IRREDUCIBLE SO(3) GEOMETRY IN DIMENSION FIVE

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Abstract. We consider the nonstandard inclusion of $\mathbf{SO}(3)$ in $\mathbf{SO}(5)$ associated with a 5-dimensional irreducible representation. The tensor Υ representing this reduction is found to be given by a ternary symmetric form with special properties. A 5-dimensional manifold (M, g, Υ) with Riemannian metric g and ternary form generated by such a tensor has a corresponding $\mathbf{SO}(3)$ structure, whose Gray-Hervella type classification is established using $\mathfrak{so}(3)$-valued connections with torsion.

Structures with antisymmetric torsions, we call them the nearly integrable $\mathbf{SO}(3)$ structures, are studied in detail. In particular, it is shown that the integrable models (those with vanishing torsion) are isometric to the symmetric spaces $M_+ = \mathbf{SU}(3)/\mathbf{SO}(3)$, $M_- = \mathbf{SL}(3,R)/\mathbf{SO}(3)$, $M_0 = \mathbb{R}^5$. We also find all nearly integrable $\mathbf{SO}(3)$ structures with transitive symmetry groups of dimension $d > 5$ and some examples for which $d = 5$.

Given an $\mathbf{SO}(3)$ structure (M, g, Υ), we define its "twistor space" $\mathbb{T}$ to be the $\mathbb{S}^2$-bundle of those unit 2-forms on M which span $\mathbb{R}^3 = \mathfrak{so}(3)$. The 7-dimensional twistor manifold $\mathbb{T}$ is then naturally equipped with several CR and G_2 structures. The ensuing integrability conditions are discussed and interpreted in terms of the Gray-Hervella type classification.

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Contents

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1. Introduction

In Cartan's list of the irreducible symmetric spaces of Type I the first entry is occupied by the family of symmetric spaces $\mathbf{SU}(n)/\mathbf{SO}(n)$. If $n=2$ the corresponding manifold is a 2-dimensional sphere $\mathbb{S}^2$, but $n=3$ already corresponds to a nontrivial manifold $M_+ = \mathbf{SU}(3)/\mathbf{SO}(3)$. This is the so called Wu space [12, 14] which has a number of interesting properties. Among them there is a fact that M_+ constitutes the lowest dimensional example of a simply connected manifold *not* admitting a $\mathbf{Spin}^c$ structure [6]. From the point of view of the present paper another property of this space is crucial: the isotropy representation of $M_+ = \mathbf{SU}(3)/\mathbf{SO}(3)$ coincides with the irreducible 5-dimensional representation of $\mathbf{SO}(3)$. Thus, this space provides a symmetric model of a 5-dimensional manifold equipped with the irreducible $\mathbf{SO}(3)$ structure. Inspecting the entire Cartan list of the irreducible symmetric spaces one finds (in Type III, again at the first entry!) another 5-dimensional space $M_- = \mathbf{SL}(3,\mathbb{R})/\mathbf{SO}(3)$ equipped with the natural irreducible $\mathbf{SO}(3)$ structure.

The aim of this paper is to study 5-dimensional geometries modelled on the spaces M_+ and M_-. By this we mean studies of 5-dimensional manifolds with the reduction of the structure group of the $\mathbf{SO}(5)$-frame bundle to the irreducible $\mathbf{SO}(3)$. This places the paper in the domain of *special geometries*, i.e. Riemannian geometries equipped with additional geometric structures. In Ref. [3] Th. Friedrich provides a general framework for analysing such geometries. He also proposes the investigation of geometries modelled on M_+ there.

The framework for analysis of special geometries consists of several steps. First, one distinguishes a geometric object, preferably of tensorial type, that reduces the structure group and defining the special geometry. Then, one introduces a metric connection which preserves this object. As the last step one determines the restrictions on the special geometry for this connection to be unique. This unique connection, its torsion and curvature are then the main tools to study the properties of the considered special geometry.

