ROBINSON MANIFOLDS AS THE LORENTZIAN ANALOGS OF HERMITE MANIFOLDS

PAWEŁ NUROWSKI AND ANDRZEJ TRAUTMAN

Abstract. A Lorentzian manifold is defined here as a smooth pseudo-Riemannian manifold with a metric tensor of signature $(2n+1, 1)$. A Robinson manifold is a Lorentzian manifold M of dimension $\geqslant 4$ with a subbundle N of the complexification of TM such that the fibers of $N \to M$ are maximal totally null (isotropic) and $[\operatorname{Sec} N, \operatorname{Sec} N] \subset \operatorname{Sec} N$. Robinson manifolds are close analogs of the proper Riemannian, Hermite manifolds. In dimension 4, they correspond to space-times of general relativity, foliated by a family of null geodesics without shear. Such space-times, introduced in the 1950s by Ivor Robinson, played an important role in the study of solutions of Einstein's equations: plane and sphere-fronted waves, the Gödel universe, the Kerr solution, and their generalizations, are among them. In this survey article, the analogies between Hermite and Robinson manifolds are presented in considerable detail.

1. Introduction and motivation from physics

There is an interesting class of Lorentzian manifolds that bear a close analogy to the Hermite manifolds of proper Riemannian geometry. They have been introduced and studied by physicists in the work on solutions of Einstein's equations, especially those representing gravitational waves. These *Robinson manifolds*, as we propose to call them, are little known to pure mathematicians. This may be due, in part, to the fact that physicists, in their work, used a local, coordinate-dependent description of those manifold and did not pay enough attention to the geometrical motivation and interpretation of their results. A good summary of this research by physicists is in [14].

In this article, which is largely an expository survey, we describe the main geometrical structures underlying Robinson manifolds and emphasize their analogies with Hermite manifolds.

1.1. **Motivation from physics.** Let $\mathbf{E}$ and $\mathbf{B}$ be the vectors representing, respectively, the electric and magnetic fields in the Minkowski space-time

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Corresponding author: Andrzej Trautman.

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$\mathbb{R}^4$ of special relativity theory. Introducing $\mathbf{F} = \mathbf{E} + \mathrm{i}\mathbf{B}$, one can write Maxwell's equations in empty space in the Riemann–Silberstein form (see [30] and p. 344 in [34])

$$\mathrm{i}\frac{\partial}{\partial t}\mathbf{F} = \operatorname{curl}\mathbf{F} \quad \text{and} \quad \operatorname{div}\mathbf{F} = 0. \tag{1}$$

Among the solutions of (1) especially simple are the *null* fields characterized by $\mathbf{F}^2 = 0$. The property of $\mathbf{F}$ to be null can be linearized: it is equivalent to the statement

(2) there exists a unit vector $\mathbf{n}$ such that $\mathbf{n} \times \mathbf{F} = \mathrm{i}\mathbf{F}$.

Introducing an orientation in $\mathbb{R}^4$ defined by the form $\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ so that Hodge duality of 2-forms is given by

$$\star(\mathrm{d}t \wedge \mathrm{d}x) = \mathrm{d}y \wedge \mathrm{d}z, \quad \star(\mathrm{d}y \wedge \mathrm{d}z) = -\,\mathrm{d}t \wedge \mathrm{d}x, \quad \text{etc.},$$

putting

$$F = F_x(\mathrm{d}t \wedge \mathrm{d}x - \mathrm{i}\,\mathrm{d}y \wedge \mathrm{d}z) + \text{cycl.} \quad \text{and} \quad \kappa = \mathrm{d}t - n_x\,\mathrm{d}x - n_y\,\mathrm{d}y - n_z\,\mathrm{d}z,$$

one has

$$\star F = \mathrm{i}F \tag{3}$$

and can write (1) and (2) in the equivalent form

$$\mathrm{d}F = 0, \tag{4}$$

and

(5) there exists a 1-form $\kappa \neq 0$ such that $\kappa \wedge F = 0$,

respectively.

The virtue of conditions (3)-(5) is that, without change of form, they are meaningful on every oriented, 4-dimensional Lorentzian manifold (M, g). (In fact, conformal geometry of Lorentzian signature is enough and one can generalize to a $2n$-dimensional manifold by assuming, in addition, that F is a decomposable n-form.) A 4-dimensional Robinson manifold can be provisionally defined as a Lorentzian manifold admitting a nowhere zero, complex-valued 2-form F such that conditions (3)-(5) hold. The vector field k associated by g with κ is *null*. (Pure mathematicians say: *isotropic*, but this is a misnomer. The term isotropic was introduced, in this context, by Ribaucour (see Ch. 4 in [11]) in the study of complex Euclidean geometry: if $\mathbb{C}^2$ is endowed with the quadratic form $(z_1, z_2) \mapsto z_1^2 + z_2^2$, then a rotation by the angle α transforms the vector $(1, \mathrm{i})$ into $(\exp \mathrm{i}\alpha, \mathrm{i}\exp \mathrm{i}\alpha)$. This vector is isotropic in the sense that its direction does not change under rotations. But null directions in higher dimensions are not invariant under rotations. Cartan had the good idea of calling such directions in $\mathbb{R}^4$ *optical*, but this name has not caught on.) The field k defines a foliation (physicists say: congruence) of M by null geodesics (Mariot's theorem; see [27] and the references given there). Ivor Robinson [26] found a necessary condition on the foliation, which is also sufficient in the analytic case, but not otherwise

[31], for the existence of a nowhere vanishing solution F of (3)-(5). In the physicists' language this condition is expressed by saying that k should generate a *shear-free null geodetic* (*sng*) congruence; see §5.3.

1.2. **Historical remarks and plan of the article.** In 1910, Harry Bateman [3] discovered a class of transformations, more general than conformal changes of the metric, that can be used to transform null solutions of Maxwell's equations into similar solutions; this work can be considered to be a precursor of the 'optical' ideas we are describing here; see [28, 32] and Theorem 2. In a short note of 1922, Élie Cartan [5] mentioned the existence of four principal optical (null) directions associated with a non-conformally flat Lorentz 4-manifold. He also pointed out that, in the case of the Schwarzschild space-time, these directions degenerate to form two pairs of double optical directions. Cartan's observations went unnoticed for almost 50 years. In the meantime and independently, A.Z. Petrov [22] devised an algebraic classification of the Weyl tensor (of conformal curvature) of a Lorentzian manifold and F.A.E. Pirani [23] clarified its physical significance. Using Weyl (two-component) spinors, Roger Penrose [17] sharpened the Petrov classification and gave a new derivation of the four null directions; this is recalled here in §3.3. This and subsequent work by Penrose (see [21] and the references given there) has had a decisive influence on the development of the subject. From the perspective of this article, most significant was the discovery by I. Robinson [26] of the shear-free property of congruences of null geodesics and their relation to null electromagnetic fields (§5.3). To make the article self-contained and moderately complete, we have included several classical theorems related to its subject, with references to literature instead of proofs. In particular, in Section 5.4 we present the Goldberg–Sachs theorem on the connection between the existence of *sng* congruences and the degeneracy of the principal null directions in Einstein manifolds, as well as its generalization to the proper Riemannian case. A theorem due to R.P. Kerr, giving all *sng* congruences in Minkowski space-time is presented in considerable detail in §6 and 7.1. In the last Section, we briefly describe twistor bundles, an important concept that emerged in connection with the study of *sng* congruences. There is a wealth of literature on Penrose's twistor ideas, in both the Lorentz and proper Riemannian cases [2, 18, 20, 21, 35]. Recent surveys are in [8].

2. NOTATION AND TERMINOLOGY

Our notation and terminology are essentially standard; see, e.g., [4, 12, 15]. The exterior algebra associated with a vector space W is $\wedge W$; the symbols $\otimes$, $\wedge$ and $\lrcorner$ denote the tensor, exterior and interior products, respectively. We use the Einstein summation convention over repeated indices. The canonical map of $W \setminus \{0\}$ onto the associated projective space $\mathrm{P}(W)$ is denoted by dir and we write $\mathbb{C}\mathrm{P}_n$ for $\mathrm{P}(\mathbb{C}^{n+1})$. A *quadratic space* is defined as a pair (V, g), where V is a finite-dimensional vector space over $\Bbbk = \mathbb{R}$

or $\mathbb{C}$, and $g : V \to V^*$ is a symmetric ($g^* = g$) isomorphism. To save on notation, we use the same letter g for the *metric tensor* $g \in V^* \otimes_{\mathrm{sym}} V^*$ associated with that isomorphism so that $g(u, v) = \langle u, g(v)\rangle$ and $v \mapsto g(v, v)$ is a quadratic form. For the symmetrized tensor product of 1-forms we use the notation of classical differential geometry, i.e., if $\alpha, \beta \in V^*$, then $2\alpha\beta = \alpha \otimes \beta + \beta \otimes \alpha$. This convention allows us to write the metric tensor as $g = g(e_\nu)e^\nu = g_{\mu\nu}e^\mu e^\nu$, where (e^μ) is the coframe dual to (e_μ) and $g_{\mu\nu} = g(e_\mu, e_\nu)$. If $N \subset V$, then $N^\perp$ is the set of all elements of V orthogonal to every element of N. The Hodge dual of α is denoted by $\star\alpha$.

