$\mathbf{GL}(2, \mathbb{R})$ geometry of 5th order ODEs

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Dedicated to Professor Oldřich Kowalski
on the occasion of a Conference in his honour

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- This $\rho(\mathbf{SO}(3))$ may be defined as a subgroup of a $\mathbf{SO}(5)$ stabilizing rank three *traceless symmetric* tensor $\Upsilon \in S^3_0\mathbb{R}^5$, which is related to the metric g via:

$$\Upsilon_{jki}\Upsilon_{lmi} + \Upsilon_{lji}\Upsilon_{kmi} + \Upsilon_{kli}\Upsilon_{jmi} = g_{jk}g_{lm} + g_{lj}g_{km} + g_{kl}g_{jm}.$$

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- An irreducible $\mathbf{SO}(3)$ structure (M^5, g, Υ) is called *nearly integrable* if Υ is a *Killing tensor* for g:

$$\overset{LC}{\nabla}_X \Upsilon(X,X,X) = 0, \qquad \forall X \in TM^5.$$

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- This sort of geometries are studied extensively by the string theorists.

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- We do not know if *nonhomogeneous* examples exist.

- Perhaps these structures are so rigid that they must be homogeneous.

Question:

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What are the possible dimensions n in which there exists a tensor Υ satisfying:

i) $\Upsilon_{ijk} = \Upsilon_{(ijk)}$, (total *symmetry*)

ii) $\Upsilon_{ijj} = 0$, (no trace)

iii) $\Upsilon_{jki}\Upsilon_{lmi} + \Upsilon_{lji}\Upsilon_{kmi} + \Upsilon_{kli}\Upsilon_{jmi} = g_{jk}g_{lm} + g_{lj}g_{km} + g_{kl}g_{jm}$?

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- Such tensor is needed to construct isoparametric hypersurfaces in spheres.

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- Coefficients a_i of a 4th order polynomial

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- A polynomial I, in variables a_i, is called an *algebraic invariant* of $w_4(x,y)$ if it changes according to

$$I \to I' = (\det b)^p\ I, \qquad b\in\mathbf{GL}(2,\mathbb{R})$$

  under the action of this 5-dimensional representation on a_is.

- The lowest order invariants of $w_4(x, y)$ are:

$$I_2 = 3a_2^2 - 4a_1a_3 + a_0a_4$$

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- Defining Υ_{ijk} and g_{ij} via

$$\Upsilon_{ijk}a_ia_ja_k = 3\sqrt{3}I_3$$

$$g_{ij}a_ia_j = I_2,$$

 one can check that the so defined g_{ij} and Υ_{ijk} satisfy the desidered relations i)-iii).

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- The stabilizer of the conformal class $[(g,\Upsilon)]$ is the irreducible $\mathbf{GL}(2,\mathbb{R})$ in dimension five.

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- An irreducible $\mathbf{GL}(2,\mathbb{R})$ structure $(M^5, [(g,\Upsilon,A)])$ is called *nearly integrable* iff tensor Υ is a *conformal* Killing tensor for $\overset{W}{\nabla}$:

$$\overset{W}{\nabla}_X\, \Upsilon(X,X,X) + \tfrac{1}{2}A(X)\Upsilon(X,X,X) = 0, \qquad \forall X \in \mathrm{T}M^5.$$

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- To achieve the uniqueness one requires the that torsion T of ∇, considered as an element of $\bigotimes^3 \mathrm{T}^* M^5$, seats in a 10-dimensional subspace $\bigwedge^3 \mathrm{T}^* M^5$.

- In terms of the connection 1-forms of the Weyl connection $\overset{W}{\Gamma}$, and the characteristic connection Γ, we have

$$\overset{W}{\Gamma} = \Gamma + \tfrac{1}{2}T,$$

where $\overset{W}{\Gamma} \in \mathfrak{co}(3,2) \otimes \mathrm{T}^*M^5$, $\Gamma \in \mathfrak{gl}(2,\mathbb{R}) \otimes \mathrm{T}^*M^5$ and $T \in \bigwedge^3 \mathrm{T}^*M^5$.

