ODE (2.2), and s is a real parameter such that the total differential vector field $D = \partial_s$. In such parametrisation $(s, \alpha^1{}_0, \alpha^1{}_1, \alpha^4{}_4)$ constitute coordinates on the leaves of $\mathcal{V}$ and (c_0, c_1, c_2, c_3) parametrise the solution space M^4.

4. GL(2, $\mathbb{R}$) geometry on the solution space

Using matrices $\Gamma = (\Gamma^i{}_j)$, $i, j = 0, 1, 2, 3$, and part $\theta^i = (\theta^0, \theta^1, \theta^2, \theta^3)$ of the invariant coframe we rewrite Eq. (2.4) in a compact form as:

$$\mathrm{d}\theta^i + \Gamma^i{}_j \wedge \theta^j = 0, \tag{4.1}$$

and Eq. (2.5) in a compact form as:

$$\mathrm{d}\Gamma^i{}_k + \Gamma^i{}_j \wedge \Gamma^j{}_k = \frac{1}{2}R^i{}_{kjl}\theta^j \wedge \theta^l. \tag{4.2}$$

The coefficients $R^i{}_{jkl}$ appearing in this last equation can be easily read off from (2.5). They are linear combinations of the coefficients $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, b_4$ of (2.5). The meaning of Eq. (4.1)–(4.2) is obvious: they constitute, respectively, the first and the second Cartan's structure equations, for a $\mathfrak{gl}(2,\mathbb{R})$-valued connection Γ on the principal fibre bundle $\mathbf{GL}(2,\mathbb{R}) \to P \to M^4$. Due to the first equation, (4.1), this connection has no torsion. The second equation, (4.2), determines the curvature of Γ; the coefficients $R^i{}_{jkl}$ are the curvature tensor coefficients for Γ.

Given the curvature tensor $R^i{}_{jkl}$ of Γ we define its 'Ricci' tensor R_{jl} by

$$R_{jl} = R^i{}_{jil}.$$

Recalling that the curvature of Γ is totally expressible in terms of $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, b_4$ and performing a purely algebraic manipulation on the curvature tensor coefficients $R^i{}_{jkl}$, we get a remarkable theorem.

Theorem 4.1. *Every fourth-order ODE satisfying conditions (2.3) uniquely defines a principal fibre bundle $\mathbf{GL}(2,\mathbb{R}) \to P \to M^4$ over the space of its solutions M^4 and a torsionless $\mathfrak{gl}(2,\mathbb{R})$-connection Γ on P with curvature $R^i{}_{jkl}$ having the Ricci tensor R_{jl} in the form*

$$R_{jl} = \begin{pmatrix} 0 & b_0 & a_0 + 2b_1 & -a_1 + b_2 \\ -b_0 & -2a_0 & a_1 + 3b_2 & a_2 + 2b_3 \\ a_0 - 2b_1 & a_1 - 3b_2 & -2a_2 & b_4 \\ -a_1 - b_2 & a_2 - 2b_3 & -b_4 & 0 \end{pmatrix}.$$

Its respective symmetric and antisymmetric parts read:

$$R_{(jl)} = \begin{pmatrix} 0 & 0 & a_0 & -a_1 \\ 0 & -2a_0 & a_1 & a_2 \\ a_0 & a_1 & -2a_2 & 0 \\ -a_1 & a_2 & 0 & 0 \end{pmatrix},$$

and

$$R_{[jl]} = \begin{pmatrix} 0 & b_0 & 2b_1 & b_2 \\ -b_0 & 0 & 3b_2 & 2b_3 \\ -2b_1 & -3b_2 & 0 & b_4 \\ -b_2 & -2b_3 & -b_4 & 0 \end{pmatrix}.$$

Thus the entire curvature tensor $R^i{}_{jkl}$ is encoded in the Ricci tensor.

Remark 4.2. Note that we also have $R^i{}_{ikl} = 2R_{[kl]}$.