All manifolds and maps among them are assumed to be smooth (of class C^∞) or real-analytic. Manifolds are finite-dimensional, but not necessarily compact. If $f : M' \to M$ is a map of manifolds, then $Tf : TM' \to TM$ is the corresponding tangent (derived) map and $T_xM \subset TM$ is the tangent vector space to M at x. The map f is an immersion (resp., submersion) if Tf, restricted to every tangent vector space, is injective (resp., surjective); an injective immersion is an embedding and defines M' as a submanifold of M. If $\pi : E \to M$ is a fiber bundle over a manifold M, then $E_p = \pi^{-1}(p) \subset E$ is the fiber over $p \in M$. A map $f : M' \to M$ gives rise to the induced bundle $f^{-1}E \to M'$ such that $(f^{-1}E)_p = E_{f(p)}$ for every $p \in M'$. If f is an immersion, then TM' is a subbundle of $f^{-1}TM$. The zero bundle is denoted by $\mathbf{0}$. A *Riemannian manifold* M is assumed to be connected; it has a metric tensor field g which is non-degenerate, but not necessarily definite; if it is, then (M, g) is said to be *proper* Riemannian. A *space-time* is a 4-dimensional manifold with a metric tensor of signature $(3, 1)$.

The module over $C^\infty(M)$ of all sections of the vector bundle $E \to M$ is denoted by $\operatorname{Sec} E$. If $X \in \operatorname{Sec} TM$, then $L(X)$ is the Lie derivative with respect to X. If α is a differential form on M and $f : M' \to M$, then $L(X)\alpha = X \lrcorner \mathrm{d}\alpha + \mathrm{d}(X \lrcorner \alpha)$ and $f^*\alpha$ is the pull-back of α to M'. We abbreviate $\partial/\partial x$ to ∂_x. In Section 4 we summarize the definitions and notions related to CR structures needed in this paper; further details can be found in [10].

To save on notation, we sometimes use the same letter to denote a vector space N with some structure and a fiber bundle $N \to M$ with fibers carrying the same structure. Local sections of $N \to M$ may be denoted by the same letters as elements of the vector space N.

3. ALGEBRAIC PRELIMINARIES

3.1. **Maximal, totally null subspaces of vector spaces.** Consider a complex quadratic space (V, g). Recall that a vector subspace N of V is said to be *null* if $N^\perp \cap N \neq \varnothing$ and *totally null* if $N \subset N^\perp$. Assume now $\dim V = 2n$; if $N \subset V$ is *maximal totally null* (*mtn*), then $N^\perp = N$ so that $\dim N = n$. An orientation having been fixed, the Hodge duality map $\star : \wedge V \to \wedge V$ can be defined so that $\star^2 = \mathrm{id}$. If $(m_1, \ldots, m_n)$ is a frame in an *mtn* subspace N, then

$$\star(m_1 \wedge \cdots \wedge m_n) = \pm m_1 \wedge \cdots \wedge m_n. \tag{6}$$

The *annihilator* of N,

$$N^0 = \{\mu \in V^* \mid \langle m, \mu \rangle = 0 \text{ for every } m \in N\}$$

is an *mtn* subspace of V^*. The set of all *mtn* subspaces of a complex, $2n$-dimensional vector space has the structure of a complex manifold, diffeomorphic to the symmetric space $\mathrm{O}_{2n} / \mathrm{U}_n$; its two connected components correspond to the two signs in (6) characterizing the *mtn* subspaces of positive and negative chiralities, respectively.

Let now (V, g) be a Euclidean quadratic space, i.e. a real quadratic space such that the form associated with g is positive-definite. Assume that V is of positive even dimension. An *mtn* subspace N of the complexification $W = \mathbb{C} \otimes V$ defines a complex orthogonal structure J on (V, g): this is so because $N \cap \bar{N} = \{0\}$ and one can put

$$J(v) = \mathrm{i}v \quad \text{and} \quad J(\bar{v}) = -\mathrm{i}\bar{v} \quad \text{for} \quad v \in N. \tag{7}$$

Conversely, an orthogonal complex structure J on (V, g) defines the *mtn* subspace $N = \{v \in W \mid J(v) = \mathrm{i}v\}$.

Consider now a *Lorentz space* (V, g), defined as a real quadratic space such that the quadratic form associated with g is of signature $(2n+1, 1)$, $n = 1, 2, \ldots$. Let $N \subset W = \mathbb{C} \otimes V$ be an *mtn* subspace. The intersection $N \cap \bar{N}$ is the complexification of a null real line $K \subset V$ and $N + \bar{N} = \mathbb{C} \otimes K^\perp$. There is a real null line L such that $V = K^\perp \oplus L$. The quotient $K^\perp / K$ inherits from (V, g) the structure of a Euclidean quadratic space of dimension $2n$ and there is an orthogonal complex structure J on $K^\perp / K$, defined by $J(v \bmod \mathbb{C} \otimes K) = \mathrm{i}v \bmod \mathbb{C} \otimes K$ for every $v \in \mathbb{C} \otimes K^\perp$. Similarly, $N^0 \cap \bar{N}^0$ is the complexification of a real null line and there is the isomorphism

$$g : K \to \operatorname{Re} N^0 \cap \bar{N}^0 \tag{8}$$

obtained by restricting $g : V \to V^*$ to K.

3.2. **Spinor algebra in dimension 4.** Spinor calculus in dimension 4 provides an economical, convenient description of many aspects of the geometry of Riemannian manifolds of this dimension [15, 21]. Since there are so many exhaustive presentations of this subject, it suffices to give here the rudiments of spinor algebra in a form adapted to our purposes.

If the dimension of the real vector space V is 4, then the complex vector space S of Dirac spinors is also four-dimensional. Let (e_μ) be an orthonormal frame in V. A representation γ of the Clifford algebra associated with (V, g) in S is given by the 'Dirac matrices' $\gamma_\mu = \gamma(e_\mu)$. The endomorphism $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ anticommutes with the Dirac matrices and $\gamma_5^2 = \mathrm{id}$ if (V, g) is Euclidean and $\gamma_5^2 = -\,\mathrm{id}$ if (V, g) is Lorentzian. Putting $\mathit{\Gamma} = \gamma_5$ in the first and $\mathit{\Gamma} = \mathrm{i}\gamma_5$ in the second case, one has $\mathit{\Gamma}^2 = \mathrm{id}$.

The spaces of 'chiral' or Weyl spinors are defined by

$$S_\pm = \{\varphi \in S \mid \varGamma\varphi = \pm\varphi\}.$$

Let $W = \mathbb{C} \otimes V$ and, for $v_1, v_2 \in V$, put $\gamma(v_1 + \mathrm{i}v_2) = \gamma(v_1) + \mathrm{i}\gamma(v_2)$, then $\gamma(w)^2 = g(w,w)\,\mathrm{id}$ for every $w \in W$. If $\varphi \in S_\pm$ and $\varphi \neq 0$, then

$$N(\varphi) = \{w \in W \mid \gamma(w)\varphi = 0\} \tag{9}$$

is an *mtn* subspace of W of the same chirality as φ.

The transposed endomorphisms γ_μ^* define the contragredient representation of the Clifford algebra in S^*, which is equivalent to γ: there is the isomorphism $B : S \to S^*$ such that $\gamma_\mu^* = B\gamma_\mu B^{-1}$ for $\mu = 1, \ldots, 5$. B restricts to a symplectic form ε on each of the spaces of Weyl spinors S_+ and S_-. If (e_A), $A = 1, 2$, is a frame in S_+ and (e^A) is the dual frame in S_+^*, then $\varepsilon(e_A) = \varepsilon_{AB}e^B$. The complex conjugate representation given by $\bar\gamma_\mu$ is also equivalent to γ: there is an isomorphism $C : S \to \bar S$ such that $\bar\gamma_\mu = C\gamma_\mu C^{-1}$ and $C\bar C = -\,\mathrm{id}$ in the Euclidean case and $C\bar C = \mathrm{id}$ for signature $(3,1)$. The spinor $\varphi_c = C^{-1}\bar\varphi$ is said (by physicists) to be the *charge conjugate* of $\varphi \in S$.