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- The converse is also true: if an irreducible $\mathbf{GL}(2,\mathbb{R})$ structure in dimension five admits a connection ∇ satisfying

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and having totally skew symmetric torsion $T \in \bigwedge^3 \mathrm{T}^*M^5$ then it is nearly integrable.

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- Can we produce examples of the nearly integrable $\mathbf{GL}(2,\mathbb{R})$ geometries in dimension five? Can we produce examples with 'pure' torsion in $\bigwedge_3$ or $\bigwedge_7$? Can we produce nonhomogeneous examples?

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- Ordinary differential equation $y^{(5)} = 0$ has $\mathbf{GL}(2,\mathbb{R}) \times_\rho \mathbb{R}^5$ as its group of contact symmetries. Here $\rho : \mathbf{GL}(2,\mathbb{R}) \to \mathbf{GL}(5,\mathbb{R})$ is the 5-dimensional irreducible representation of $\mathbf{GL}(2,\mathbb{R})$.

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- This, in particular, means that $y^{(5)} = 0$ may be described in terms of a *flat* $\mathfrak{gl}(2,\mathbb{R})$-valued connection on the principal fibre bundle $\mathbf{GL}(2,\mathbb{R}) \to P \to M^5$ over the solution space M^5 of the ODE.

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- What about more complicated 5th order ODEs?

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- Suppose that the equation satsifies three, contact invariant conditions:

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- Consider a 5th order ODE $y^{(5)} = F(x, y, y', y'', y^{(3)}, y^{(4)})$ modulo *contact* transformation of the variables.

- Let $D = \partial_x + y'\partial_y + y''\partial_{y'} + y^{(3)}\partial_{y''} + y^{(4)}\partial_{y^{(3)}} + F\partial_{y^{(4)}}$.

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$$50D^2F_4 - 75DF_3 + 50F_2 - 60F_4DF_4 + 30F_3F_4 + 8F_4^3 = 0$$

$$375D^2F_3 - 1000DF_2 + 350DF_4^2 + 1250F_1 - 650F_3DF_4 + 200F_3^2 -$$

$$150F_4DF_3 + 200F_2F_4 - 140F_4^2DF_4 + 130F_3F_4^2 + 14F_4^4 = 0$$

$$
\begin{aligned}
&1250D^2F_2 - 6250DF_1 + 1750DF_3DF_4 - 2750F_2DF_4 - \\
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&550F_2F_4^2 - 280F_4^3DF_4 + 210F_3F_4^3 + 28F_4^5 + 18750F_y = 0,
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- We call the three conditions on F the Wünschmann-like conditions.

Examples of F satisfying the Wünschmann-like conditions

The three differential equations

$$y^{(5)} = c\Big(\frac{5y^{(3)3}(5-27cy''^2)}{9(1+cy''^2)^2} + 10\frac{y''y^{(3)}y^{(4)}}{1+cy''^2}\Big),$$

with $c=+1,0,-1,$ represent the only three contact nonequivalent classes of Wünschmann-like ODEs having the corresponding nearly integrable $\mathbf{GL}(2,\mathbb{R})$ structures $(M^5,[g,\Upsilon,A])$ with the characteristic connection with vanishing torsion.

In all three cases the holonomy of the Weyl connection $\overset{W}{\Gamma}$ of structures $(M^5, [g, \Upsilon, A])$ is reduced to the $\mathbf{GL}(2,\mathbb{R})$. For all the three cases the Maxwell 2-form $\mathrm{d}A \equiv 0$. The corresponding Weyl structure is flat for $c = 0$. If $c = \pm 1$, then in the conformal class $[g]$ there is an Einstein metric of positive $(c = +1)$ or negative $(c = -1)$ Ricci scalar. In case $c = 1$ the manifold M^5 can be identified with the homogeneous space $\mathbf{SU}(1,2)/\mathbf{SL}(2,\mathbb{R})$ with an Einstein g descending from the Killing form on $\mathbf{SU}(1,2)$. Similarly in $c = -1$ case the manifold M^5 can be identified with the homogeneous space $\mathbf{SL}(3,\mathbb{R})/\mathbf{SL}(2,\mathbb{R})$ with an Einstein g descending from the Killing form on $\mathbf{SL}(3,\mathbb{R})$. In both cases with $c \neq 0$ the metric g is not conformally flat.