It is instructive to illustrate this procedure on the well known example of a *nearly* Kähler geometry. Our choice of nearly Kähler geometry for this illustration is motivated by the fact that its behaviour is remarkably close [7] to all the phenomena we want to discuss in the context of the irreducible $\mathbf{SO}(3)$ geometries in dimension five.

A Riemannian geometry (M,g) on a $2n$-dimensional manifold M can be made more special by an introduction of a metric compatible almost complex structure. This is a tensor field $J: TM \to TM$ which satisfies $J^2 = -\mathrm{id}$ and $g(JX,JY) = g(X,Y)$. The tensor J reduces the structure group from $\mathbf{SO}(2n)$ to $\mathbf{U}(n)$ and induces the distinguished inclusion of the Lie algebra $\mathfrak{u}(n)$ in $\mathfrak{so}(2n)$. This inclusion defines a class of a metric compatible connections Γ which preserve J. Here and in the following we will represent connections by means of Lie-algebra-valued 1-forms on manifolds so, in the considered case, $\Gamma \in \mathfrak{u}(n) \otimes \Omega^1(M)$, where $\mathfrak{u}(n) \subset \mathfrak{so}(2n)$. The connections Γ are highly not unique. However, since all of them may be considered as elements of $\mathfrak{so}(2n) \otimes \Omega^1(M)$, i.e. as elements of the space in which the Levi-Civita connection $\overset{LC}{\Gamma}$ resides, one can try to make Γ unique by the requirement that in the decomposition

$$\overset{LC}{\Gamma} = \Gamma + \tfrac{1}{2}T \tag{1.1}$$

the T-part has some special properties. In the considered case the uniqueness of Γ is achieved by the demand that in the above decomposition

$$T \in \Omega^3(M). \tag{1.2}$$

The 3-form T is then interpreted as a skew-symmetric torsion of the connection Γ. It follows that the decomposition (1.1)-(1.2) is possible only for a subclass of metric compatible almost complex structures. They may be characterised by the condition

$$(\overset{LC}{\nabla}_v J)(v) = 0 \qquad \forall v \in TM.$$

The metric compatible almost complex structures satisfying this condition are called *nearly* Kähler. Their geometric properties are described in terms of the properties of the unique $\mathfrak{u}(n)$-valued connection Γ defined by (1.1)-(1.2). In particular, the torsion-free case, $T \equiv 0$, corresponds to Kähler geometries. Another types of the nearly Kähler structures may be distinguished by specifying that the curvature of Γ belongs to a particular $\mathbf{U}(n)$-irreducible component of the tensor representation $\mathfrak{u}(n) \otimes \Omega^2(M)$.

Our treatment of the irreducible $\mathbf{SO}(3)$ geometries in dimension five imitates the above approach to the nearly-Kähler geometries. We first introduce an object, the (3,0)-rank tensor Υ, which reduces the $\mathbf{SO}(5)$ structure to the irreducible $\mathbf{SO}(3)$. Although this tensor has a different rank then J its geometric characterisation, which is a certain algebraic quadratic identity on Υ, resembles very much the quadratic condition $J^2 = -\,\mathrm{id}$. Using Υ we distinguish an inclusion of $\mathfrak{so}(3)$ in $\mathfrak{so}(5)$. This maximal inclusion is used on a Riemannian manifold endowed with Υ to distinguish a class of $\mathfrak{so}(3)$-valued metric connections Γ. These are such that, in the decomposition (1.1), they have the skew-symmetric T-part. It follows that such connections, if exist, are unique. Their existence is only possible for a particular class of tensors Υ characterised by the condition

$$(\overset{LC}{\nabla}_v \Upsilon)(v, v, v) = 0 \qquad \forall v \in TM.$$

The organisation of the paper is reflected in the table of contents. The notation is standard. However, depending on the context and esthetics of the presentation, we use both the Schouten notation with the indices of tensors as well as the geometric, index-free notation. Since all the time we are in the Riemannian category, we do not distinguish between covariant and contravariant tensors. This convention, when used in the formulae written in the Schouten notation, enables as to identify tensors with upper and lower indices. We will write them in the both positions depending on convenience. In the entire text the Einstein summation convention is assumed.