3.3. **The algebraic classification of Weyl tensors.** The spaces $S_+^4 = \otimes_{\mathrm{sym}}^4 S_+^*$ and $S_-^4 = \otimes_{\mathrm{sym}}^4 S_-^*$ are isomorphic to spaces of tensors of rank 4 over $W = \mathbb{C}^4$, with symmetries of self-dual and anti-self-dual Weyl (conformal curvature) tensors, denoted by C_+ and C_-, respectively. Consider $0 \neq \psi \in S_+^4$: there is a frame (e_A), $A = 1, 2$, in S_+ such that the component $\psi_{1\ldots1} = \psi(e_1, \ldots, e_1)$ is not zero. Given such a frame, let $\varphi(z) = ze_1 + e_2 \in S_+$, $z \in \mathbb{C}$, and consider the complex polynomial p_ψ of degree 4,

$$p_\psi(z) = \psi(\varphi(z), \ldots, \varphi(z)) = \psi_{1\ldots1}z^4 + \cdots + \psi_{2\ldots2}.$$

Let $\{z_1, \ldots, z_4\}$ be the set of all roots of this polynomial; a root of multiplicity s appears s times in the set. Then

$$\psi = \psi_{1\ldots1}\varphi^1 \otimes_{\mathrm{sym}} \cdots \otimes_{\mathrm{sym}} \varphi^4, \quad \text{where} \quad \varphi_A^i = \varepsilon_{AB}\varphi(z_i)^B, \ \ i = 1, \ldots, 4.$$

The spinors φ^i are *eigenspinors* (with eigenvalue 0) of ψ. The *algebraic type* of ψ is the sequence $[s_1 \ldots s_k]$, $1 \leqslant s_1 \leqslant \cdots \leqslant s_k \leqslant 4$, $s_1 + \cdots + s_k = 4$, of the multiplicities of the roots of p_ψ. In the generic case, all roots are simple, $s_1 = \cdots = s_4 = 1$. Otherwise, one says that ψ is *algebraically degenerate*. An eigenspinor is said to be *repeated* if its multiplicity s is larger than 1.

The enumeration of the possible degeneracies can be traced back to Cartan [5]; physicists use it now in a form due to Penrose [17]:

(i) Type I (non-degenerate) $[1111]$,
(ii) Type II $[112]$,
(iii) Type III $[13]$,
(iv) Type D ('degenerate') $[22]$,
(v) Type N ('null') $[4]$.

$$\begin{tikzcd}
& & \mathrm{I} \arrow[dl] \arrow[d] \\
& \mathrm{II} \arrow[r] \arrow[dl] \arrow[d] & \mathrm{D} \arrow[dl] \arrow[d] \\
\mathrm{III} \arrow[r] & \mathrm{N} \arrow[r] & 0
\end{tikzcd}$$

The 0 in the Penrose diagram above represents a vanishing ψ. The arrows point towards more special cases. This classification of complex, self-dual Weyl tensors is often associated with the name of Petrov, who, however, recognized only three types (I, II and III). The Weyl tensor of a complex Riemannian manifold decomposes into its self-dual and anti-self-dual parts; their algebraic types are independent.

In the case of real manifolds, one has to consider separately each signature. We restrict ourselves to the proper Riemannian and Lorentzian cases.

1. In the proper Riemannian case, the Weyl tensor decomposes into the real, self-dual and anti-self-dual parts; they are independent. The self-dual part is represented by a spinor $\psi \in S^4_+$ that satisfies a suitable reality condition which implies that the eigenspinors of ψ occur in pairs (φ, φ_c). Therefore, there are only two types of $\psi \neq 0$: either these two pairs are distinct (type I) or they coincide (type D). Similar remarks apply to the anti-self-dual part of the Weyl tensor. Therefore, the complete algebraic classification of the Weyl tensor of a proper Riemannian 4-dimensional manifold contains 9 cases; (I,I) is the most general case and (0,0) represents conformally flat manifolds. The cases $(*, 0)$ and $(0, *)$ are referred to as self-dual and anti-self-dual, respectively.

2. In the Lorentzian case, the real Weyl tensor decomposes into its self- and anti-self-dual parts, which are complex, $\mathsf{C} = \mathsf{C}_+ + \mathsf{C}_-$, where $\star\mathsf{C}_\pm = \pm \mathrm{i}\mathsf{C}_\pm$ so that $\bar{\mathsf{C}}_+ = \mathsf{C}_-$. Therefore, the classification is given by that of the complex, self-dual Weyl tensor presented above.

4. Cauchy–Riemann manifolds

4.1. Almost CR manifolds.

Definition 1. An *almost Cauchy–Riemann manifold* $\mathcal{M}$ of dimension $2n+1$ is defined as a manifold with a distinguished subbundle $\mathcal{N}$ of $\mathbb{C}\otimes T\mathcal{M}$, with fibers of complex dimension n, such that $\bar{\mathcal{N}} \cap \mathcal{N} = \mathbf{0}$.

One also says that $\mathcal{M}$ has an almost CR structure. The direct sum $\bar{\mathcal{N}} \oplus \mathcal{N}$ is the complexification of a bundle $\mathcal{H} \subset T\mathcal{M}$ with $2n$-dimensional fibers, endowed with $J \in \operatorname{Sec}\operatorname{End}\mathcal{H}$ such that $J^2 = -\operatorname{id}_{\mathcal{H}}$; namely, $J(w + \bar{w}) = \mathrm{i}(w - \bar{w})$ for every $w \in \mathcal{N}$.

The annihilator $\mathcal{N}^0 \subset \mathbb{C}\otimes T^*\mathcal{M}$ has fibers of complex dimension $n+1$ and $\overline{\mathcal{N}^0} \cap \mathcal{N}^0$ is the complexification of a real line bundle. The *canonical bundle* [9] of the almost CR structure, $\Omega = \wedge^{n+1}\mathcal{N}^0$, is a complex line bundle over $\mathcal{M}$ and

$$\mathcal{N}_p = \{w \in \mathbb{C}\otimes T_p\mathcal{M} \mid w \lrcorner \omega = 0,\ 0 \neq \omega \in \Omega_p,\ p \in \mathcal{M}\}.$$

There is a convenient, equivalent description of an almost CR structure by an atlas of CR compatible charts: every point of $\mathcal{M}$ has a neighborhood $\mathcal{U}$ admitting a collection of 1-forms

$$(\kappa, \mu^1, \dots, \mu^n) \text{ with } \kappa \text{ real and } \kappa \wedge \mu^1 \wedge \dots \wedge \mu^n \wedge \overline{\mu^1} \wedge \dots \wedge \overline{\mu^n} \neq 0 \tag{10}$$

such that

$$\mathcal{N}^0_p = \mathrm{span}_p\{\kappa, \mu^1, \ldots, \mu^n\} \quad \text{for every} \quad p \in \mathcal{U}. \tag{11}$$

The pair

$$(\mathcal{U}, (\kappa, \mu^1, \ldots, \mu^n)) \tag{12}$$

is a *CR chart*. Given any other CR chart $(\mathcal{U}', (\kappa', \mu'^1, \ldots, \mu'^n))$, on the overlap $\mathcal{U} \cap \mathcal{U}'$ one has

$$\kappa' = a\kappa, \quad \mu'^\alpha = b^\alpha\kappa + b^\alpha{}_\beta \mu^\beta, \quad \alpha, \beta = 1, \ldots, n, \tag{13}$$

where a is a real function, the bs are complex and $a \det b \neq 0$, where $b = (b^\alpha{}_\beta)$. An almost CR manifold can be defined as an odd-dimensional manifold with an atlas of compatible CR charts, their compatibility being defined by (13). The $(n+1)$-form

$$\omega = \kappa \wedge \mu^1 \wedge \cdots \wedge \mu^n, \tag{14}$$

is a nowhere vanishing local section of $\Omega \to \mathcal{M}$ defined on $\mathcal{U}$.

Given (10), one puts

$$\mathrm{d}\kappa = \mathrm{i}h_{\alpha\beta}\mu^\alpha \wedge \bar{\mu}^\beta + \ldots,$$

where the dots stand for exterior products of pairs of the local basis 1-forms other than the products $\mu^\alpha \wedge \bar{\mu}^\beta$, $1 \leqslant \alpha, \beta \leqslant n$. The transformation (13) induces the change

$$h'_{\alpha\beta} = a h_{\gamma\delta} c^\gamma{}_\alpha \bar{c}^\delta_\beta, \quad 1 \leqslant \alpha, \beta, \gamma, \delta \leqslant n,$$

where $c = (c^\alpha{}_\beta)$ is the inverse of the matrix b. The matrix $h = (h_{\alpha\beta})$ is Hermitean and the signature of the associated Hermitean *Levi form* is well-defined: it does not change under the replacement (13). The almost CR structure is said to be *non-degenerate* if $\det h \neq 0$; it is called *pseudo-convex* (sometimes: strongly pseudo-convex) if the associated Hermitean form is definite.

If the distribution $\ker \kappa = \mathcal{H}$ is integrable, $\kappa \wedge \mathrm{d}\kappa = 0$, then the CR structure is said to be *trivial* and, locally, $\mathcal{M} = \mathbb{R} \times \mathbb{C}^n$. In dimension three, non-triviality of a CR structure is equivalent to its pseudo-convexity.

4.2. **CR manifolds.**

Definition 2. A *Cauchy–Riemann manifold* $(\mathcal{M}, \mathcal{N})$ is an almost CR manifold characterized by the bundle $\mathcal{N} \to \mathcal{M}$, satisfying the integrability condition $[\mathrm{Sec}\,\mathcal{N}, \mathrm{Sec}\,\mathcal{N}] \subset \mathrm{Sec}\,\mathcal{N}$.

