INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **20** (2003) 3029–3035 PII: S0264-9381(03)62813-3

Conformal Einstein equations and Cartan conformal connection

Carlos Kozameh[1], **Ezra T Newman**[2] **and Pawel Nurowski**[3]

[1] FaMAF Universidad Nacional de Cordoba, Ciudad Universitaria, Cordoba 5000, Argentina
[2] Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA, USA
[3] Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, ul. Hoza 69, Warszawa, Poland

E-mail: newman@pitt.edu and nurowski@fuw.edu.pl

Received 29 April 2003
Published 18 June 2003
Online at stacks.iop.org/CQG/20/3029

Abstract

Necessary and sufficient conditions for a spacetime to be conformal to an Einstein spacetime are interpreted in terms of curvature restrictions for the corresponding Cartan conformal connection.

PACS numbers: 02.40.−k, 02.30.Hq

1. Introduction

For many years after its inception, the basic equations of general relativity were largely for the determination of metrics, g' that satisfied the vacuum Einstein equations

$$R'_{ab} = 0. \tag{1}$$

However, in more recent years, for a variety of reasons, there has been an increasing interest in metrics g that do not satisfy the Einstein equations but are conformal to Einstein metrics, i.e., metrics such that

$$g = \mathrm{e}^{2\phi} g', \tag{2}$$

with ϕ being a certain nonvanishing function.

A major issue has been to find all the consequences of the requirement that g is conformal to an Einstein g'. Two (closely related) points of view (or sets of equations) emerge concerning this issue: in one view, the final goal is to obtain conditions entirely expressible in terms of g, that would guarantee the existence of the conformal factor $\mathrm{e}^{2\phi}$ scaling g to the Einstein metric g'. Often, for many applications, ϕ is not needed, but if needed it is to be determined separately. In the other view [2, 3, 11], in addition to the equations for g there are further coupled equations (often with boundary conditions) or geometric conditions for the determination of ϕ. The differential equations, *in both cases*, are often called the conformal Einstein equations or the conformal field equations. The second point of view is often associated with the problems of infinity; how to locate it and how to describe its local and global structure. In fact, via the pioneering work of Roger Penrose and Helmut Friedrich, it has played a fundamental role in general relativity in recent years.

Our interest here is mainly with the first point of view. Though the second point of view has been of far greater use in applications of GR, the first point of view does have an interest of its own, both for its intrinsic value (better understanding of the structure of the Einstein equations) and for applications. Long after Brinkman's [1] and Schouten's [13] very general approach to the problem in n-dimensions, Kozameh *et al* [6] derived relatively simple necessary and sufficient conditions for a four-dimensional metric to be conformal to an Einstein metric. Baston and Mason [8], working with a twistorial formulation of the Einstein equations, needed and developed a different version of the first view. Penrose found that there was a natural $SU(2, 2)$, Cartan normal conformal connection associated with twistor bundles [12] over spacetime that could be used to encode, via the second point of view, the conformal Einstein equations. The null-surface formulation of GR [4, 5], since it was based on null surfaces, was naturally associated with conformal metrics. Though there was a very cumbersome method for imposing the conformal Einstein equations on these metrics, a more suitable method is needed. Recently the null-surface formulation of GR was found to be intimately associated with an $SO(4, 2)$ Cartan normal conformal connections and it seemed again a natural question to be asked; how can the conformal Einstein equations, from the first point of view, be stated in the language of a Cartan conformal connection.

In [6], we gave necessary and sufficient conditions for a four-dimensional metric to be conformal to an Einstein metric. One of these conditions, the vanishing of the Bach tensor of the metric, has been discussed by many authors [6–9]. In particular, it was interpreted as being equivalent to the vanishing of the Yang–Mills current of the corresponding Cartan conformal connection. The other condition, which is given in terms of a rather complicated equation on the Weyl tensor of the metric, has not been analysed from the point of view of the corresponding Cartan conformal connection. The purpose of this paper is to fill this gap.

2. The result

Let M be a four-dimensional manifold equipped with the conformal class of metrics $[g]$. Here we will be assuming that g has Lorentzian signature, but our results are also valid in the other two signatures.