Now we can use matrix m of the previous section to find explicit formulae for the coefficients $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, b_4$. It follows that if we evaluate R_{ij} for $(\alpha^1{}_0, \alpha^1{}_1, \alpha^4{}_4) = (0, 1, 1)$, denoting the calculated R_{ij} by R^0_{ij}, then the full Ricci tensor R_{ij} is related to R^0_{ij} via

$$R_{ij} = R^0_{kl} {m^{-1}}^k{}_i {m^{-1}}^l{}_j. \tag{4.3}$$

Here $m^{-1} = ({m^{-1}}^j{}_i)$ is the inverse matrix to m. From this expression we can calculate the explicit form of $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3, b_4$. The resulting formulae involve coefficients $a^0_0, a^0_1, a^0_2, a^0_3, b^0_0, b^0_1, b^0_2, b^0_3, b^0_4$ of the previous section and parameters $\alpha^1{}_0, \alpha^1{}_1, \alpha^4{}_4$ and read:

$$\begin{aligned}
b_4 &= \frac{(\alpha^4{}_4)^3}{(\alpha^1{}_1)^2} b^0_4 \\
b_3 &= \frac{(\alpha^4{}_4)^2}{(\alpha^1{}_1)^2} b^0_3 + \frac{\alpha^1{}_0(\alpha^4{}_4)^2}{3(\alpha^1{}_1)^3} b^0_4 \\
b_2 &= \frac{\alpha^4{}_4}{(\alpha^1{}_1)^2} b^0_2 + \frac{2\alpha^1{}_0\alpha^4{}_4}{3(\alpha^1{}_1)^3} b^0_3 + \frac{(\alpha^1{}_0)^2\alpha^4{}_4}{9(\alpha^1{}_1)^4} b^0_4 \\
b_1 &= \frac{1}{(\alpha^1{}_1)^2} b^0_1 + \frac{\alpha^1{}_0}{(\alpha^1{}_1)^3} b^0_2 + \frac{(\alpha^1{}_0)^2}{(3\alpha^1{}_1)^4} b^0_3 + \frac{(\alpha^1{}_0)^3}{27(\alpha^1{}_1)^5} b^0_4 \\
b_0 &= \frac{1}{(\alpha^1{}_1)^2_4{}^{\alpha^4}} b^0_0 + \frac{4\alpha^1{}_0}{3(\alpha^1{}_1)^3\alpha^4{}_4} b^0_1 + \frac{2(\alpha^1{}_0)^2}{3(\alpha^1{}_1)^4\alpha^4{}_4} b^0_2 + \frac{4(\alpha^1{}_0)^3}{27(\alpha^1{}_1)^5\alpha^4{}_4} b^0_3 + \frac{(\alpha^1{}_0)^4}{81(\alpha^1{}_1)^6\alpha^4{}_4} b^0_4, \\
a_2 &= \frac{(\alpha^4{}_4)^2}{(\alpha^1{}_1)^2} a^0_2 \\
a_1 &= \frac{\alpha^4{}_4}{(\alpha^1{}_1)^2} a^0_1 - \frac{\alpha^1{}_0\alpha^4{}_4}{3(\alpha^1{}_1)^3} a^0_2 \\
a_0 &= \frac{1}{(\alpha^1{}_1)^2} a^0_0 - \frac{2\alpha^1{}_0}{3(\alpha^1{}_1)^3} a^0_1 + \frac{(\alpha^1{}_0)^2}{9(\alpha^1{}_1)^4} a^0_2.
\end{aligned} \tag{4.4}$$

This, in particular, means that the respective spaces consisting of $(b^0_0, b^0_1, b^0_2, b^0_3, b^0_4)$ and of (a^0_0, a^0_1, a^0_2) constitute a 5-dimensional and 3-dimensional representation of G and, as a consequence of $\mathbf{GL}(2,\mathbb{R})$.

