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IS DARK ENERGY MEANINGLESS?

Abstract. We show that there are isometrically nonequivalent Robertson–Walker metrics which have the same set of geodesics. While one of these metrics satisfies the Einstein equations of pure dust without a cosmoological constant, all the other describe pure dust with additional energy momentum tensor of cosmological constant type. Since each of these metrics have the same geodesics it is not clear how to distinguish experimentally between the Universes whose energy momentum tensor includes or not the cosmological constant type term.

To interpret the cosmological data one has to assume a model of space-time, which according to the current paradigm, is a 4-dimensional manifold M equipped with the Robertson–Walker metric

$$(1) \qquad g = -\mathrm{d}t^2 + R^2 \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{\left(1 + \frac{\kappa}{4}(x^2 + y^2 + z^2)\right)^2}, \qquad \kappa = +1, 0, -1.$$

Here $R = R(t)$ is a real function (the scale factor), of the cosmic time t. In the following we use an orthonormal coframe θ^μ, $\mu = 0, 1, 2, 3$, for g. This is given by

$$(2) \qquad \theta^0 = \mathrm{d}t, \qquad \theta^i = \frac{R\mathrm{d}x^i}{1 + \frac{\kappa}{4}(x^2 + y^2 + z^2)}, \qquad x^i = (x, y, z),$$

and in it the metric g reads:

$$g = g_{\mu\nu}\theta^\mu\theta^\nu = -\theta^{0^2} + \theta^{1^2} + \theta^{2^2} + \theta^{3^2}.$$

In this letter we observe that each Robertson–Walker spacetime (M, g) admits a 1-parameter family of metrics $\tilde{g}$, which are not isometric to g, but which have the same set of geodesics as (M, g). Then we speculate about the consequences of using $\tilde{g}$ rather than g to interpret the cosmological data. In particular, we show that a pure dust without a cosmological constant in the Robertson–Walker metric g, can be interpreted as a pure dust with energy momentum tensor of cosmological constant type (dark energy),

$$\overset{DE}{T}_{\mu\nu} = -\frac{1}{8\pi G}\tilde{\Lambda}\tilde{g}_{\mu\nu},$$

in the corresponding metric $\tilde{g}$.

To see this we proceed as follows:

Consider a 1-parameter family of metrics $\tilde{g}$ on M related to g in (1) by:

$$(3) \qquad \tilde{g} = -\frac{\theta^{0^2}}{(1 - sR^2)^2} + \frac{\theta^{1^2}}{1 - sR^2} + \frac{\theta^{2^2}}{1 - sR^2} + \frac{\theta^{3^2}}{1 - sR^2},$$

where s is a real constant. Then we have the following theorem.

THEOREM 1. *For each value of the real parameter s the metric (3) has on M the same unparametrised geodesics as the Roberston–Walker metric (1).*

Proof. It is well known [1, 2, 3, 4, 10, 11] that two metrics g and $\tilde{g}$ have the same unparametrized geodesics if and only if their respective Levi–Civita connections ∇ and $\tilde{\nabla}$ are related via

$$\tilde{\nabla}_X Y = \nabla_X Y + A(X)Y + A(Y)X, \qquad \forall X, Y \in TM$$

with some 1-form A on M.

For our purposes, it is convenient to describe a Levi–Civita connection ∇ of a metric $g = g_{\mu\nu}\theta^\mu\theta^\nu$ in terms of the connection 1-forms Γ^μ_ν associated to the coframe θ^μ via

$$d\theta^\mu + \Gamma^\mu_\nu \wedge \theta^\nu = 0, \qquad dg_{\mu\nu} - \Gamma_{\mu\nu} - \Gamma_{\nu\mu} = 0, \qquad \Gamma_{\mu\nu} = g_{\mu\rho}\Gamma^\rho_\nu.$$

In particular we have

$$\Gamma_{\mu\nu} = g(X_\mu, \nabla X_\nu),$$

where X_μ is a frame dual to θ^ν so that $X_\mu \lrcorner \theta^\nu = \delta^\nu_\mu$.