Given a conformal class $[g]$ on M, we choose a representative g for the metric. Let θ^μ, $\mu = 1, 2, 3, 4$, be a null (or orthonormal) coframe for g on M. This, in particular, means that $g = \eta_{\mu\nu}\theta^\mu\theta^\nu$, with all the coefficients $\eta_{\mu\nu}$ being constants. We define $\eta^{\mu\nu}$ by $\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho$ and we will use $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ to raise and lower the Greek indices, respectively. The metric

$$g' = \mathrm{e}^{-2\phi} g, \tag{3}$$

conformally related to g will be represented by a coframe

$$\theta'^\mu = \mathrm{e}^{-\phi}\theta^\mu, \tag{4}$$

so that

$$g' = \eta_{\mu\nu}\theta'^\mu\theta'^\nu, \tag{5}$$

with the same $\eta_{\mu\nu}$ as in expression (3). Given g and θ^μ, we consider the 1-forms $\Gamma^\mu{}_\nu$ uniquely determined on M by the equations

$$\begin{aligned} &\mathrm{d}\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu = 0, \\ &\Gamma_{\mu\nu} = \Gamma_{[\mu\nu]}, \qquad \text{where} \qquad \Gamma_{\mu\nu} = \eta_{\mu\rho}\Gamma^\rho{}_\nu. \end{aligned} \tag{6}$$

Using $\Gamma^\mu{}_\nu$, we calculate the Riemann tensor 2-forms

$$\Omega^\mu{}_\nu = \tfrac{1}{2}\Omega^\mu{}_{\nu\rho\sigma}\theta^\rho \wedge \theta^\sigma = \mathrm{d}\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu, \tag{7}$$

and the Ricci part of the Riemann tensor

$$K_{\nu\sigma} = \Omega^\mu{}_{\nu\mu\sigma}, \qquad K = \eta^{\nu\sigma}K_{\nu\sigma} \qquad \text{and} \qquad S_{\nu\sigma} = K_{\nu\sigma} - \tfrac{1}{4}K\eta_{\nu\sigma}. \tag{8}$$

We recall that the metric g is called *Einstein* iff

$$S_{\mu\nu} = 0, \tag{9}$$

and that it is called *conformal to Einstein* iff there exists ϕ such that the metric $g' = \mathrm{e}^{-2\phi}g$ is Einstein.

In the following we will also need 1-forms:

$$\tau_\nu = \left(-\tfrac{1}{2}S_{\nu\rho} - \tfrac{1}{24}K\eta_{\nu\rho}\right)\theta^\rho, \tag{10}$$

and 2-forms

$$C^\mu{}_\nu = \mathrm{d}\Gamma^\mu{}_\nu + \theta^\mu \wedge \tau_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu + \tau^\mu \wedge \theta_\nu, \qquad \tau^\mu = \eta^{\mu\nu}\tau_\nu. \tag{11}$$

It follows that the 2-forms

$$C^\mu{}_\nu = \tfrac{1}{2}C^\mu{}_{\nu\rho\sigma}\theta^\rho \wedge \theta^\sigma, \tag{12}$$

are the Weyl 2-forms associated with the Weyl tensor $C^\mu{}_{\nu\rho\sigma}$ of the metric g. It is known that the Weyl tensor obeys the following identity:

$$C_{\alpha\mu\nu\rho}C^{\beta\mu\nu\rho} = \tfrac{1}{4}C^2\delta^\beta{}_\alpha, \tag{13}$$

where

$$C^2 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}. \tag{14}$$

Using the Bianchi identities one shows that the tensor $\tau_{\mu\nu\rho}$ defined on M via

$$D\tau_\mu = \tfrac{1}{2}\tau_{\mu\nu\rho}\theta^\nu \wedge \theta^\rho = \mathrm{d}\tau_\mu + \tau_\nu \wedge \Gamma^\nu{}_\mu, \tag{15}$$

is

$$\tau_{\nu\rho\sigma} = \nabla_\mu C^\mu{}_{\nu\rho\sigma}, \tag{16}$$

where ∇_μ is the covariant derivative operator associated with D.