Due to (4.3) the vanishing of any of the two determinants:

$$\det(R_{(ij)}) \quad \text{and} \quad \det(R_{[ij]})$$

is a contact invariant property of the corresponding fourth-order ODE (1.2). These two determinants, when expressed in terms of the eight curvature coefficients $a_0, a_1, a_2, b_0, b_1, b_2, b_3, b_4$, are

$$\det(R_{(ij)}) = (a_1^2 - a_0a_2)^2$$

and

$$\det(R_{[ij]}) = (3b_2^2 - 4b_1b_3 + b_0b_4)^2.$$

Thus they are expressible in terms of the two well-known $\mathbf{GL}(2,\mathbb{R})$-invariant polynomials

$$I_2 = a_1^2 - a_0a_2 \quad \text{and} \quad I_3 = 3b_2^2 - 4b_1b_3 + b_0b_4.$$

Remark 4.3. In this context it is interesting to note that function $F = (y_3)^{(4/3)}$ of the well-known example (1.8), provides a contact equivalent class of ODEs that has both invariants I_2 and I_3 vanishing.

Interestingly, the next $\mathbf{GL}(2,\mathbb{R})$-invariant polynomial

$$I_4 = -3(\theta^1)^2(\theta^2)^2 + 4\theta^0(\theta^2)^3 + 4(\theta^1)^3\theta^3 - 6\theta^0\theta^1\theta^2\theta^3 + (\theta^0)^2(\theta^3)^2,$$

when considered as defined on P in terms of forms $(\theta^0, \theta^1, \theta^2, \theta^3)$ of the invariant coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \Omega_+, \Omega_-, \Omega_0, \Omega)$, has the following property:

$$\mathcal{L}_X I_4 = 12(X \lrcorner \Omega) I_4,$$

where $X \in \mathcal{V}$ is any vertical vector field on $\mathbf{GL}(2,\mathbb{R}) \to P \to M^4$. Thus I_4 descends to a well-defined conformal symmetric tensor of fourth degree on the solution space M^4 of the ODE [12]. Let us denote the descended to M^4 tensor I_4 by Υ. It is also worth mentioning that, for the vertical vectors $X \in \mathcal{V}$, we have

$$\mathcal{L}_X \Omega = \mathrm{d}(X \lrcorner \Omega).$$

This means that on the solution space M^4 the form Ω is defined up to a *gradient*. It is convenient to rescale Ω and to define a 1-form A on P equal to

$$A = -12\Omega.$$

This form is also defined up to a gradient on the solution space M^4. Thus, a solution space M^4 of any fourth-order ODE satisfying (2.3) is equipped with a sort of Weyl geometry $[\Upsilon, A]$. This consists of class of pairs (Υ, A), in which Υ is a fourth-order symmetric tensor field, A is a 1-form on M^4, and two pairs (Υ, A) and (Υ', A') represent the same class iff

$$\Upsilon' = \mathrm{e}^{4\phi}\Upsilon, \quad A' = A - 4\mathrm{d}\phi.$$

In the context of this gauge freedom, it is worth noting that the vanishing of $R_{[ij]}$ corresponds to the $[\Upsilon, A]$ geometries on M^4 with form A that can be gauged to $A = 0$. Such a situation occurs if and only if $b_i = 0$ for all $i = 0, 1, 2, 3, 4$.

Remark 4.4. In terms of the Weyl-like geometry $[\Upsilon, A]$ on the solution space M^4, the $\mathfrak{gl}(2,\mathbb{R})$-valued connection may be defined as the unique torsionless connection satisfying

$$\nabla_X \Upsilon = -A(X)\Upsilon.$$

Thus we have the following theorem.

Theorem 4.5. *Every fourth-order ODE $y^{(4)} = F(x, y, y', y'', y^{(3)})$ satisfying Bryant's conditions (2.3) uniquely defines a conformal Weyl-like geometry $[\Upsilon, A]$ on its solution space M^4. The Weyl-like geometry $[\Upsilon, A]$ consists of a symmetric fourth rank tensor Υ and a 1-form A given up to transformations*

$$\Upsilon' = \mathrm{e}^{4\phi}\Upsilon, \quad A' = A - 4\mathrm{d}\phi.$$

Its corresponding $\mathfrak{gl}(2,\mathbb{R})$-valued connection has no torsion and very special curvature tensor described by Theorem 4.1.