In terms of the connection 1-forms the two connections $\tilde{\nabla}$ and ∇ have the same unparametrised geodesics if and only if there exists a coframe θ and a 1-form $A = A_\mu\theta^\mu$ on M, such that the corresponding connection 1-forms $\tilde{\Gamma}^\mu_\nu$ and Γ^μ_ν are related via

$$(4) \qquad\qquad \tilde{\Gamma}^\mu_\nu = \Gamma^\mu_\nu + \delta^\mu_\nu A + A_\nu\theta^\mu.$$

in this coframe[5].

Thus, to prove the theorem, it is enough to find a common coframe and a 1-form A such that the Levi–Civita connection 1-forms for metrics (1) and (3) satisfy (4).

It turns out that such a coframe is given by (2). Calculating the Levi–Civita connection 1-forms Γ^μ_ν for g as in (1) in this coframe we find that

$$(5) \qquad \Gamma^\mu_\nu = \begin{pmatrix} 0 & \frac{\dot{R}\theta^1}{R} & \frac{\dot{R}\theta^2}{R} & \frac{\dot{R}\theta^3}{R} \\[2mm] \frac{\dot{R}\theta^1}{R} & 0 & \frac{-\kappa y\theta^1 + \kappa x\theta^2}{2R} & \frac{-\kappa z\theta^1 + \kappa x\theta^3}{2R} \\[2mm] \frac{\dot{R}\theta^2}{R} & \frac{\kappa y\theta^1 - \kappa x\theta^2}{2R} & 0 & \frac{-\kappa z\theta^2 + \kappa y\theta^3}{2R} \\[2mm] \frac{\dot{R}\theta^3}{R} & \frac{\kappa z\theta^1 - \kappa x\theta^3}{2R} & \frac{\kappa z\theta^2 - \kappa y\theta^3}{2R} & 0 \end{pmatrix}.$$

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[5]The transformation of Levi–Civita connections $\Gamma \to \tilde{\Gamma}$ is called a projective transformation. To see that two connections which are transformable to each other via projective transformations have the same geodesics is very easy: the connection coeefficients $\tilde{\Gamma}^\mu_{\nu\rho}$ defined by the connection 1-forms via $\tilde{\Gamma}^\mu_\nu = \tilde{\Gamma}^\mu_{\nu\rho}\theta^\rho$ define the geodesic equation: $\frac{dv^\mu}{dt} + \tilde{\Gamma}^\mu_{\nu\rho}v^\nu v^\rho = 0$. If we insert $\tilde{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} + \delta^\mu_\nu A_\rho + \delta^\mu_\rho A_\nu$ in this equation we get $\frac{dv^\mu}{dt} + \Gamma^\mu_{\nu\rho}v^\nu v^\rho = -2(v \cdot A)v^\mu$, i.e. again a geodesics equation, but now for the connection Γ and in a different parametrization.

Calculations of $\tilde{\Gamma}^{\mu}_{\nu}$ for (3) in this coframe gives:

$$
(6) \qquad \tilde{\Gamma}^{\mu}_{\nu} = \begin{pmatrix} \frac{2sR\dot{R}\theta^0}{1-sR^2} & \frac{\dot{R}\theta^1}{R} & \frac{\dot{R}\theta^2}{R} & \frac{\dot{R}\theta^3}{R} \\[2ex] \frac{\dot{R}\theta^1}{R(1-sR^2)} & \frac{sR\dot{R}\theta^0}{1-sR^2} & \frac{-\kappa y\theta^1+\kappa x\theta^2}{2R} & \frac{-\kappa z\theta^1+\kappa x\theta^3}{2R} \\[2ex] \frac{\dot{R}\theta^2}{R(1-sR^2)} & \frac{\kappa y\theta^1-\kappa x\theta^2}{2R} & \frac{sR\dot{R}\theta^0}{1-sR^2} & \frac{-\kappa z\theta^2+\kappa y\theta^3}{2R} \\[2ex] \frac{\dot{R}\theta^3}{R(1-sR^2)} & \frac{\kappa z\theta^1-\kappa x\theta^3}{2R} & \frac{\kappa z\theta^2-\kappa y\theta^3}{2R} & \frac{sR\dot{R}\theta^0}{1-sR^2} \end{pmatrix}.
$$