The integrability condition is equivalent to

$$\mathrm{dSec}\,\mathcal{N}^0 \subset \mathrm{Sec}\,\mathcal{N}^0 \wedge \mathrm{Sec}(\mathbb{C} \otimes T^*\mathcal{M}).$$

In terms of a CR chart (12) of $\mathrm{Sec}\,\mathcal{N}^0$ this is equivalent to

$$\mathrm{d}\kappa \wedge \omega = 0 \quad \text{and} \quad \mathrm{d}\mu^\alpha \wedge \omega = 0 \quad \text{for} \quad \alpha = 1, \ldots, n. \tag{15}$$

Clearly, every 3-dimensional almost CR manifold is a CR manifold; we refer to it as a *CR space*.

If the canonical bundle Ω admits, for every $\mathcal{U}$ in the atlas, a closed local section ω nowhere zero on $\mathcal{U}$, then the integrability conditions (15) follow from $\kappa \wedge \omega = 0$ and $\mu^\alpha \wedge \omega = 0$, $\alpha = 1, \dots, n$.

The chart (12) is said to be locally *embeddable* (sometimes: realizable) if the *tangential CR equation*

$$\mathrm{d}z \wedge \omega = 0 \tag{16}$$

has $n+1$ solutions $z_1, \dots, z_{n+1}$ such that

$$\operatorname{span}_p\{\mathrm{d}z_1, \dots, \mathrm{d}z_{n+1}, \mathrm{d}\bar{z}_1, \dots, \mathrm{d}\bar{z}_{n+1}\} = \mathbb{C} \otimes T_p^*\mathcal{M} \quad \text{for every } p \in \mathcal{U}.$$

One then has the exact local section $\omega = \mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_{n+1}$ of the canonical bundle and the map $z : \mathcal{U} \to \mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$, $z = (z_1, \dots, z_{n+1})$, is an immersion. A CR manifold is locally embeddable if it has a CR atlas of locally embeddable charts. Every analytic CR manifold is locally embeddable [1].

Let $\mathcal{M}$ be now an embeddable CR space so that there are two solutions z_1 and z_2 of (16) and a real-valued smooth function G on $\mathbb{C}^2$ such that

$$G(z_1, z_2, \bar{z}_1, \bar{z}_2) = 0 \quad \text{and} \quad \mathrm{d}G \neq 0. \tag{17}$$

One can then take

$$\kappa = \mathrm{i}(\frac{\partial G}{\partial z_1}\mathrm{d}z_1 + \frac{\partial G}{\partial z_2}\mathrm{d}z_2), \quad \mu = \overline{\frac{\partial G}{\partial z_2}}\mathrm{d}z_1 - \overline{\frac{\partial G}{\partial z_1}}\mathrm{d}z_2. \tag{18}$$

4.3. CR submanifolds.

Definition 3. Let $(\mathcal{M}, \mathcal{N})$ and $(\mathcal{M}', \mathcal{N}')$ be CR manifolds of dimension $2n+1$ and $2n-1$, respectively. If $\mathcal{M}'$ is a submanifold of $\mathcal{M}$ with an embedding $f : \mathcal{M}' \to \mathcal{M}$ and $\mathcal{N}' = (\mathbb{C} \otimes T\mathcal{M}') \cap f^{-1}\mathcal{N}$, then one says that $\mathcal{M}'$ is a *CR submanifold* of $\mathcal{M}$ [7].

There is a convenient characterization of CR submanifolds in terms of an atlas of CR charts:

Proposition 1. *Let $f : \mathcal{M}' \to \mathcal{M}$ define $\mathcal{M}'$ as a submanifold of the CR manifold $(\mathcal{M}, \mathcal{N})$. Let (12) be a CR chart on $\mathcal{M}$ and ω the corresponding local section of the canonical bundle. If, for every such chart,*

$$f^*\omega = 0 \tag{19}$$

and one can find $n-1$ linear combinations $(\mu'^1, \dots, \mu'^{n-1})$ of the forms $(\mu^1, \dots, \mu^n)$ such that $\omega' = f^*(\kappa \wedge \mu'^1 \wedge \dots \wedge \mu'^{n-1}) \neq 0$, then

$$\mathcal{N}' = \{w \in \mathbb{C} \otimes T\mathcal{M}' \mid w \lrcorner \omega' = 0\}$$

defines on $\mathcal{M}'$ the structure of a CR submanifold of $(\mathcal{M}, \mathcal{N})$.

Proof. For every $p \in \mathcal{M}'$ the monomorphism T_pf, after extension to $\mathbb{C} \otimes T_p\mathcal{M}' \to \mathbb{C} \otimes T_{f(p)}\mathcal{M}$, restricts to a monomorphism $\mathcal{N}'_p \to \mathcal{N}_{f(p)}$ and the epimorphism $(T_pf)^*$ restricts to an epimorphism $\mathcal{N}^0_{f(p)} \to \mathcal{N}'^0_p$. Note that

$(T_pf)^*(\mathcal{N}^0_{f(p)} \cap \overline{\mathcal{N}^0}_{f(p)})$ is the complexification of a real line bundle: it coincides with $\mathcal{N}'^0_p \cap \overline{\mathcal{N}'^0_p}$. Therefore, given a local basis as in (12), one has

$$(T_pf)^*(\kappa \wedge \mu^1 \wedge \cdots \wedge \mu^n) = 0$$

and one can choose n linear combinations of the forms (12) at $f(p)$, κ being one of them, which are mapped by $(T_pf)^*$ to a basis of $\mathcal{N}'^0_p$. □

5. Hermite and Robinson structures

5.1. Almost Hermite and almost Robinson structures.

Definition 4. An *N-structure* on a Riemannian manifold (M, g) of even dimension $\geqslant 4$, is a complex vector subbundle N of the complexified tangent bundle $\mathbb{C} \otimes TM$ such that, for every $p \in M$, the fiber N_p is *mtn*.

It is known that, if (M, g) is proper Riemannian, then an N-structure on M is equivalent to that of an *almost Hermite* manifold; the orthogonal almost complex structure J on M is defined as in (7) (see, e.g., Ch. IX §4 in [12]).

Definition 5. An *almost Robinson* manifold is a Lorentzian manifold with an N-structure.

In this case, the intersection $N \cap \bar{N}$ is the complexification of a line bundle $K \subset TM$; its fibers are null; they are tangent to a foliation of M by null curves. An almost Robinson structure on M is said to be *regular* if the set $\mathcal{M}$ of the leaves of the foliation defined by K has the structure of a manifold such that the natural map $\pi : M \to \mathcal{M}$ is a submersion. ¿From now on, only such regular structures will be considered.

5.2. The integrability condition.

Definition 6. The N-structure $N \to M$ on a Riemannian manifold (M, g) is said to be *integrable* if

$$[\operatorname{Sec} N, \operatorname{Sec} N] \subset \operatorname{Sec} N. \tag{20}$$

Dually, the integrability condition is

$$\mathrm{d} \operatorname{Sec} N^0 \subset \operatorname{Sec} N^0 \wedge \operatorname{Sec}(\mathbb{C} \otimes T^*M). \tag{21}$$

In the proper Riemannian case, condition (20) is equivalent to the vanishing of the Nijenhuis (torsion) tensor of the almost complex structure J and, by the celebrated Newlander–Nirenberg theorem, it implies that M is a Hermite manifold; see Ch. IX §2 and 4 in [12].

Definition 7. A *Robinson manifold* is an almost Robinson manifold with an integrable N-structure.

Let ω be defined as in (14). It characterizes N,

$$N_p = \{w \in \mathbb{C} \otimes T_pM \mid w \lrcorner \omega = 0\}. \tag{22}$$

In view of (11), the integrability condition (21) of Robinson manifolds is of the same *form* (15) as for CR structures.

Theorem 1. *Consider a Robinson manifold M of dimension $2n+2$. Let (ϕ_t) be the flow generated by a vector field $k : M \to K$, where $K \subset TM$ is the null line bundle defined by $N \cap \bar{N} = \mathbb{C} \otimes K$, then*

(i) *the N-structure on M is invariant with respect to the action of the flow (ϕ_t) and the trajectories of (ϕ_t) are null geodesics;*

(ii) *the N-structure on M defines a Cauchy–Riemann structure on the quotient manifold $\mathcal{M}$;*

(iii) *the $2n$-dimensional fibers of the bundle $K^\perp/K \to M$ have a complex structure and a positive-definite quadratic form, induced by g.*

Proof. (i) Let $(\kappa, \mu^1, \ldots, \mu^n)$ be as in (11); in view of the reality of κ, the integrability condition (21) is equivalent to

$$\mathrm{d}\kappa = \kappa \wedge \rho + \mathrm{i}\sigma_{\alpha\beta}\mu^\alpha \wedge \bar{\mu}^\beta, \tag{23}$$

and

$$\mathrm{d}\mu^\alpha = \kappa \wedge \varsigma^\alpha + \mu^\beta \wedge \tau_\beta{}^\alpha, \quad \alpha = 1, \ldots, n, \tag{24}$$

where ρ, ς^α and $\tau_\beta{}^\alpha$ are one-forms and the σs are functions such that $\sigma_{\alpha\beta} = \overline{\sigma_{\beta\alpha}}$. It follows from (22) that the invariance of N with respect to (ϕ_t) is equivalent to $L(k)\omega \| \omega$; this relation follows from (23) and (24). Moreover, equation (23) implies

$$\kappa \wedge L(k)\kappa = 0. \tag{25}$$

In view of (8) one can take $\kappa = g(k)$ so that $L(k)\kappa = (L(k)g)(k) = \nabla_k \kappa$; this shows that (25) is equivalent to the geodetic condition $\nabla_k k \| k$.