In the following we will also need the Bach tensor:

$$B_{\mu\nu} = \nabla^\rho\nabla^\sigma C_{\mu\rho\nu\sigma} + \tfrac{1}{2}C_{\mu\rho\nu\sigma}K^{\rho\sigma}, \tag{17}$$

and a tensor

$$N_{\nu\rho\sigma} = \left(\nabla_\alpha C^\alpha{}_{\beta\gamma\delta}\right)C^{\mu\beta\gamma\delta}C_{\mu\nu\rho\sigma} - \tfrac{1}{4}C^2\nabla_\mu C^\mu{}_{\nu\rho\sigma}. \tag{18}$$

It is known that the vanishing of the Bach tensor is a conformally invariant property. If $C^2 \neq 0$, the vanishing of $N_{\mu\nu\rho}$ is also conformally invariant. The relevance of both these tensors in the context of this note is given by the following theorem [6]:

Theorem 1. *Assume that the metric g satisfies the genericity condition*

$$C^2 \neq 0,$$

on M. Then the metric is locally conformally equivalent to the Einstein metric if and only if

$$\text{(i)}\, B_{\mu\nu} = 0 \qquad \text{and} \qquad \text{(ii)}\, N_{\nu\rho\sigma} = 0, \tag{19}$$

on M.

Remarks.

- Conditions (i) and (ii) are independent. In particular, metrics with vanishing Bach tensor and not conformal to Einstein metrics are known [10].
- If $C^2 = 0$, the condition (ii) must be replaced by another condition for the above theorem to be true. This other condition depends on the algebraic type of the Weyl tensor and is given in [6].
- Baston and Mason [8] gave another version of the above theorem in which condition (ii) was replaced by the vanishing of a different tensor from $N_{\nu\rho\sigma}$. Unlike $N_{\nu\rho\sigma}$, which is *cubic* in the Weyl tensor, the Baston–Mason tensor $E_{\nu\rho\sigma}$, is only *quadratic* in $C^{\mu}{}_{\nu\rho\sigma}$. However, they need the operation of dualing on the curvature.
- Merkulov [9] interpreted condition (ii) as the vanishing of the Yang–Mills current of the *Cartan normal conformal connection* ω associated with the metric g. Following him, Baston and Mason [8] interpreted the condition $E_{\nu\rho\sigma} = 0$ in terms of the curvature condition for ω.

Although in the context of theorem 1, conditions (ii) and $E_{\nu\rho\sigma} = 0$ are equivalent, the tensors $N_{\mu\nu\rho}$ and $E_{\mu\nu\rho}$ are quite different. In addition to cubic versus quadratic dependence on the Weyl tensor, one can mention the fact that it is quite easy to express tensor $E_{\nu\rho\sigma}$ in spinorial language and quite complicated in tensorial language. The totally opposite situation occurs for the tensor $N_{\nu\rho\sigma}$. One of the motivations for the present paper is the existence of the normal conformal connection interpretation for the condition $E_{\nu\rho\sigma} = 0$. As far as we know such an interpretation of $N_{\nu\rho\sigma} = 0$ has not been discussed. To fill this gap, we first give the formal definition of the Cartan normal conformal connection. In order to do this we first introduce the 6×6 matrix

$$Q_{AB} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \eta_{\mu\nu} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{20}$$

and then define the $\mathbf{so}(2,4)$-valued 1-form $\tilde{\omega}$ on M by

$$\tilde{\omega} = \begin{pmatrix} 0 & \tau_\mu & 0 \\ \theta^\nu & \Gamma^\nu{}_\mu & \eta^{\nu\rho}\tau_\rho \\ 0 & \eta_{\mu\rho}\theta^\rho & 0 \end{pmatrix}. \tag{21}$$