5. Examples

5.1. Equations with symmetric Ricci tensor

There is only one contact equivalence class of ODEs (2.2) having an 8-dimensional group of contact symmetries. This is equivalent to $y^{(4)} = 0$ and the symmetry group is $\mathbf{GL}(2,\mathbb{R}) \times_\rho \mathbb{R}^4$. For this class of equations the $\mathfrak{gl}(2,\mathbb{R})$-valued connection of Theorem 4.1 is flat. In this section we focus on the equivalence classes of ODEs (2.2) for which the Maxwell form $\mathrm{d}A = -12\mathrm{d}\Omega$ of this connection is flat $\mathrm{d}F = 0$. In such a case we have $b_0 = b_1 = b_2 = b_3 = b_4 = 0$.

Let us assume that we are in this situation.

Looking at the transformation properties (4.4) of the curvature coefficient a_2, we see that there are essentially two distinct cases distinguished by the *vanishing or not* of the expression $a_2^0 = \frac{1}{45}(-18DF_{333} - 24F_{233} - 4F_{33}^2 - 27F_3F_{333})$.

We analyse the more easy case $a_2^0 = 0$ first.

If $a_2^0 = 0$, then also $a_2 = 0$. Thus we have $a_2 = 0$ everywhere on P with the full system (2.4) and (2.5) of eight independent 1-forms $\theta^1, \theta^2, \theta^3, \Omega_+, \Omega_0, \Omega_-, \Omega$ there. Imposing $(\mathrm{d}^2\Omega_+) \wedge \theta^1 \wedge \theta^2 = 0$ on (2.4) and (2.5) quickly leads to $a_1 = 0$ and, consequently, by imposition of $(\mathrm{d}^2\Omega_+) \wedge \theta^1 = 0$, to $a_0 = 0$. This shows that if $a_2^0 = 0$ then the corresponding ODEs (2.2) are contact equivalent to $y^{(4)} = 0$.

Now we assume that $a_2^0 \neq 0$. Then the choice

$$\alpha^1{}_1 = \frac{\sqrt{2}}{4}\alpha^4{}_4\sqrt{|a_2^0|}$$

brings a_2 to the form

$$a_2 = 8\epsilon_1,$$

where $\epsilon_1 = \mathrm{sgn}(a_2^0)$. Then the choice

$$\alpha^1{}_0 = \frac{3\sqrt{2}}{4}\epsilon_1\alpha^4{}_4\frac{a_1^0}{\sqrt{|a_2^0|}}$$

makes

$$a_1 = 0.$$

After these two normalisations we get

$$a_0 = \frac{8\epsilon_1}{(\alpha^4{}_4 a_2^0)^2}(a_0^0 a_2^0 - (a_1^0)^2).$$

Thus again we have two cases, depending on the vanishing or not of the invariant $I_2^0 = (a_1^0)^2 - a_0^0 a_2^0$.

It follows that the $I_2^0 = 0$ case, which under our assumptions is the same as $a_0 = 0$, corresponds to only *one* nonequivalent class of equations. They are defined by $\epsilon_1 = 1$ (the $\epsilon_1 = -1$ case is not compatible with system (2.4) and (2.5)), and are described by the following theorem.

Theorem 5.1. *All ODEs $y^{(4)} = F(x, y, y', y'', y^{(3)})$ satisfying Bryant's conditions (2.3), having symmetric Ricci tensor, and invariants $I_2 = 0$ and $a_2 \neq 0$, are in local one-to-one correspondence with coframes $(\theta^0, \theta^1, \theta^2, \theta^3, \Omega_+, \Omega)$ on a 6-manifold satisfying:*

$$\begin{aligned}
\mathrm{d}\theta^0 &= 12\Omega \wedge \theta^0 - 3\Omega_+ \wedge \theta^1 + \frac{3\sqrt{2}}{2}\theta^0 \wedge \theta^2 \\
\mathrm{d}\theta^1 &= 6\Omega \wedge \theta^1 - 2\Omega_+ \wedge \theta^2 + \frac{\sqrt{2}}{2}(\theta^0 \wedge \theta^3 + \theta^1 \wedge \theta^2) \\
\mathrm{d}\theta^2 &= -\Omega_+ \wedge \theta^3 + \sqrt{2}\theta^1 \wedge \theta^3 \\
\mathrm{d}\theta^3 &= -6\Omega \wedge \theta^3 + 3\sqrt{2}\theta^2 \wedge \theta^3 \\
\mathrm{d}\Omega_+ &= 6\Omega \wedge \Omega_+ + \sqrt{2}\Omega_+ \wedge \theta^2 + \theta^0 \wedge \theta^3 - 5\theta^1 \wedge \theta^2 \\
\mathrm{d}\Omega &= 0.
\end{aligned}$$