(ii) It follows from (i) that the distribution $N \subset \mathbb{C} \otimes TM$ descends to a distribution $\mathcal{N} \subset \mathbb{C} \otimes T\mathcal{M}$; its fibers are of complex dimension n and $\mathcal{N} \cap \overline{\mathcal{N}} = \mathbf{0}$. Moreover, the integrability of N implies that of $\mathcal{N}$.

(iii) Only the complex structure requires a construction: since

$$K^\perp = \mathrm{Re}(N + \bar{N}),$$

one can put $J(w + \bar{w} \bmod K) = \mathrm{i}(w - \bar{w}) \bmod K$ for $w \in N$. □

Note that if k and k' are two sections of $K \to M$, nowhere vanishing on open subsets U and U' of M, respectively, then $k' = fk$, where f is a nowhere zero function on $U \cap U'$. If (ϕ_t) and (ϕ'_t) are the flows generated by k and k', respectively, then, on $U \cap U'$, the invariance of N with respect to (ϕ_t) is equivalent to that with respect to (ϕ'_t) and the trajectories of these two flows coincide.

There is a local converse to Theorem 1. Let $\mathcal{M}$ be a $(2n+1)$-dimensional CR manifold characterized by differential forms as described in Section 4. Put

$$\pi = \mathrm{pr}_1 : M = \mathcal{M} \times \mathbb{R} \to \mathcal{M}. \tag{26}$$

and denote by κ, $\mu^1,\ldots,\mu^n$ the pull-backs by π to M of the corresponding forms on $\mathcal{M}$. Let v be the canonical coordinate on $\mathbb{R}$ and $k = \partial_v \in \operatorname{Sec} TM$. The collection of forms

$$(\kappa, \mathrm{d}v, \mu^1, \ldots, \mu^1, \bar{\mu}^1, \ldots, \bar{\mu}^n) \tag{27}$$

is a (local) basis of $\operatorname{Sec}(\mathbb{C} \otimes T^*M)$; let

$$(l, k, \bar{Z}_1, \ldots, \bar{Z}_n, Z_1, \ldots, Z_n)$$

be the dual basis. We shall construct a Robinson manifold (M, g, N) so that (11) holds. With respect to the basis (27), the metric is

$$g = g(l)\kappa + g(k)\,\mathrm{d}v + g(\bar{Z}_\alpha)\mu^\alpha + g(Z_\alpha)\bar{\mu}^\alpha.$$

Note that since $k \in \operatorname{Sec}(N+\bar{N})^\perp$, one has $g(k) = g(k,l)\kappa$; therefore $g(k,l) \neq 0$. Defining $\lambda = g(l) + g(k,l)\,\mathrm{d}v + g(\bar{Z}_\alpha, l)\mu^\alpha + g(Z_\alpha, l)\bar{\mu}^\alpha$ so that $k \lrcorner \lambda = 2g(k,l)$, putting $g_{\alpha\beta} = 2g(\overline{Z}_\alpha, Z_\beta) = \bar{g}_{\beta\alpha}$, one obtains

$$g = \kappa\lambda + g_{\alpha\beta}\mu^\alpha\bar{\mu}^\beta. \tag{28}$$

This concludes the proof of

Proposition 2. *Locally, every Robinson $(2n+2)$-manifold (M, g, N), having $\mathcal{M}$ as the associated CR manifold, is of the form (26) with a metric given by (28), where the forms κ, $\mu^1,\ldots,\mu^n$ are obtained by pull-back of the corresponding forms on $\mathcal{M}$, the functions $g_{\alpha\beta} : M \to \mathbb{C}$ are such that, for every $p \in M$, the form $g_{\alpha\beta}(p)z^\alpha\bar{z}^\beta$ is Hermitean positive-definite, λ is any real 1-form on M such that $k \lrcorner \lambda$ is nowhere 0 and $N^0 = \operatorname{span}\{\kappa, \mu^1, \ldots, \mu^n\}$.*

5.3. **Four-dimensional Robinson manifolds: space-times with a non-distorting foliation by null geodesics.** The case of dimension 4 is well known, but, since it is also the most important one, it is worth-while to review it briefly here. In a sense made precise below, in this case, unlike as in higher dimensions, all information about the Robinson structure is encoded in the properties of the bundle K.

Let (M, g) be a space- and time-oriented Robinson manifold of dimension 4 with the bundle $N \to M$ of *mtn* spaces. The fibers of the bundle $K^\perp/K \to M$ are two-dimensional 'screen spaces'. According to part (iii) of Theorem 1, each screen space has a complex structure, which, *in this case*, is equivalent to a conformal structure and an orientation; this being preserved by the flow is equivalent to [28]

$$L(k)g = \rho g + \kappa \otimes \xi + \xi \otimes \kappa \tag{29}$$

for some function ρ and 1-form ξ. Physicists say that k generates a shear-free congruence of null geodesics. The expression 'shear-free' reflects the non-distorting property property of the flow: it preserves the conformal structure of the screen spaces. Conversely, given a bundle K of null directions, the space and time orientations of M induce an orientation in the screen spaces; together with the induced Euclidean metric this determines a complex structure J in each screen space. This complex structure defines the bundle $N = \{w \in \mathbb{C} \otimes K^\perp \mid J(w \bmod \mathbb{C} \otimes K) = \mathrm{i}w \bmod \mathbb{C} \otimes K\}$ with *mtn* fibers. Equation (29) implies $[\mathrm{Sec}\, K, \mathrm{Sec}\, N] \subset \mathrm{Sec}\, N$; in dimension 4 this is enough to establish the validity of (15). In view of this, we shall often denote by (M, g, K) a Robinson space-time determined by the bundle K of null lines satisfying (29). As a consequence of Proposition 2 one has

Corollary 1. *Let $\mathcal{M}$ be a CR space. Put $M = \mathcal{M} \times \mathbb{R}$, denote by v a coordinate on $\mathbb{R}$, put $k = \partial_v$, $K = \mathrm{span}\, k$, pull-back to M the forms characterizing the CR structure on $\mathcal{M}$ to obtain the pair (κ, μ). Let $p : M \to \mathbb{R}^+$ and let λ be a 1-form on M such that $k \lrcorner \lambda \neq 0$. If*

$$g = \kappa\lambda + p\mu\bar{\mu}, \tag{30}$$

then (M, g, K) is a Robinson space-time and every Robinson space-time can be locally so described, as a lift *of $\mathcal{M}$.*

Problem 1. Characterize the CR spaces that admit lifts to Einstein–Robinson space-times.

Theorem 2. *Let (M, g, K) be a Robinson space-time so that g is of the form (30) and the N-structure is characterized by $N^0 = \mathrm{span}\{\kappa, \mu\}$. Given a function $\rho : M \to \mathbb{R}^+$ and a 1-form ξ on M such that*

$$k \lrcorner (\lambda + \xi) \neq 0, \tag{31}$$

define

$$g' = \rho(g + \kappa\xi).$$

Then

(i) *(M, g', K) is a Robinson manifold,*

(ii) *if F satisfies (3)-(5) on (M, g, K), then it also satisfies these equations on (M, g', K).*

Proof. (i) One has $g' = \rho(\kappa\lambda' + p\mu\bar{\mu})$, where $\lambda' = \lambda + \xi$ and $\kappa \wedge \lambda' \wedge \mu \wedge \bar{\mu} \neq 0$ by virtue of (31). Moreover, the bundle $N \to M$ does not change under the replacement of g by g'.
(ii) The properties (3)-(5) of the form $F = A\kappa \wedge \mu$ also do not change. □

The theorem originates with work of Bateman [3]; see also [28]. The geometry of (M, g') may be rather different from that of (M, g); the electromagnetic fields defined by F in these two space-times may also be physically distinct. This is illustrated by the following

Example 1. Let $\mathbb{R}^4$ be the Minkowski space-time. It is convenient to use a global coordinate system (u, v, w), where the coordinates u, v are real and w is complex so that

$$g = \mathrm{d}u\, \mathrm{d}v + \mathrm{d}w\, \mathrm{d}\bar{w}. \tag{32}$$

Consider the N-structure corresponding to $\operatorname{span}\{\mathrm{d}u, \mathrm{d}w\}$. If $A(u,w)$ is a function complex-analytic in w, smoothly depending on u, then the complex 2-form

$$F = A(u,w)\,\mathrm{d}u \wedge \mathrm{d}w \tag{33}$$

satisfies equations (3)-(5) with $\kappa = \mathrm{d}u$; it describes a *plane-fronted* electromagnetic wave. If A depends on u only, then F is a plane wave.