2. Tensor Υ reducing $\mathbf{O}(5)$ to the irreducible $\mathbf{SO}(3)$

The two obvious examples $M_+ = \mathbf{SU}(3)/\mathbf{SO}(3)$ and $M_- = \mathbf{SL}(3,\mathbb{R})/\mathbf{SO}(3)$ of the irreducible $\mathbf{SO}(3)$ structures should be supplemented by still another one, which in a certain sense, is the simplest. One achieves this example by identifying vectors A in $\mathbb{R}^5$ with 3×3 symmetric traceless real matrices $\sigma(A)$,

$$\mathbb{M}^5 = \{\ \sigma(A) \in M_{3\times 3}(\mathbb{R}):\ \sigma(A)^{\mathrm{T}} = \sigma(A),\ \mathrm{tr}(\sigma(A)) = 0\ \}, \tag{2.1}$$

and defining the unique irreducible 5-dimensional representation ρ of $\mathbf{SO}(3)$ in $\mathbb{R}^5$ by

$$\rho(h)A \ = \ h\ \sigma(A)\ h^{\mathrm{T}}, \qquad \forall\ h \in \mathbf{SO}(3),\ A \in \mathbb{R}^5. \tag{2.2}$$

Then $M_0 = (\mathbf{SO}(3)\times_\rho \mathbb{R}^5)/\mathbf{SO}(3)$ also has an irreducible $\mathbf{SO}(3)$ structure.

From now on we identify $\mathbb{R}^5$ with matrices $\mathbb{M}^5$ as in (2.1). Given an element $A\in\mathbb{R}^5$ we consider its characteristic polynomial

$$P_A(\lambda) = \det(\sigma(A) - \lambda I) = -\lambda^3 + g(A,A)\lambda + \frac{2\sqrt{3}}{9}\Upsilon(A,A,A).$$

This polynomial is invariant under the $\mathbf{SO}(3)$-action given by the representation ρ of (2.2),

$$P_{\rho(h)A}(\lambda) = P_A(\lambda).$$

Thus, all the coefficients of $P_A(\lambda)$, which are multilinear in A, are $\mathbf{SO}(3)$-invariant. It is convenient to choose a basis $\mathbf{e}_i$ in $\mathbb{R}^5$ in such a way that the identification σ is given by

$$\mathbb{R}^5 \ni A = a_i\mathbf{e}_i \longmapsto \sigma(A) = \begin{pmatrix} \frac{a^1}{\sqrt{3}} - a^4 & a^2 & a^3 \\ a^2 & \frac{a^1}{\sqrt{3}} + a^4 & a^5 \\ a^3 & a^5 & -2\frac{a^1}{\sqrt{3}} \end{pmatrix} \in \mathbb{M}^5. \tag{2.3}$$

After this convenient choice, the bilinear form g simply becomes

$$g(A,A) = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2, \tag{2.4}$$

and the ternary one Υ is given by

$$\Upsilon(A,A,A) = \frac{1}{2}a_1\big(\ 6a_2^2+6a_4^2-2a_1^2-3a_3^2-3a_5^2\ \big)+\frac{3\sqrt{3}}{2}a_4(a_5^2-a_3^2)+3\sqrt{3}a_2a_3a_5. \tag{2.5}$$

Both g and Υ are obviously $\mathbf{SO}(3)$-invariant. Since g is the usual Riemannian metric on $\mathbb{R}^5$ the action ρ of (2.2) gives a nonstandard *irreducible* inclusion

$$\iota : \mathbf{SO}(3) \hookrightarrow \mathbf{O}(5). \tag{2.6}$$

Remark 2.1. Although it is obvious we remark that

$$\Upsilon(A,A,A) = \frac{3\sqrt{3}}{2}\det(\sigma(A)).$$

In the following we consider a tensor $\Upsilon_{ijk} \in \bigodot^3 \mathbb{R}^5$ such that

$$\Upsilon(A,A,A) = \Upsilon_{ijk}a^ia^ja^k.$$

A simple algebra leads to the following proposition.