The forms Ω and Ω_0 are given by

$$\Omega_- = \frac{\sqrt{2}}{2}\theta^3, \qquad \Omega_0 = 3\Omega - \frac{\sqrt{2}}{2}\theta^2.$$

All the equations having such invariant forms are equivalent to an ODE defined by

$$F = \frac{4}{3}\frac{y_3^2}{y_2}.$$

This class has strictly 6-dimensional group of contact symmetries.

Now we pass to the $I_2^0 \neq 0$ case. We introduce $\epsilon_2 = \pm 1$, which encodes the sign of I_2^0. This is defined by $\epsilon_1\epsilon_2(a_0^0 a_2^0 - (a_1^0)^2) > 0$. Now we choose

$$\alpha^4{}_4 = \sqrt{\frac{\epsilon_1\epsilon_2(a_0^0 a_2^0 - (a_1^0)^2)}{(a_2^0)^2}}.$$

This normalises a_0 to

$$a_0 = 8\epsilon_2.$$

Under such normalisations system (2.4) and (2.5) descends from P to the 5-dimensional jet space J. There, it reads:

$$\begin{aligned}
\mathrm{d}\theta^0 &= 3(\Omega + \Omega_0) \wedge \theta^0 - 3\Omega_+ \wedge \theta^1 \\
\mathrm{d}\theta^1 &= -\Omega_- \wedge \theta^0 + (3\Omega + \Omega_0) \wedge \theta^1 - 2\Omega_+ \wedge \theta^2 \\
\mathrm{d}\theta^2 &= -2\Omega_- \wedge \theta^1 + (3\Omega - \Omega_0) \wedge \theta^2 - \Omega_+ \wedge \theta^3 \\
\mathrm{d}\theta^3 &= -3\Omega_- \wedge \theta^2 + 3(\Omega - \Omega_0) \wedge \theta^3 \\
\mathrm{d}\Omega_+ &= 2\Omega_0 \wedge \Omega_+ - 2\epsilon_2\theta^0 \wedge \theta^1 + \epsilon_1(\theta^0 \wedge \theta^3 - 5\theta^1 \wedge \theta^2) \\
\mathrm{d}\Omega_- &= -2\Omega_0 \wedge \Omega_- + \epsilon_2(-\theta^0 \wedge \theta^3 + 5\theta^1 \wedge \theta^2) + 2\epsilon_1\theta^2 \wedge \theta^3 \\
\mathrm{d}\Omega_0 &= \Omega_+ \wedge \Omega_- - \epsilon_2\theta^0 \wedge \theta^2 - \epsilon_1\theta^1 \wedge \theta^3 \\
\mathrm{d}\Omega &= 0.
\end{aligned}$$

To close this system it is convenient to eliminate form Ω. This can be achieved by an introduction of new forms $(\sigma^0, \sigma^1, \sigma^2, \sigma^3)$ related to $(\theta^0, \theta^1, \theta^2, \theta^3)$ via:

$$\sigma^0 = \mathrm{e}^w\theta^0, \quad \sigma^1 = \mathrm{e}^w\theta^1, \quad \sigma^2 = \mathrm{e}^w\theta^2, \quad \sigma^3 = \mathrm{e}^w\theta^3,$$

where w is a function on J such that $\Omega = -\frac{1}{3}\mathrm{d}w$. The local existence of such a function is guaranteed by $\mathrm{d}\Omega = 0$. In terms of the new variables $(\sigma^0, \sigma^1, \sigma^2, \sigma^3)$, w, the reduced system takes a form in which the 1-form Ω is not present:

$$\begin{aligned}
\mathrm{d}\sigma^0 &= 3\Omega_0 \wedge \sigma^0 - 3\Omega_+ \wedge \sigma^1 \\
\mathrm{d}\sigma^1 &= -\Omega_- \wedge \sigma^0 + \Omega_0 \wedge \sigma^1 - 2\Omega_+ \wedge \sigma^2 \\
\mathrm{d}\sigma^2 &= -2\Omega_- \wedge \sigma^1 - \Omega_0 \wedge \sigma^2 - \Omega_+ \wedge \sigma^3 \\
\mathrm{d}\sigma^3 &= -3\Omega_- \wedge \sigma^2 - 3\Omega_0 \wedge \sigma^3 \\
\mathrm{d}\Omega_+ &= 2\Omega_0 \wedge \Omega_+ + \mathrm{e}^{-2w}\left(-2\epsilon_2\sigma^0 \wedge \sigma^1 + \epsilon_1(\sigma^0 \wedge \sigma^3 - 5\sigma^1 \wedge \sigma^2)\right) \\
\mathrm{d}\Omega_- &= -2\Omega_0 \wedge \Omega_- + \mathrm{e}^{-2w}\left(\epsilon_2(-\sigma^0 \wedge \sigma^3 + 5\sigma^1 \wedge \sigma^2) + 2\epsilon_1\sigma^2 \wedge \sigma^3\right) \\
\mathrm{d}\Omega_0 &= \Omega_+ \wedge \Omega_- - \mathrm{e}^{-2w}\left(\epsilon_2\sigma^0 \wedge \sigma^2 + \epsilon_1\sigma^1 \wedge \sigma^3\right).
\end{aligned} \tag{5.1}$$

As we can see, the price paid for the elimination of Ω is the introduction of a nonconstant function w appearing explicitly in these equations.

Now the remarkable fact is that system (5.1) closes on J and is described by the following theorem.

Theorem 5.2. *All ODEs $y^{(4)} = F\left(x, y, y', y'', y^{(3)}\right)$ satisfying Bryant's conditions (2.3), having symmetric Ricci tensor, and invariants $I_2 \neq 0$ and $a_2 \neq 0$, are in local one-to-one correspondence with coframes $(\sigma^0, \sigma^1, \sigma^2, \sigma^3, \Omega_+)$ on a 5-manifold satisfying system (5.1) with:*

$$\begin{aligned}
\Omega_0 &= w_0\sigma^0 - (w_1 + 4\epsilon_1\epsilon_2 w_3)\sigma^1 + (4\epsilon_1\epsilon_2 w_0 + w_2)\sigma^2 - w_3\sigma^3 \\
\Omega_- &= -\epsilon_1\epsilon_2\Omega_+ - 2(\epsilon_1\epsilon_2 w_1 + 2w_3)\sigma^0 + 2w_0\sigma_1 + 2\epsilon_1\epsilon_2 w_3\sigma^2 - 2(2\epsilon_1\epsilon_2 w_0 + w_2)\sigma^3.
\end{aligned} \tag{5.2}$$

Functions w, w_0, w_1, w_2, w_3 appearing here are defined by:

$$\mathrm{d}w = w_0\sigma^0 + w_1\sigma^1 + w_2\sigma^2 + w_3\sigma^3. \tag{5.3}$$