Then, we use a Lie subgroup $\mathbf{H}$ of $\mathbf{SO}(2,4)$, generated by the 6×6 matrices of the form

$$b = \begin{pmatrix} \mathrm{e}^{-\phi} & \mathrm{e}^{-\phi}\xi_\mu & \frac{1}{2}\mathrm{e}^{-\phi}\xi_\mu\xi_\nu\eta^{\mu\nu} \\ 0 & \Lambda^\nu{}_\mu & \Lambda^\nu{}_\rho\eta^{\rho\sigma}\xi_\sigma \\ 0 & 0 & \mathrm{e}^{\phi} \end{pmatrix}, \qquad \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\eta_{\mu\nu} = \eta_{\rho\sigma}, \tag{22}$$

to lift the form $\tilde{\omega}$ to an $\mathbf{so}(2,4)$-valued 1-form ω on $M \times H$. Explicitly, if b is a generic element of $\mathbf{H}$, we put

$$\omega = b^{-1}\tilde{\omega}b + b^{-1}\mathrm{d}b,$$

so that

$$\omega = \begin{pmatrix} -\frac{1}{2}A & \tau'_\mu & 0 \\ \theta'^\nu & \Gamma'^\nu{}_\mu & \eta^{\nu\sigma}\tau'_\sigma \\ 0 & \theta'^\sigma\eta_{\sigma\mu} & \frac{1}{2}A \end{pmatrix}, \tag{23}$$

with

$$
\begin{aligned}
&\theta'^{\nu} = \mathrm{e}^{-\phi}\Lambda^{-1\nu}{}_{\rho}\theta^{\rho},\\
&A = 2\xi_{\mu}\theta'^{\mu} + 2\,\mathrm{d}\phi,\\
&\Gamma'^{\nu}{}_{\mu} = \Lambda^{-1\nu}{}_{\rho}\Gamma^{\rho}{}_{\sigma}\Lambda^{\sigma}{}_{\mu} + \Lambda^{-1\nu}{}_{\rho}\,\mathrm{d}\Lambda^{\rho}{}_{\mu} + \theta'^{\nu}\xi_{\mu} - \xi^{\nu}\eta_{\mu\rho}\theta'^{\rho},\\
&\tau'_{\mu} = \mathrm{e}^{\phi}\tau_{\nu}\Lambda^{\nu}{}_{\mu} - \xi_{\alpha}\Lambda^{-1\alpha}{}_{\rho}\Gamma^{\rho}{}_{\nu}\Lambda^{\nu}{}_{\mu} + \tfrac{1}{2}\mathrm{e}^{-\phi}\eta^{\alpha\beta}\xi_{\alpha}\xi_{\beta}\tau_{\nu}\Lambda^{\nu}{}_{\mu} - \tfrac{1}{2}\xi_{\mu}A + \mathrm{d}\xi_{\mu} - \xi_{\alpha}\Lambda^{-1\alpha}{}_{\rho}\mathrm{d}\Lambda^{\rho}{}_{\mu}.
\end{aligned}
\tag{24}
$$

The so-defined form ω is the Cartan normal conformal connection associated with the conformal class $[g]$ written in a particular trivialization of an appropriately defined H-bundle over M. It can be viewed as a useful tool for encoding conformal properties of the metrics on manifolds. Indeed, if given g on M one calculates the quantities θ^{μ}, $\Gamma^{\mu}{}_{\nu}$, τ_{μ}, then the corresponding quantities for the conformally rescaled metric $g' = \mathrm{e}^{-2\phi}g$ are given by (23) with $A = 0$.[4]

The curvature of ω is

$$R = \mathrm{d}\omega + \omega \wedge \omega,$$

and has the rather simple form

$$
R = \begin{pmatrix} 0 & (D\tau_{\mu})' & 0 \\ 0 & C'^{\nu}{}_{\mu} & g^{\nu\mu}(D\tau_{\mu})' \\ 0 & 0 & 0 \end{pmatrix},
\tag{25}
$$

with the 2-forms $C'^{\nu}{}_{\mu}$ and $(D\tau_{\mu})'$ defined by

$$
\begin{aligned}
&C'^{\nu}{}_{\mu} = \Lambda^{-1\nu}{}_{\rho}C^{\rho}{}_{\sigma}\Lambda^{\sigma}{}_{\mu},\\
&(D\tau_{\mu})' = \mathrm{e}^{\phi}D\tau_{\nu}\Lambda^{\nu}{}_{\mu} - \xi_{\alpha}\Lambda^{-1\alpha}{}_{\rho}\mathbf{C}^{\rho}{}_{\nu}\Lambda^{\nu}{}_{\mu}.
\end{aligned}
\tag{26}
$$