Proposition 2.2. *The tensor Υ_{ijk} has the following properties*

i) *it is totally symmetric, $\Upsilon_{ijk} = \Upsilon_{(ijk)}$,*

ii) *it is trace-free,* $\Upsilon_{ijj}=0$,

iii) *it satisfies the following identity*

$$\Upsilon_{jki}\Upsilon_{lni}+\Upsilon_{lji}\Upsilon_{kni}+\Upsilon_{kli}\Upsilon_{jni}=g_{jk}g_{ln}+g_{lj}g_{kn}+g_{kl}g_{jn},$$

where $g(A,A)=g_{ij}a^ia^j$.

Remark 2.3. It is worth noting that property iii) after contraction with g_{kn} and Υ_{mkn}, respectively, implies

$$\begin{aligned} 4\Upsilon_{ijk}\Upsilon_{mjk} &= 14g_{im},\\ 4\Upsilon_{ilm}\Upsilon_{jln}\Upsilon_{kmn} &= -3\Upsilon_{ijk}. \end{aligned}$$

Group $\mathbf{O}(5)$ naturally acts on $\bigodot^3\mathbb{R}^5$ by

$$\Upsilon_{ijk}\mapsto H^l{}_iH^m_{\ j}H^n{}_k\Upsilon_{lmn},\qquad H\in\mathbf{O}(5).$$

Our aim now is to find the stabiliser G_Υ of tensor Υ_{ijk} under this action. We know that $\mathbf{SO}(3)\subset G_\Upsilon$. In the following we show that it is actually equal to $\mathbf{SO}(3)$. To see this we take a 1-parameter subgroup $H(s)=\mathrm{e}^{sX}$ of $\mathbf{SO}(5)$ generated by an element X of the Lie algebra $\mathfrak{so}(5)$ in the standard 5-dimensional representation of skew symmetric matrices. Taking $\frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}$ of the stabilising equation $\Upsilon_{ijk}=H(s)^l{}_iH(s)^m_{\ j}H(s)^n{}_k\Upsilon_{lmn}$ we get the following linear equation

$$\Upsilon_{ljk}X^l{}_i+\Upsilon_{ilk}X^l{}_j+\Upsilon_{ijl}X^l{}_k=0 \tag{2.7}$$

for the elements of the Lie algebra of the stabiliser. Its general solution is

$$X=(X^i{}_j)=x^1E_1+x^2E_2+x^3E_3=x^IE_I,$$

where (x^I), $I=1,2,3$, are real parameters and the matrices

$$E_1=\begin{pmatrix}0&0&0&0&\sqrt{3}\\0&0&1&0&0\\0&-1&0&0&0\\0&0&0&0&1\\-\sqrt{3}&0&0&-1&0\end{pmatrix},\qquad E_2=\begin{pmatrix}0&0&\sqrt{3}&0&0\\0&0&0&0&1\\-\sqrt{3}&0&0&1&0\\0&0&-1&0&0\\0&-1&0&0&0\end{pmatrix},\qquad E_3=\begin{pmatrix}0&0&0&0&0\\0&0&0&2&0\\0&0&0&0&1\\0&-2&0&0&0\\0&0&-1&0&0\end{pmatrix}, \tag{2.8}$$

satisfy the $\mathfrak{so}(3)$ commutation relations

$$[E_1,E_2]=E_3,\qquad [E_3,E_1]=E_2,\qquad [E_2,E_3]=E_1,$$

or $[E_J,E_K]=\epsilon^I_{JK}E_I$, for short. Thus, the intersection of the stabiliser with the $\mathbf{SO}(5)$ component of $\mathbf{O}(5)$ is equal to the irreducible $\mathbf{SO}(3)$. Actually the stabiliser does not intersect with the complement of $\mathbf{SO}(5)$ in $\mathbf{O}(5)$, as it is explained in the following lemma.