They satisfy

$$\begin{aligned} dw_0 &= -\epsilon_1\epsilon_2 w_1 \Omega_+ + \frac{1}{4}(-\epsilon_2 e^{-2w} + 4w_0^2 + 16w_1w_3 + 32\epsilon_1\epsilon_2 w_3^2)\sigma^0 + 3w_0w_1\sigma^1 \\ &\quad - (-\epsilon_1\epsilon_2 w_{13} - 11w_0w_2 - 4\epsilon_1\epsilon_2 w_2^2 + 5\epsilon_1\epsilon_2 w_1w_3 + 12w_3^2)\sigma^2 + (11w_0 + 4\epsilon_1\epsilon_2 w_2)w_3\sigma^3 \\ dw_1 &= (3w_0 - 2\epsilon_1\epsilon_2 w_2)\Omega_+ + (-3w_0w_1 - 4\epsilon_1\epsilon_2 w_1w_2 - 12\epsilon_1\epsilon_2 w_0w_3 - 8w_2w_3)\sigma^0 \\ &\quad - \frac{1}{4}(3\epsilon_1 e^{-2w} + 24\epsilon_1\epsilon_2 w_0^2 - 20w_1^2 + 8\epsilon_1\epsilon_2 w_{13} + 64w_0w_2 + 32\epsilon_1\epsilon_2 w_2^2 - 120\epsilon_1\epsilon_2 w_1w_3 - 192w_3^2)\sigma^1 \\ &\quad - (12w_0w_1\epsilon_1\epsilon_2 + w_1w_2 + 30w_0w_3 + 4\epsilon_1\epsilon_2 w_2w_3)\sigma^2 + w_{13}\sigma^3 \\ dw_2 &= (2w_1 - 3\epsilon_1\epsilon_2 w_3)\Omega_+ + \frac{1}{2}(24\epsilon_1\epsilon_2 w_0^2 + 2\epsilon_1\epsilon_2 w_{13} + 30w_0w_2 + 8\epsilon_1\epsilon_2 w_2^2 - 26\epsilon_1\epsilon_2 w_1w_3 - 48w_3^2)\sigma^0 \\ &\quad - (8\epsilon_1\epsilon_2 w_0w_1 + w_1w_2 + 24w_0w_3 + 12\epsilon_1\epsilon_2 w_2w_3)\sigma^1 \\ &\quad + \frac{1}{4}(-3\epsilon_2 e^{-2w} + 96w_0^2 - 8w_{13} - 12w_2^2 + 40w_1w_3 + 96\epsilon_1\epsilon_2 w_3^2)\sigma^2 - 3(8\epsilon_1\epsilon_2 w_0 + 3w_2)w_3\sigma^3 \\ dw_3 &= w_2\Omega_+ + (4\epsilon_1\epsilon_2 w_0w_1 + 2w_1w_2 + 11w_0w_3 + 4\epsilon_1\epsilon_2 w_2w_3)\sigma^0 \\ &\quad + (w_{13} + 8\epsilon_1\epsilon_2 w_0w_2 + 4w_2^2 - 4w_1w_3 - 12\epsilon_1\epsilon_2 w_3^2)\sigma^1 + w_2w_3\sigma^2 \\ &\quad + \frac{1}{4}(-\epsilon_1 e^{-2w} + 32\epsilon_1\epsilon_2 w_0^2 + 32w_0w_2 + 8\epsilon_1\epsilon_2 w_2^2 + 4w_3^2)\sigma^3, \end{aligned} \tag{5.4}$$

with a function w_{13} satisfying

$$\begin{aligned} dw_{13} &= (-12\epsilon_1\epsilon_2 w_0w_1 - w_1w_2 + 45w_0w_3 + 30\epsilon_1\epsilon_2 w_2w_3)\Omega_+ \\ &\quad + \frac{1}{2}(-6\epsilon_2 w_0 e^{-2w} - 240w_0^3 + 40\epsilon_1\epsilon_2 w_0w_1^2 - 16w_0w_{13} + 5\epsilon_1 w_2 e^{-2w} \\ &\quad - 472\epsilon_1\epsilon_2 w_0^2 w_2 + 20w_1^2 w_2 - 16\epsilon_1\epsilon_2 w_{13}w_2 - 304w_0w_2^2 - 64\epsilon_1\epsilon_2 w_2^3 \\ &\quad + 384w_0w_1w_3 + 192\epsilon_1\epsilon_2 w_1w_2w_3 + 552\epsilon_1\epsilon_2 w_0w_3^2 + 272w_2w_3^2)\sigma^0 \\ &\quad - \frac{1}{4}(20\epsilon_2 w_1 e^{-2w} - 256w_0^2 w_1 - 28w_1w_{13} - 416\epsilon_1\epsilon_2 w_0w_1w_2 - 144w_1w_2^2 \\ &\quad + 15\epsilon_1 w_3 e^{-2w} - 840\epsilon_1\epsilon_2 w_0^2 w_3 + 20w_1^2 w_3 - 24\epsilon_1\epsilon_2 w_{13}w_3 - 1440w_0w_2w_3 \\ &\quad - 480\epsilon_1\epsilon_2 w_2^2 w_3 - 40\epsilon_1\epsilon_2 w_1w_3^2 - 192w_3^3)\sigma^1 - \frac{1}{2}(-15\epsilon_1 w_0 e^{-2w} + 480\epsilon_1\epsilon_2 w_0^3 - 24\epsilon_1\epsilon_2 w_0w_{13} \\ &\quad - 2\epsilon_2 w_2 e^{-2w} + 544w_0^2 w_2 - 16w_{13}w_2 + 184\epsilon_1\epsilon_2 w_0w_2^2 + 16w_2^3 + 240\epsilon_1\epsilon_2 w_0w_1w_3 + 80w_1w_2w_3 \\ &\quad + 588w_0w_3^2 + 184\epsilon_1\epsilon_2 w_2w_3^2)\sigma^2 - \frac{1}{4}(5\epsilon_1 w_1 e^{-2w} - 160\epsilon_1\epsilon_2 w_0^2 w_1 - 160w_0w_1w_2 - 40\epsilon_1\epsilon_2 w_1w_2^2 \\ &\quad + 36\epsilon_2 w_3 e^{-2w} - 1152w_0^2 w_3 - 28w_{13}w_3 - 1152\epsilon_1\epsilon_2 w_0w_2w_3 - 288w_2^2 w_3 + 20w_1w_3^2)\sigma^3. \end{aligned} \tag{5.5}$$