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$$F = \frac{5(8y_3^3 - 12y_2y_3y_4 + 3y_1y_4^2)}{6(2y_1y_3 - 3y_2^2)},$$

$$F = \frac{5y_4^2}{3y_3} \pm y_3^{5/3},$$

represent four nonequivalent nearly integrable $\mathbf{GL}(2, \mathbb{R})$ structures corresponding to the different signs in the second expression and to the different signs of the denominator in the first expression. These structures have 6-dimensional symmety group.

$$F = \frac{1}{9(y_1^2 + y_2)^2} \times$$

$$\Big(5w\big(y_1^6 + 3y_1^4 y_2 + 9y_1^2 y_2^2 - 9y_2^3 - 4y_1^3 y_3 + 12y_1 y_2 y_3 + 4y_3^2 - 3y_4(y_1^2 + y_2)\big) +$$

$$45y_4(y_1^2 + y_2)(2y_1 y_2 + y_3) - 4y_1^9 - 18y_1^7 y_2 - 54y_1^5 y_2^2 - 90y_1^3 y_2^3 + 270y_1 y_2^4 +$$

$$15y_1^6 y_3 + 45y_1^4 y_2 y_3 - 405y_1^2 y_2^2 y_3 + 45y_2^3 y_3 + 60y_1^3 y_3^2 - 180y_1 y_2 y_3^2 - 40y_3^3\Big),$$

where

$$w^2 = y_1^6 + 3y_1^4 y_2 + 9y_1^2 y_2^2 - 9y_2^3 - 4y_1^3 y_3 + 12y_1 y_2 y_3 + 4y_3^2 - 3y_1^2 y_4 - 3y_2 y_4.$$

This again has 6-dimensional symmetry group.

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in which $z = \frac{y_4^3}{y_3^4}$.

This equation may be solved explicitely giving example of ODEs having its nearly integrable structure being nonhomogeneous.

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- This conformal structure in dimension *three* is related to the quadratic $\mathbf{GL}(2,\mathbb{R})$ invariant $\Delta = a_0a_2 - a_1^2$ of $w_2(x,y) = a_0x^2 + 2a_1xy + a_2y^2$.

- If a 4th order ODE $y^{(4)} = F(x, y, y', y'', y''')$ satisfies the Wünschmann-like conditions

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$$D = \partial_x + y_1\partial_y + y_2\partial_{y_1} + y_3\partial_{y_2} + F\partial_{y_3},$$

  then it defines an irreducible $\mathbf{GL}(2,\mathbb{R})$ structure on the 4-dimensional space M^4 of its solutions.

- This $\mathbf{GL}(2,\mathbb{R})$ structure in dimension *four* may be understood in terms of a *conformal* Weyl-like structure associated with the *quartic* $\mathbf{GL}(2,\mathbb{R})$ invariant

$$I_4 = -3a_1^2a_2^2 + 4_0a_2^3 + 4a_1^3a_3 - 6a_0a_1a_2a_3 + a_0^2a_3^2,$$

of

$$w_3(x,y) = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$$

and a certain 1-form A on M^4.

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- These $\mathbf{GL}(2,\mathbb{R})$ structures can be understood in terms of a certain Weyl-like conformal geometries $[(\Upsilon_1, \Upsilon_2, ..., \Upsilon_k, A)]$ of $\mathbf{GL}(2,\mathbb{R})$-invariant symmetric conformal tensors Υ_μ and a certain 1-form A given up to a gradient.

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This is a report on a *joint* work with my student Michał Godliński.