Similar to the properties of ω, the curvature R can be used to extract the transformations of $D\tau_{\mu}$ and $C^{\nu}{}_{\mu}$ under the conformal rescaling of the metrics. If $g \to g' = \mathrm{e}^{-2\phi}g$, these transformations are given by (26) with $\xi_{\mu} = -\nabla'_{\mu}\phi$. In particular, if we freeze the Lorentz transformations of the tetrad, $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$, then we see that the Weyl 2-forms $C^{\nu}{}_{\mu}$ constitute the conformal invariant.

The curvature R of the Cartan normal connection ω is horizontal which, in other words, means that it has only $\theta^{\mu} \wedge \theta^{\nu}$ terms in the decomposition onto the basis of forms $\left(\theta^{\mu}, \mathrm{d}\phi, \Lambda^{-1\mu}{}_{\nu}\,\mathrm{d}\Lambda^{\nu}{}_{\rho}, \mathrm{d}\xi_{\mu}\right)$. Thus, the Hodge $*$ operator associated with g on M is well defined acting on R and in consequence the Yang–Mills equations for R can be written

$$D * R = \mathrm{d} * R - *R \wedge \omega + \omega \wedge *R = 0. \tag{27}$$

The following theorem is well known (see, e.g., [7]):

Theorem 2. *The metric g on a four-dimensional manifold M satisfies the Bach equations $B_{\mu\nu} = 0$ if and only if it satisfies the Yang–Mills equations $D * R = 0$ for the Cartan normal conformal connection associated with g.*

In view of this theorem, it is natural to ask about the normal conformal connection interpretation of $N_{\mu\nu\rho} = 0$, which together with $B_{\mu\nu} = 0$ are sufficient for g to be conformal

[4] Note that $A = 0$ means that $\xi_{\mu} = -\nabla'_{\mu}\phi$, where $\nabla'_{\mu} = \mathrm{e}^{\phi}\Lambda^{\nu}{}_{\mu}\nabla_{\nu}$. Admitting ξ_{μ} which are not gradients in the definition of $\mathbf{H}$ allows for transformations between different Weyl geometries $[(g, A)]$.

to Einstein. To answer this question, we introduce indices $A, B, C, \ldots$ which run from 0 to 5 and attach them to any 6×6 matrix. In this way the elements of matrix R are 2-forms

$$R^A{}_B = \tfrac{1}{2} R^A{}_{B\mu\nu} \theta'^\mu \wedge \theta'^\nu. \tag{28}$$

We define the 6×6 matrix of 2-forms $\hat{R}^3$, which is the appropriately contracted triple product of R, with matrix elements

$$\hat{R}^3_{AF} = \tfrac{1}{2} Q_{AE} R^E{}_{B\alpha\beta} R^B_{C\gamma\delta} R^C{}_{F\mu\nu} \eta^{\alpha\gamma} \eta^{\beta\delta} \theta'^\mu \wedge \theta'^\nu. \tag{29}$$

The symmetric part of this matrix $(\hat{R}^3)^T + \hat{R}^3$ is of the form

$$(\hat{R}^3)^T + \hat{R}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & P_\mu \\ 0 & P_\mu & P \end{pmatrix}, \tag{30}$$

where P_μ and P are appropriate 2-forms on $M \times H$. It follows that under the assumption that

$$C^2 \neq 0, \tag{31}$$

this matrix has a particularly simple form

$$(\hat{R}^3)^T + \hat{R}^3 = \frac{1}{2} \mathrm{e}^{6\phi} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-\phi} \Lambda^\alpha{}_\sigma N_{\alpha\beta\gamma} \theta^\beta \wedge \theta^\gamma \\ 0 & \mathrm{e}^{-\phi} \Lambda^\alpha{}_\sigma N_{\alpha\beta\gamma} \theta^\beta \wedge \theta^\gamma & V^\alpha N_{\alpha\beta\gamma} \theta^\beta \wedge \theta^\gamma \end{pmatrix}, \tag{32}$$