System (5.1)–(5.5) is closed, meaning that $d^2 = 0$ *does not imply any further relations between forms* $\sigma^0, \sigma^1, \sigma^2, \sigma^3, \Omega_+$ *and functions* $w, w_0, w_1, w_2, w_3, w_{13}$.

We easily see that the assumption that *all* $w, w_0, w_1, w_2, w_3, w_{13}$ are *constant* is incompatible with system (5.1)–(5.5). Finding *any* solution to system (5.1)–(5.5) is a difficult task.

5.2. Inhomogeneous examples

Here we present examples of contact equivalent classes of fourth-order ODEs satisfying Bryant's conditions (2.3) which are *not homogeneous*. By this we mean they do *not* admit a transitive contact symmetry group of dimension greater than *four*. We consider an ansatz in which function F depends in a special way on only two coordinates y_2 and y_3. Explicitly:

$$F = (y_2)^2\, q\left(\frac{y_3^2}{y_2^3}\right), \tag{5.6}$$

where $q = q(z)$ is a sufficiently differentiable real function of its argument

$$z = \frac{y_3^2}{y_2^3}.$$

Imposing Bryant's conditions (2.3) on (5.6) we find the following proposition.

Proposition 5.3. *Function F of (5.6) satisfies Bryant's conditions (2.3) if and only if*

(a) *either:*

$$6z(3z-2q)q''+3zq'^2-6qq'+4q=0,$$

(b) *or:*

$$6z(3z-2q)q''+3zq'^2-6qq'+14q-15z=0.$$

The special solutions of (a) are: $q(z)=0$ and $q(z)=\frac{4}{3}z$. In case (b), we have $q(z)=3z$ and $q(z)=\frac{5}{3}z$ as special solutions. Writing these four solutions as $q(z)=cz$, we remark that in cases $c=0$ and $c=3$, function F defines a fourth-order ODE which is contact equivalent to $y^{(4)}=0$. Cases $c=\frac{4}{3}$ and $c=\frac{5}{3}$ define two different F's, but the corresponding fourth-order ODEs are contact equivalent. They both are equivalent to the ODE described by Theorem 5.1.

We emphasise that apart from the singular solutions $q=cz$, equation (a) or (b) admits a two-parameter family of solutions. Every solution $q=q(z)$ from these two families leads to a fourth-order ODE which satisfies Bryant's conditions (2.3) and which is *inhomogeneous*. Remarkably all Bryant's F's which are defined by ansatz (5.6) have $I_3=I_4=0$, but $a_2\neq 0$ and $b_4\neq 0$. Thus, in particular, $\mathrm{d}A\neq 0$ for them.

We were unable to find any example of Bryant's ODEs for which at least one of I_2 or I_3 is not vanishing.

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