Lemma 2.4. *The stabiliser of* Υ_{ijk} *is contained in* $\mathbf{SO}(5)$ *component of* $\mathbf{O}(5)$.

Proof. Since the complement of $\mathbf{SO}(5)$ in $\mathbf{O}(5)$ consists of elements of the form $-g$ such that $g\in\mathbf{SO}(5)$ it is enough to prove that $-g$ with $g\in\mathbf{SO}(5)$ can not be in G_Υ. Assuming the opposite i.e. that $g\in\mathbf{SO}(5)$ and $-g\in G_\Upsilon$ we get the contradiction by the following steps. The adjoint map Ad_g preserves $\mathfrak{so}(3)$. Thus it provides an orthogonal (with respect to the Killing form) transformation of $\mathfrak{so}(3)$

$$\mathrm{Ad}_g\,|_{\mathfrak{so}(3)}\in\mathbf{SO}(\mathfrak{so}(3)),\qquad \mathfrak{so}(3)=\mathrm{Span}(E_1,E_2,E_3).$$

On the other hand, any orthogonal transformation of our $\mathfrak{so}(3)$ has the form Ad_h for an element $h\in\iota(\mathbf{SO}(3))$. So, g has its corresponding $h\in\iota(\mathbf{SO}(3))$ such that,

$\mathrm{Ad}_g|_{\mathfrak{so}(3)} = \mathrm{Ad}_h|_{\mathfrak{so}(3)}$. Thus, $\mathrm{Ad}_{gh^{-1}}|_{\mathfrak{so}(3)} = \mathrm{Id}$, so that the element $gh^{-1} \in \mathbf{SO}(5)$ must satisfy

$$gh^{-1}X = Xgh^{-1}, \quad \forall X \in \mathrm{Span}(E_1, E_2, E_3).$$

Forcing gh^{-1} to satisfy this condition on the basis E_J for $J = 1, 2, 3$, we find that $gh^{-1} = I$. Thus $g = h$ is in G_Υ which means that also $-gg^{-1} = -I$ is in G_Υ. But $-I \in \mathbf{O}(5)$ sends Υ_{ijk} to $-\Upsilon_{ijk}$, which gives the contradiction and finishes the proof.

□

Thus we have the following proposition.

Proposition 2.5.
The stabiliser of tensor Υ_{ijk} is the irreducible $\mathbf{SO}(3)$ included by ι in $\mathbf{O}(5)$.

2.1. **The $\mathbf{O}(5)$ invariant characterisation of tensor Υ.** Since the stabiliser of Υ_{ijk} is the irreducible $\mathbf{SO}(3)$, its orbit under the $\mathbf{O}(5)$ action is a 7-dimensional homogeneous space $\mathbf{O}(5)/\iota(\mathbf{SO}(3))$. In this section we fully characterise this orbit among all the orbits of $\mathbf{O}(5)$ action in $\bigodot^3\mathbb{R}^5$. On doing this we view Υ_{ijk} as a linear map

$$\mathbb{R}^5 \ni v \mapsto \Upsilon_v \in \mathrm{End}(\mathbb{R}^5), \qquad (\Upsilon_v)_{ij} = \Upsilon_{ijk}v_k.$$

Using this map we can rewrite the property iii) of Proposition 2.2 characterising Υ_{ijk} to the equivalent form

$$\forall v \in \mathbb{R}^5 \qquad \Upsilon_v^2 v = g(v, v)v.$$

The importance of this reformulation is justified by the following theorem.