where

$$V^\alpha = \frac{4}{C^2} \left(\nabla_\rho C^\rho{}_{\lambda\tau\sigma} \right) C^{\alpha\lambda\tau\sigma} - \mathrm{e}^{-\phi} \xi_\rho \Lambda^{-1\rho}{}_\lambda \eta^{\lambda\alpha}, \tag{33}$$

and $N_{\alpha\beta\gamma}$ is given by (18). The proof of this fact consists of a straightforward but lengthy calculation which uses the identities (13) and (16). Formula (32), which shows that the vanishing of $N_{\alpha\beta\gamma}$ is equivalent to the vanishing of $(\hat{R}^3)^T + \hat{R}^3$, enables us to formulate the following theorem:

Theorem 3. *Assume that the metric g satisfies*

$$C^2 \neq 0,$$

on M. Let ω be its Cartan normal conformal connection with curvature R and the matrix $\hat{R}^3$ as above. Then the metric is locally conformally equivalent to the Einstein metric if and only if

$$\text{(i)} \quad D * R = 0 \qquad \text{and} \qquad \text{(ii)} \quad (\hat{R}^3)^T + \hat{R}^3 = 0. \tag{34}$$

The above condition (ii) can now be compared to the Baston–Mason conformal connection interpretation of the condition $E_{\mu\nu\rho} = 0$. According to them, this condition is [8]

$$\text{(ii}'\text{)} \quad \left[R^+_{\mu\nu}, R^-_{\rho\sigma} \right] = 0, \tag{35}$$

where $R^+ = \frac{1}{2} R^+{}_{\mu\nu} \theta^\mu \wedge \theta^\nu$ and $R^- = \frac{1}{2} R^-{}_{\mu\nu} \theta^\mu \wedge \theta^\nu$ denote, respectively, the self-dual and anti-self-dual parts of the curvature R, i.e., $*R^\pm = \pm \mathrm{i} R^\pm$ and $R = R^+ \oplus R^-$.

Acknowledgments

We acknowledge support from NSF grant no PHY-0088951 and the Polish KBN grant no 2 P03B 12724.

References

[1] Brinkman H W 1924 Riemann spaces conformal to Einstein spaces *Math. Ann.* **91** 269–78
[2] Friedrich H 1977 Twistor connection and normal conformal Cartan connection *Gen. Rel. Grav.* **8** 303–12
[3] Friedrich H 1995 Einstein equations and conformal structure *J. Geom. Phys.* **17** 125–84
[4] Frittelli S, Kozameh C and Newman E 1995 Lorentzian metrics from characteristic surfaces *J. Math. Phys.* **36** 4975
[5] Frittelli S, Kozameh C and Newman E 1995 GR via characteristic surfaces *J. Math. Phys.* **36** 4984
[6] Kozameh C N, Newman E T and Tod K P 1985 Conformal Einstein spaces *Gen. Rel. Grav.* **17** 343–52
[7] Korzynski M and Lewandowski J 2003 The normal conformal Cartan connection and the Bach tensor *Preprint* gr-qc/0301096
[8] Baston R J and Mason L J 1987 Conformal gravity, the Einstein equations and spaces of complex null geodesics *Class. Quantum Grav.* **4** 815–26
[9] Merkulov S A 1984 A conformally invariant theory of gravitation and electromagnetism *Class. Quantum Grav.* **1** 349–54
[10] Nurowski P and Plebanski J F 2001 Non-vacuum twisting type N metrics *Class. Quantum Grav.* **18** 341–51
[11] Penrose R P 1965 Zero rest-mass fields including gravitation *Proc. R. Soc.* **284** 159–203
[12] Penrose R P and Rindler W 1984 *Spinors and Space-Time* vol 2 (Cambridge: Cambridge University Press)
[13] Schouten J A 1954 *Ricci Calculus* 2nd edn (Berlin: Springer) p 312