Consider now the open submanifold M of $\mathbb{R}^4$ defined by $v > 0$ and put, for $m \in \mathbb{R}^+$,

$$\rho = v^2(1 + \tfrac{1}{4}w\bar{w})^{-2}, \quad \mathrm{d}v + \xi = \rho^{-1}(1 - 2mv^{-1})\,\mathrm{d}u + 2\rho^{-1}\,\mathrm{d}v.$$

Then

$$g' = (1 - 2mv^{-1})\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}v + \rho\,\mathrm{d}w\,\mathrm{d}\bar{w}$$

and (M, g') describes the Schwarzschild space-time. The form (33) corresponds now to a wave with spherical fronts; its amplitude decreases as $1/v$ along the null lines of the expanding congruence generated by $k = \partial_v$.

If the CR structure underlying a Robinson space-time (M, g, K) is trivial, then one can choose coordinates so that $\kappa = \mathrm{d}u$ and $\mu = \mathrm{d}w$, as in the last Example. In such a case physicists say that K defines an *sng* congruence *without twist*. There are many Einstein–Robinson space-times of this kind. For example, if the function $f(u,x,y)$ satisfies the Laplace equation, $\partial_x^2 f + \partial_y^2 f = 0$, then the *plane-fronted gravitational wave*,

$$g = f(u,x,y)\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2,$$

has vanishing Ricci tensor, but is not flat unless f is linear in x and y. Its Weyl tensor is of type N. The plane-fronted waves are among Lorentzian analogs of Kähler manifolds of proper Riemannian geometry: their bundle $N \to M$ is invariant with respect to parallel transport.

Problem 2. In dimension $\geqslant 4$, develop a theory of Robinson manifolds analogous to Kähler manifolds.

'Twisting' congruences, characterized by $\mathrm{d}\kappa \wedge \kappa \neq 0$, are more interesting; the Kerr space-time, describing a black hole arising from the collapse of a rotating star, is a Robinson manifold with a twisting congruence.

Example 2. In Minkowski space-time, one of the first twisting shear-free congruences of null lines was described by Robinson around 1963; it played a major role in the emergence of Penrose's twistors [18, 19]. Robinson established that the metric tensor

$$g = (\mathrm{d}u + \mathrm{i}(z\,\mathrm{d}\bar{z} - \bar{z}\,\mathrm{d}z))\,\mathrm{d}v + (v^2 + 1)\,\mathrm{d}z\,\mathrm{d}\bar{z}, \quad z = x + \mathrm{i}y \tag{34}$$

is flat and the *sng* congruence generated by ∂_v is twisting. The complex 2-form $F = A(x,y,u,v)\kappa \wedge (\mathrm{d}x + \mathrm{i}\,\mathrm{d}y)$ is self-dual and Maxwell's equations $\mathrm{d}F = 0$ reduce to $\partial A/\partial v = 0$ and the equation $Z \lrcorner \mathrm{d}A = 0$, where $Z = \partial_x + \mathrm{i}\partial_y - \mathrm{i}(x + \mathrm{i}y)\partial_u$ is an operator on $\mathbb{R}^3$ introduced by Hans Lewy in 1957.

He constructed a smooth function h such that the equation $Z \lrcorner \mathrm{d}A = h$ has no solution, even locally.

The underlying CR geometry on $\mathcal{M} = \mathbb{R}^3$ with coordinates $u, z = x + \mathrm{i}y$ is given by the pair $(\kappa = \mathrm{d}u + \mathrm{i}(z\,\mathrm{d}\bar{z} - \bar{z}\,\mathrm{d}z), \mu = \mathrm{d}x + \mathrm{i}\,\mathrm{d}y)$. Two solutions of (16) are $z_1 = x + \mathrm{i}y$ and $z_2 = u + \frac{1}{2}\mathrm{i}(x^2 + y^2)$ so that equation (17) is now that of the hyperquadric, $\mathrm{i}(\bar{z}_2 - z_2) - |z_1|^2 = 0$. The biholomorphic map

$$w_1 = \sqrt{2}\frac{z_1}{z_2 + \mathrm{i}}, \quad w_2 = \frac{z_2 - \mathrm{i}}{z_2 + \mathrm{i}}$$

transforms the hyperquadric into the 3-sphere of equation

$$|w_1|^2 + |w_2|^2 = 1.$$

This is the most symmetric, non-trivial, 3-dimensional CR geometry: its group of automorphisms is $\mathrm{SU}_{2,1}$. The CR structure on $\mathbb{S}_3$ can be viewed as obtained from the complex structure of $\mathbb{S}_2 = \mathbb{C}\mathrm{P}_1$ via the Hopf map.

Several solutions of Einstein's equations admit this congruence. As an example, we show this for the *Gödel universe* [13]. Take its metric in the form given in [28],

$$(\mathrm{d}X^2 + \mathrm{d}Y^2 - 2(Y\,\mathrm{d}U - \mathrm{d}X)(Y\,\mathrm{d}V - \mathrm{d}X))/Y^2.$$

Its Weyl tensor is of type D: the null vector fields $k = \partial_V$ and $l = \partial_U$ generate each an *sng* congruence. Consider k; the corresponding CR structure on $\mathbb{R}^3$ with coordinates (U, X, Y) is given by $\kappa = \mathrm{d}X - Y\,\mathrm{d}U$ and $\mu = \mathrm{d}X + \mathrm{i}\,\mathrm{d}Y$. Introduce new local coordinates (u, x, y) in $\mathbb{R}^3$ by $u = X, \quad z = x + \mathrm{i}y = \sqrt{Y}\exp(-\frac{1}{2}\mathrm{i}U)$. One then obtains $\kappa = \kappa', \quad \mu = \kappa' + 2\mathrm{i}\bar{z}\mu'$, where

$$\kappa' = \mathrm{d}u + \mathrm{i}(z\,\mathrm{d}\bar{z} - \bar{z}\,\mathrm{d}z), \quad \mu' = \mathrm{d}z.$$

The pair (κ', μ') defines the same CR structure as the pair (κ, μ): it is that of the hyperquadric.

5.4. **The Goldberg–Sachs theorem.** Consider a 4-manifold (M, g) that is either proper Riemannian or Lorentzian. An N-structure on M can be (locally) given by a field φ of chiral spinors: one uses 'point by point' the definition (9).

Theorem 3. (i) *If the N-structure $N(\varphi)$ is integrable, then the chiral spinor φ is an eigenspinor of the Weyl tensor.*
(ii) *If (M, g) is conformal to an Einstein manifold, then $N(\varphi)$ is integrable if, and only if, the chiral spinor field φ is a repeated eigenspinor of the Weyl tensor.*

For space-times, the theorem was established by Goldberg and Sachs [6]. Its extension to the proper Riemannian case is due to Plebański, Hacyan, Przanowski and Broda [24, 25].

Problem 3. Find a generalization of the Goldberg–Sachs theorem to manifolds of dimension > 4.

In the Lorentzian case, it follows from Theorem 3 and the algebraic classification of Weyl tensors that a space-time which is conformally Einstein, but not conformally flat, can have at most 2 distinct *sng* congruences (type D). The following example shows that there are non-conformally flat space-times admitting 3 such distinct congruences; we do not know whether there are space-times with $\mathsf{C} \neq 0$ and 4 distinct congruences of this type.

Example 3. Consider a space-time $M = \mathbb{R}^4$ with the real coordinates u, v and a complex coordinate w. Let the metric tensor be $g = \lambda\kappa + \mu\bar{\mu}$, where

$$\kappa = \mathrm{d}u + \tfrac{1}{2}\mathrm{i}(w\,\mathrm{d}\bar{w} - \bar{w}\,\mathrm{d}w), \quad \lambda = \mathrm{d}v - \tfrac{1}{2}\mathrm{i}(w\,\mathrm{d}\bar{w} - \bar{w}\,\mathrm{d}w), \quad \mu = (w + \bar{w})\,\mathrm{d}w.$$

This space-time admits three congruences of shear-free null geodesics : those generated by the vector fields $k_1 = \partial_u$ and $k_2 = \partial_v$ are twisting and are both equivalent to the Robinson congruence. The congruence generated by

$$k_3 = \partial_v - \partial_u + 2\mathrm{i}(w + \bar{w})^{-1}(\partial_{\bar{w}} - \partial_w)$$

is *sng* and has vanishing twist. The space-time (M, g) has a Weyl tensor of type I and does not admit any other *sng* congruences.

5.5. **Remarks on the embeddability problem.** The property of a CR space $\mathcal{M}$ to be embeddable is relevant to the local existence of a non-zero, null solution of Maxwell's equations on space-times obtained as lifts of $\mathcal{M}$. If $\mathcal{M}$ is embeddable, if the forms κ and μ are as in (18), and g is given by (30), then $F = A(z_1, z_2)\kappa \wedge \mu$ satisfies (3)-(5) for every function A holomorphic in its two arguments. In fact, less is required for the local existence of such an F: if the canonical bundle of $\mathcal{M}$ admits a locally defined closed section ω, then its pull-back to M can be taken as F.