It is a matter of checking that the 1-form

$$
A = \frac{sR\dot{R}}{1-sR^2}\theta^0
$$

is such that (4) holds for $\tilde{\Gamma}^{\mu}_{\nu}$ and Γ^{μ}_{ν}. This finishes the proof. $\qquad\square$

REMARK 1. Note that if $s = 0$ the metric $\tilde{g}$ coincides with g. Observe also that the metrics $\tilde{g}$ belong to the Robertson–Walker class for all values of s: one can bring them in the form (1) by an appropriate redefinition of the coordinate t and the function R. Thus, associated with each Robertson–Walker metric g, there is an entire one-parameter family of Robertson–Walker metrics $\tilde{g}$, which includes g, with the property that all its metrics have the same unparametrised geodesics on M. The metrics $\tilde{g}$, being Robertson–Walker metrics, are all conformally flat. However for different values of s, such as e.g. $s = 0$ and $s = 1$, they are not isometric: their curvature, totally encoded in the Einstein tensor, has different properties.

Calculation of the curvatures $R^{\mu}_{\nu\rho\sigma}$ and $\tilde{R}^{\mu}_{\nu\rho\sigma}$, and the Ricci tensors $R_{\nu\sigma} = R^{\mu}_{\nu\mu\sigma}$ and $\tilde{R}_{\nu\sigma} = \tilde{R}^{\mu}_{\nu\mu\sigma}$, for the metrics g and $\tilde{g}$, still using the same coframe (2), yields the following proposition.

PROPOSITION 1. *The respective Einstein tensors $E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ and $\tilde{E}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu}$, in coframe (2), read:*

$$
E_{\mu\nu} = \begin{pmatrix} E_{00} & 0 \\ 0 & E_{ij} \end{pmatrix}
$$

with

$$
E_{00} = \frac{3(\kappa+\dot{R}^2)}{R^2}, \qquad E_{ij} = -\frac{\kappa+\dot{R}^2+2R\ddot{R}}{R^2}\delta_{ij},
$$

and

$$
\tilde{E}_{\mu\nu} = \begin{pmatrix} \tilde{E}_{00} & 0 \\ 0 & \tilde{E}_{ij} \end{pmatrix},
$$

with

$$
\tilde{E}_{00} = \frac{1}{(1-sR^2)^2}\left(E_{00}-3s\kappa\right), \quad \tilde{E}_{ij} = \frac{1}{1-sR^2}\left(E_{ij}+s(\kappa+2\dot{R}^2+2R\ddot{R})\delta_{ij}\right).
$$

Now we assume that the metric g satisfies the Einstein equations

$$(7) \qquad\qquad E_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where $T_{\mu\nu} = \rho u_\mu u_\nu$ is the energy-momentum tensor of pure dust with energy density ρ and the 4-velocity $u = u^\mu X_\mu$, orthogonal to the hypersurfaces $t = const$. This in particular means that in the frame X_μ dual to the coframe (2), we have

$$u^\mu = (1,0,0,0),$$

so that the Einstein equations (7) are

$$(8) \qquad \begin{aligned} E_{00} &= \frac{3(\kappa + \dot{R}^2)}{R^2} = 8\pi G\rho \\ E_{ij} &= -\frac{\kappa + \dot{R}^2 + 2R\ddot{R}}{R^2}\delta_{ij} = 0. \end{aligned}$$

Each solution to these equations satisfies the Friedmann equation

$$(9) \qquad\qquad \dot{R}^2 = \frac{2GM}{R} - \kappa,$$

with a constant $M = \frac{4}{3}\pi\rho R^3$. From now on we assume the equations (8)–(9) to be satisfied.

Thus we have a Friedmann–Robertson–Walker Universe (M,g) filled with the comoving dust with 4-velocity u.

Now if we forget about the parametrization of geodesics in this Universe and would like to reconstruct the metric from an analysis of unparametrized geodesics, we could equally use any metric $\tilde{g}$ and any value of the parameter s. But if we decided to use a metric $\tilde{g}$ with $s \neq 0$, we would noticed that now our Universe satsifies quite different Einstein equations than those in (7).