Theorem 2.6. *The $\mathbf{O}(5)$ orbit of tensor Υ_{ijk} consists of all tensors $\varUpsilon_{ijk}$ for which the associated linear map*

$$\mathbb{R}^5 \ni v \mapsto \varUpsilon_v \in \mathrm{End}(\mathbb{R}^5), \qquad (\varUpsilon_v)_{ij} = \varUpsilon_{ijk}v_k$$

satisfies the following three conditions

1. *it is totally symmetric, i.e.* $g(u, \varUpsilon_v w) = g(w, \varUpsilon_v u) = g(u, \varUpsilon_w v)$,
2. *it is trace free* $\mathrm{tr}(\varUpsilon_v) = 0$,
3. *for any vector* $v \in \mathbb{R}^5$

$$\varUpsilon_v^2 v = g(v, v)v. \tag{2.9}$$

Remark 2.7. The $\mathbf{O}(5)$ orbit of Υ_{ijk}, described invariantly in the above theorem, consists of two disjoint $\mathbf{SO}(5)$ orbits: the orbit of Υ_{ijk} and the orbit of $-\Upsilon_{ijk}$. Indeed, both tensors $\pm\Upsilon_{ijk}$ satisfy the three conditions characterising the $\mathbf{O}(5)$ orbit and Υ_{ijk} can not be sent to $-\Upsilon_{ijk}$ via an element $h \in \mathbf{SO}(5)$. Otherwise the element $-h$ preserves Υ_{ijk} and as such belongs to G_Υ which contradicts Lemma 2.4.

Proof of Theorem. Let us consider tensor Υ_{ijk} for which $\Upsilon_{ijk}a^ia^ja^k$ has the standard form (2.5). Then its corresponding map Υ_v in the g-orthonormal basis $\mathbf{e}_i$ of (2.3), is represented by the following matrices

$$\begin{gathered}
\Upsilon_{\mathbf{e}_1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & s \end{pmatrix} \qquad \Upsilon_{\mathbf{e}_2} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \end{pmatrix} \\
\Upsilon_{\mathbf{e}_3} = \begin{pmatrix} 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & c \\ s & 0 & 0 & -b & 0 \\ 0 & 0 & -b & 0 & 0 \\ 0 & c & 0 & 0 & 0 \end{pmatrix} \qquad \Upsilon_{\mathbf{e}_4} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -b & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b \end{pmatrix} \qquad \Upsilon_{\mathbf{e}_5} = \begin{pmatrix} 0 & 0 & 0 & 0 & s \\ 0 & 0 & c & 0 & 0 \\ 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b \\ s & 0 & 0 & b & 0 \end{pmatrix},
\end{gathered} \tag{2.10}$$

where $s=-\frac{1}{2}$, $b=c=\frac{\sqrt{3}}{2}$. The advantage of introducing additional constant b will be clear later in the proof.

Now, let us take an arbitrary tensor Υ_{ijk} satisfying the three assumptions of Theorem 2.6. The theorem will be proven if we manage to construct an orthonormal basis $(e_1,\dots,e_5)$ in $\mathbb{R}^5$ in which the matrices Υ_{e_j} take the same form (2.10) as the matrices $\Upsilon_{\mathbf{e}_i}$.

Lemma 2.8. *For any pair of orthogonal vectors v,w the following identity holds*

$$g(v,v)w=2\,\Upsilon_v^2\,w+\Upsilon_w\,\Upsilon_v\,v.$$

Proof of Lemma. Applying (2.9) for the vector $v+rw$ ($r\in\mathbb{R}$) we get

$$rg(v,v)w+r^2g(w,w)v=r\Upsilon_v^2w+r^2\Upsilon_w^2v+r\Upsilon_v\Upsilon_wv+r^2\Upsilon_v\Upsilon_ww+r\Upsilon_w\Upsilon_vv+r^2\Upsilon_w\Upsilon_vw.$$

The linear in r term of this identity when compared with the symmetry $\Upsilon_wv=\Upsilon_vw$ yields the thesis.