It is now known that there are CR spaces that are non-embeddable, but have *one* solution of (16) [29]; by the results of [31], extended to higher dimensions in [9], such CR spaces do not admit closed, non-zero sections of their canonical bundle. Therefore, space-times constructed as lifts of these CR spaces do not admit any associated non-zero null solutions of Maxwell's equations. There are examples of non-embeddable 7-dimensional CR manifolds that have non-zero, closed, sections of their canonical bundle, but it is not clear whether there are such examples in dimensions 3 and 5. Further remarks on this subject are in [33].

Lewandowski, Nurowski and Tafel [16] established the following

Theorem 4. *If the CR space $\mathcal{M}$ lifts to an Einstein–Robinson space-time, then $\mathcal{M}$ is locally embeddable.*

6. THE KERR THEOREM

The Kerr theorem provides a method for constructing all integrable analytic N-structures in Minkowski space-time (M, g); even though it is well-known, we present it here because of its importance. See [20, 21, 31] for further details and references. Consider the coordinate system and metric (32) as given in Example 1. The manifold of all *mtn* subspaces of one chirality of the complexified Minkowski space $\mathbb{C}^4$ is $\mathrm{SO}_4 / \mathrm{U}_2 = \mathbb{C}\mathrm{P}_1$.

Let $z \in \mathbb{C}$ and define

$$k_z = \partial_v - z\partial_w - \bar{z}\partial_{\bar{w}} - z\bar{z}\partial_u, \tag{35a}$$

$$\kappa_z = \mathrm{d}u - z\,\mathrm{d}\bar{w} - \bar{z}\,\mathrm{d}w - z\bar{z}\,\mathrm{d}v, \tag{35b}$$

$$\mu_z = \mathrm{d}w + z\,\mathrm{d}v, \quad \text{and} \quad \lambda_z = \mathrm{d}v. \tag{35c}$$

The map $(\kappa_0, \mu_0, \lambda_0) \mapsto (\kappa_z, \mu_z, \lambda_z)$ is a proper Lorentz transformation. It is induced by the homomorphisms $\mathbb{C} \to \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{3,1}$. The pair (κ_z, μ_z) defines an *mtn* subspace N_z such that $\mathrm{Re}(N_z \cap \bar{N}_z) = \mathrm{dir}\, k_z$. The subspace corresponding to the 'point at infinity' of $\mathbb{C}\mathrm{P}_1 = \mathbb{C} \cup \{\infty\}$ is defined by the pair $(\mathrm{d}v, \mathrm{d}\bar{w})$ and $k_\infty = \partial_u$. Assume now z to be a complex *function* on M such that its real and imaginary parts are real-analytic functions of the coordinates u, v, $\mathrm{Re}\, w$ and $\mathrm{Im}\, w$. At every point p of M the pair $(\kappa_{z(p)}, \mu_{z(p)})$ defines an *mtn* subspace of $\mathbb{C} \otimes T_pM$. According to (15), the N-structure defined by (κ_z, μ_z) is integrable if, and only if,

$$\mathrm{d}\kappa_z \wedge \kappa_z \wedge \mu_z = 0 \quad \text{and} \quad \mathrm{d}\mu_z \wedge \kappa_z \wedge \mu_z = 0. \tag{36}$$

A simple calculation shows that equations (36) reduce to

$$\mathrm{d}v \wedge \mathrm{d}z \wedge \mathrm{d}(u - z\bar{w}) \wedge \mathrm{d}(w + zv) = 0,$$
$$\mathrm{d}\bar{w} \wedge \mathrm{d}z \wedge \mathrm{d}(u - z\bar{w}) \wedge \mathrm{d}(w + zv) = 0,$$

and are thus equivalent to

$$\mathrm{d}(u - z\bar{w}) \wedge \mathrm{d}(w + zv) \wedge \mathrm{d}z = 0. \tag{37}$$

By the implicit function theorem, equation (37) implies, locally, the existence of a holomorphic function $H(z_1, z_2, z_3)$ of three complex variables such that

$$H(u - z\bar{w}, w + zv, z) = 0. \tag{38}$$

This proves a theorem attributed to Kerr:

Theorem 5. *Locally, every integrable analytic N-structure in Minkowski space-time $\mathbb{R}^4$ is given either by the pair $(\mathrm{d}v, \mathrm{d}\bar{w})$ or by (35), where $z : \mathbb{R}^4 \to \mathbb{C}$ is a solution of (38) and H is a holomorphic function of three complex variables such that $\mathrm{d}H \neq 0$.*

Denoting $H_1 = \partial H / \partial z_1$, etc., one obtains by differentiation of (38)

$$H_1 \kappa_z + (H_2 + \bar{z}H_1)\mu_z + (H_3 - \bar{w}H_1 + vH_2)\,\mathrm{d}z = 0.$$

The condition $\mathrm{d}H \neq 0$ implies $H_3 - \bar{w}H_1 + vH_2 \neq 0$. If $H_1 = H_2 = 0$, then $z = \text{const.}$ and the N-structure is trivial, i.e. reducible, by a Lorentz transformation of the coordinates, to $\kappa_0 = \mathrm{d}u$ and $\mu_0 = \mathrm{d}w$. Define

$$u_z = u - z\bar{w} - \bar{z}w - z\bar{z}v \quad \text{and} \quad w_z = w + zv. \tag{39}$$

Since

$$L(k_z)u_z = 0 \quad \text{and} \quad L(k_z)w_z = 0, \tag{40}$$

the functions u_z and w_z descend to the CR manifold $\mathcal{M}$ obtained from M as described in Theorem 1. Moreover, the pair $(\kappa_z, \mathrm{d}w_z)$ defines the same N-structure on M as the pair (κ_z, μ_z). The pair $(\kappa_z, \mathrm{d}w_z)$ defines the CR structure on $\mathcal{M}$.

Assume now that H_1 and/or $H_2 \neq 0$. Equation (38) can be written as

$$H(u_z + \bar{z}w_z, w_z, z) = 0$$

and shows that w_z is a function of $z, \bar{z}$ and u_z only. The integrability condition $\mathrm{d}\kappa_z \wedge \kappa_z \wedge \mu_z = 0$ is now satisfied identically and $\mathrm{d}\mu_z \wedge \kappa_z \wedge \mu_z = 0$ is equivalent to

$$\frac{\partial w_z}{\partial \bar{z}} - w_z \frac{\partial w_z}{\partial u_z} = 0. \tag{41}$$

Using (41) one obtains

$$\mathrm{d}w_z = \frac{\partial w_z}{\partial u_z}\kappa_z + \left(\frac{\partial w_z}{\partial z} - \bar{w}_z \frac{\partial w_z}{\partial u_z}\right) \mathrm{d}z.$$

This shows that the pair $(\kappa_z, \mathrm{d}z)$ defines on $\mathcal{M}$ the same CR structure as the pair $(\kappa_z, \mathrm{d}w_z)$. Let $(\partial_{u_z}, \bar{Z}, Z)$ be the frame on $\mathcal{M}$ dual to the coframe $(\kappa_z, \mathrm{d}z, \mathrm{d}\bar{z})$ so that

$$Z = \frac{\partial}{\partial \bar{z}} - w_z \frac{\partial}{\partial u_z}.$$

Equation (41) is now interpreted as a tangential Cauchy–Riemann equation, $Z \lrcorner \mathrm{d}w_z = 0$.

The map $(u, v, w) \mapsto (u_z, v, z)$ is a local diffeomorphism. This is seen by computing the volume form on M,

$$\mathrm{i}\, \mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}w \wedge \mathrm{d}\bar{w} = \mathrm{i}|\bar{Z} \lrcorner \mathrm{d}w_z - v|^2\, \mathrm{d}u_z \wedge \mathrm{d}v \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z},$$

where use has been made of (41). The distribution $\ker \kappa_z$ is integrable if, and only if, $\bar{Z} \lrcorner \mathrm{d}w_z$ is real. Dropping the subscripts z, one has

Corollary 2. *Let (u, v, z) be a local coordinate system on M, let $w(u, z, \bar{z})$ be a smooth, complex-valued function satisfying*

$$\partial_{\bar{z}} w - w \partial_u w = 0$$

and put $\kappa = \mathrm{d}u + \bar{w}\, \mathrm{d}z + w\, \mathrm{d}\bar{z}$, $\mu = \mathrm{d}w - v\, \mathrm{d}z$. The metric

$$g = \kappa\, \mathrm{d}v + \mu\bar{\mu} \tag{42}$$

is flat and the vector field $k = \partial_v$ generates an expanding $(\operatorname{div} k \neq 0)$ sng *congruence.*

Example 4. If $w = \mathrm{i}z$, then (42) assumes the form (34) and corresponds to the Robinson congruence of Example 2.