This is because of the folllowing line of arguments:

The vector field u is no longer a unit vector field in the metric $\tilde{g}$. Actually

$$\tilde{g}(u,u) = -\frac{1}{(1 - sR^2)^2}.$$

So obviously we cannot use u as the 4-velocity of the fluid in the metric $\tilde{g}$. Instead of u we now take a rescaled vector field

$$\tilde{u} = (1 - sR^2)u,$$

which at each point is in the direction of u and has a unit norm, $\tilde{g}(\tilde{u},\tilde{u}) = -1$, in the metric $\tilde{g}$. Surprisingly $\tilde{g}$ with such $\tilde{u}$ satisfies the Einstein equations with energy momentum tensor being a sum of the energy momentum tensor of a dust moving along $\tilde{u}$ and the energy momentum of the cosmological constant type

$$\overset{DE}{T}_{\mu\nu} = -\frac{1}{8\pi G}\tilde{\Lambda}\tilde{g}_{\mu\nu}.$$

More precisely we have the following theorem.

THEOREM 2. *Consider Robertson–Walker metrics g as in (1) and $\tilde{g}$ as in (3). If g satisfies the Friedmann equations (8)–(9) for the pure dust moving with the 4-velocity u in M, and having the energy density in the comoving frame equal to ρ, then the metric $\tilde{g}$, which in M has the same unparametrized geodesics as g, satisfies the Einstein equations*

$$(10) \qquad \tilde{E}_{\mu\nu} + \tilde{\Lambda}\tilde{g}_{\mu\nu} = 8\pi G\tilde{T}_{\mu\nu}$$

for a pure dust, $\tilde{T}_{\mu\nu} = \tilde{\rho}\tilde{u}_\mu\tilde{u}_\nu$, with 4-velocity $\tilde{u} = (1 - sR^2)u$, the energy density

$$\tilde{\rho} = \rho + \frac{s}{8\pi G}\Big(\frac{2GM}{R} - 4\kappa\Big),$$

and the cosmological "constant"

$$(11) \qquad \tilde{\Lambda} = s\Big(\kappa - \frac{2GM}{R}\Big).$$

Proof. We use Proposition 1. According to it, the nonvanishing components of the Einstein equations (10) are the diagonal ones: $\{00\}$ and $\{ij\}$. The $\{00\}$ component gives:

$$E_{00} - 3s\kappa - \tilde{\Lambda} = 8\pi G\tilde{\rho},$$

and the $\{ij\}$ components give:

$$E_{ij} + s(\kappa + 2\dot{R}^2 + 2R\ddot{R})\delta_{ij} + \tilde{\Lambda}\delta_{ij} = 0.$$

Inserting in these equations the values of E_{00} and E_{ij} from (8) we get:

$$(12) \qquad \begin{aligned} 8\pi G\rho - 3s\kappa - \tilde{\Lambda} &= 8\pi G\tilde{\rho} \\ s(\kappa + 2\dot{R}^2 + 2R\ddot{R}) + \tilde{\Lambda} &= 0. \end{aligned}$$

Now we insert the value of $\ddot{R}$ from the second equation (8) and the value of $\dot{R}$ from the Friedman equation (9) in the second equation (12). After some simple algebra, this proves the formula (11) for $\tilde{\Lambda}$. Inserting this in the first of equations (12) proves the formula for $\tilde{\rho}$. This finishes the proof. $\qquad\square$

Using this theorem we address the following issue:

REMARK 2. Since the measurments in cosmology are based on observations of photons, other elementary particles, or massive bodies, and since all of them move along geodesics, it is not clear why, based only on observations of geodesics, astronomers decide to use the Robertson–Walker metric *g* to interpret their data. According to our analysis they can equally use any metric $\tilde{g}$ with any value of the parameter *s*, because in all of these metrics the geodesics look the same: whatever choice of *s* in $\tilde{g}$ we make the Universe is always identified with the same manifold *M*, and the geodesics, i.e. the trajectories of all particles and massive bodies, are the same for all of these choices. But if we accept that we can use the metrics $\tilde{g}$ with $s = 0$ and $s \neq 0$

on equal footing, we encounter the problem what is really the energy content of the Universe. In particular the celebrated notion of the dark energy becomes meaningless in such case: the dark energy content is absent in the metric $\tilde{g}$ with $s = 0$ and present in the metric $\tilde{g}$ with $s \neq 0$.

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