7. Twistor bundles

Recall a general idea in geometry: if one wishes to study a structure, but there is no distinguished structure, then it is appropriate to consider the set of all such structures.

Given an oriented Riemannian $2n$-manifold (M, g) (conformal geometry suffices), define its *twistor bundles* $P_\pm$ to have, as the total sets, the collections of all *mtn* subspaces of $\mathbb{C} \otimes TM$ of the $\pm$ chiralities. These are bundles with fiber $\mathrm{SO}_{2n} / \mathrm{U}_n$, which has a canonical metric and complex structure. If $\star^2 = -\,\mathrm{id}$, then complex conjugation in $\mathbb{C} \otimes TM$ changes the chirality of the *mtn* subspaces; this induces an isomorphism of the bundles P_+ and P_-. They are then identified and denoted by P: such is the case when (M, g) is a space-time. The Levi-Civita connection on M induces a horizontal distribution on $P_\pm$; together with the canonical metric on the fibers, this defines a metric and a canonical N-structure on $P_\pm$, which need not be integrable. If (M, g) is proper Riemannian (resp., Lorentzian), then so is $P_\pm$ and its canonical N-structure defines on $P_\pm$ the structure of an almost Hermite (resp., almost Robinson) manifold.

Theorem 6. *If M is a space-time, then the integrability of the canonical N-structure on its twistor bundle P is equivalent to $\mathsf{C} = 0$. If M is a 4-dimensional proper Riemannian manifold, then the canonical N-structure on $P_\pm$ is integrable if, and only if, $\mathsf{C}_\pm = 0$.*

In the Lorentzian case, the theorem was established by Penrose in the course of work that led to his fundamental twistor programme; see [21] and the references given there. The proof in the proper Riemannian case is due to Atiyah, Hitchin and Singer [2].

7.1. **The Kerr theorem revisited.** Let $(M = \mathbb{R}^4, g)$ be the Minkowski space-time. According to Theorem 6, its twistor bundle P is a Robinson manifold so that there is the associated 5-dimensional CR manifold $\mathcal{P}$. The twistor bundle P is identified with the set of null directions in the tangent spaces at all points of M. Its typical fiber is the 'celestial sphere' $\mathbb{S}_2 \approx \mathbb{CP}_1$ so that $P = M \times \mathbb{CP}_1$. Locally, the bundle $P \to M$ can be conveniently described as follows. Let (u, v, w) be a coordinate system on M, as in (32). A number $z \in \mathbb{C}$ defines a null direction $\operatorname{dir} k_z$ at (u, v, w), parallel to the vector k_z given in (35). A point of P is given by the sequence $(u, v, w, \operatorname{dir} k_z)$ or, equivalently, by the sequence (u, v, w, z), i.e. by a sequence of 6 real functions; they provide a convenient coordinate system on P. In these coordinates, the metric tensor on P is $\mathrm{d}u\,\mathrm{d}v + \mathrm{d}w\,\mathrm{d}\bar{w} + (1 + \frac{1}{4} z\bar{z})^{-2}\,\mathrm{d}z\,\mathrm{d}\bar{z}$. The canonical N-structure on P is given by $N_P^0 = \operatorname{span}\{\kappa_z, \mu_z, \mathrm{d}z\}$. Its integrability is easily checked by computing $\omega_z = \kappa_z \wedge \mu_z \wedge \mathrm{d}z$ and verifying that equations (15) are satisfied. The line bundle $N_P \cap \bar{N}_P \to P$ is spanned by $\operatorname{dir} k_z$.

Consider now the CR manifold $\mathcal{P}$ associated with P as in Theorem 1 and the functions defined in (39). In view of (40) and $L(k_z)z = 0$, the sequence

(u_z, w_z, z) of 5 real functions descends to $\mathcal{P}$ and provides a coordinate system on that manifold. Its CR structure is embeddable: three solutions of (16) are $z_1 = u - z\bar{w}$, $z_2 = w + zv$ and $z_3 = z$. Consider a regular congruence K of null lines on M *which need not be shear-free*. The set $\mathcal{M}$ of these lines is a 3-dimensional manifold. There is the map $f : \mathcal{M} \to \mathcal{P}$ that sends an element of the congruence on M to its lift to $\mathcal{P}$,

$$\begin{array}{ccc} P & \xrightarrow{\text{can}} & \mathcal{P} \\ {\scriptstyle\text{can}}\downarrow & & \uparrow{\scriptstyle f} \\ M & \xrightarrow[\pi]{} & \mathcal{M} \end{array}$$

Theorem 7. *The congruence K of null lines on Minkowski space-time is shear-free if, and only if, the map $f : \mathcal{M} \to \mathcal{P}$ defines on $\mathcal{M}$ the structure of a CR submanifold of $\mathcal{P}$.*

Proof. Let $z : M \to \mathbb{C}$ be the function defining the congruence K of null lines. The map $f \circ \pi : M \to \mathcal{P}$ sends (u, v, w) to (u_z, w_z, z) with z evaluated at (u, v, w). A section of the canonical bundle of the CR manifold $\mathcal{P}$ is $\omega = \mathrm{d}(u - z\bar{w}) \wedge \mathrm{d}w_z \wedge \mathrm{d}z$. According to (37), the pull-back $(f \circ \pi)^*\omega$ vanishes if, and only if, the null geodetic congruence K is shear-free. Since π is a surjective submersion, this holds only whenever (19) is satisfied. □

The image of $\mathcal{P}$ in $\mathbb{C}^3$ is the hypersurface ('generalized hyperquadric') of equation

$$z_3 - \bar{z}_3 + z_1\bar{z}_2 - \bar{z}_1 z_2 = 0. \tag{43}$$

Every point of this hypersurface corresponds to a null line $l : \mathbb{R} \to M$ given, in the coordinate system (u, v, w) on M, by

$$l(t) = \left(\tfrac{1}{2}(z_3 + \bar{z}_3 + z_1\bar{z}_2 + \bar{z}_1 z_2) - z_1\bar{z}_1 t,\ t,\ z_2 - z_1 t\right)$$

so that $l(v) = (u, v, w)$ and $\mathrm{d}l/\,\mathrm{d}t = k_z$. All null lines in M, except those parallel to ∂_u, can be obtained by this 'Penrose correspondence' between M and $\mathcal{P}$. Consider now the embedding

$$f : \mathbb{C}^3 \to \mathbb{C}\mathrm{P}_3, \quad f(z_1, z_2, z_3) = \mathrm{dir}(1 + \mathrm{i}z_3, z_1 - \mathrm{i}z_2, 1 - \mathrm{i}z_3, z_1 + \mathrm{i}z_2).$$

The image of $\mathbb{C}^3$ by f is $\mathbb{C}\mathrm{P}_3$ with a $\mathbb{C}\mathrm{P}_2$ removed. The image of the hypersurface (43) by f is an open and dense submanifold of the manifold $\mathcal{P}_0$ of *null twistor* directions

$$\{\mathrm{dir}(w_1, w_2, w_3, w_4) \in \mathbb{C}\mathrm{P}_3 \mid |w_1|^2 + |w_2|^2 - |w_3|^2 - |w_4|^2 = 0\}. \tag{44}$$

Penrose [20] proved the following fundamental

Theorem 8. *If $M = (\mathbb{S}_1 \times \mathbb{S}_3)/\mathbb{Z}_2$ is the conformally compactified Minkowski space-time, then $P = \mathbb{C}\mathrm{P}_3$. Every analytic CR 3-manifold, defining a Robinson structure in M, is obtained as the intersection of the 5-dimensional CR manifold of projective null twistors (44) with a complex analytic 2-dimensional submanifold of $\mathbb{C}\mathrm{P}_3$.*

According to Penrose, a non-analytic, shear-free and twisting congruence of null geodesics in (compactified) Minkowski space-time can be described as corresponding to a complex surface Σ in $\mathbb{CP}_3$ that 'touches only one side' of the manifold of projective null twistors $\mathcal{P}_0$ so that the real dimension of $\mathcal{P}_0 \cap \Sigma$ is 3, but the surface cannot be holomorphically extended to the other side of $\mathcal{P}_0$, see pp. 220–222 in [21].

7.2. **The Kerr theorem in the proper Riemannian setting.** There is an analog of the Kerr theorem for proper Riemannian self-dual (or anti-self-dual) 4-manifolds. We only sketch the idea of the theorem in the *local* setting. According to Theorem 6, the twistor bundle P_+ of such a self-dual manifold has a canonical integrable N-structure defining there the structure of a complex 3-manifold so that there is the fibration $\mathbb{CP}_1 \to P_+ \xrightarrow{\pi} M$. Let U be an open subset of M and $s : U \to P_+$ a local section of π such that $s(U)$ is a complex submanifold of P_+. The restriction of π to $s(U)$ induces on U the structure of a Hermite manifold and all local Hermite structures on M can be so obtained. The insistence on locality is essential: for example, the 4-sphere has no global complex structure, but it has local Hermite structures.

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INSTYTUT FIZYKI TEORETYCZNEJ, HOŻA 69, 00681 WARSZAWA, POLAND