□

The 5-th order homogeneous polynomial $\det(\Upsilon_v)$ considered on the unit sphere $\{v:g(v,v)=1\}$ satisfies $\det(\Upsilon_{-v})=-\det(\Upsilon_v)$. Thus, it can not have a fixed sign everywhere on the sphere and there exists a unit vector e_2 such that

$$\det(\Upsilon_{e_2})=0.$$

Let

$$e_1:=\Upsilon_{e_2}e_2$$

and let e_4 be the unit vector in the kernel of Υ_{e_2}:

$$\Upsilon_{e_2}e_4=0.$$

Lemma 2.9. *The vectors (e_1,e_2,e_4) are unit and pairwise orthogonal.*

Proof.

$$g(e_4,e_1)=g(e_4,\Upsilon_{e_2}e_2)=g(e_2,\Upsilon_{e_2}e_4)=0,$$
$$g(e_4,e_2)=g(e_4,\Upsilon_{e_2}^2e_2)=g(e_2,\Upsilon_{e_2}^2e_4)=0.$$

Using Lemma 2.8 for the unit orthogonal vectors $w=e_2$ and $v=e_4$ we get $e_2=\Upsilon_{e_2}\,\Upsilon_{e_4}\,e_4$ and so

$$g(e_2,e_1)=g(e_2,\Upsilon_{e_2}\,\Upsilon_{e_2}\,\Upsilon_{e_4}e_4)=g(\Upsilon_{e_2}^2e_2,\Upsilon_{e_4}e_4)=0.$$

Finally, the vector e_1 is unit:

$$g(e_1,e_1)=g(\Upsilon_{e_2}e_2,\Upsilon_{e_2}e_2)=g(\Upsilon_{e_2}^2e_2,e_2)=1.$$

□

The space $\mathrm{Span}(e_1,e_2,e_4)$ is Υ_{e_2}-invariant and Υ_{e_2} restricted to this invariant space is trace-free; the same is true for the restriction of Υ_{e_2} to the orthogonal complement $\mathrm{Span}(e_1,e_2,e_4)^\perp$. So, there exists a number $c\geq 0$ and a pair of unit vectors (e_3,e_5) such that

$$\Upsilon_{e_2}e_3=c\,e_5,\quad \Upsilon_{e_2}e_5=c\,e_3,\quad c\geq 0$$

and the system (e_1,e_2,e_3,e_4,e_5) is the orthonormal basis of $\mathbb{R}^5$. The matrix of Υ_{e_2} in this basis has the form as in (2.10), but the constant c is not fixed.

Now, the use of the assumed properties of (Υ_{ijk}) and the successive application of Lemma 2.8 proves that the matrices $\Upsilon_{e_1}, \dots, \Upsilon_{e_5}$ have the form of (2.10) with the following restrictions to the constants (b, c, s):

$$s = -\frac{1}{2} \qquad c^2 = \frac{3}{4} \qquad b^2 = c^2.$$

If $b = -c$ then one can perform the following change of basis

$$(e_1, e_2, e_3, e_4, e_5) \longmapsto (e_1, e_2, -e_3, -e_4, -e_5)$$

resulting the change $b \mapsto (-b)$ in the matrices (2.10).

This finishes the proof of Theorem 2.6.

□

Corollary 2.10. *The tensor Υ_{ijk} is fully determined by its properties listed in Proposition 2.2.*

3. The $\mathbf{SO}(3)$ structure in $\mathbb{R}^5$ and the representations of $\mathbf{SO}(3)$

The last corollary motivates the following definition.

Definition 3.1. An $\mathbf{SO}(3)$ structure on $\mathbb{R}^5$ is a pair (g, Υ) where g is a Riemannian metric $g(A, A) = g_{ij}a^i a^j$ and Υ is a ternary form $\Upsilon(A, A, A) = \Upsilon_{ijk}a^i a^j a^k$ such that

i) $\Upsilon_{ijk} = \Upsilon_{(ijk)}$,

ii) $\Upsilon_{ijj} = 0$,

iii) $\Upsilon_{jki}\Upsilon_{lni} + \Upsilon_{lji}\Upsilon_{kni} + \Upsilon_{kli}\Upsilon_{jni} = g_{jk}g_{ln} + g_{lj}g_{kn} + g_{kl}g_{jn}$.

In this section we will use an $\mathbf{SO}(3)$ structure to define representations of $\mathbf{SO}(3)$ in $\bigotimes^2 \mathbb{R}^5$. First, we recall the following well known theorem .

Theorem 3.2. *All the irreducible finite-dimensional representations of* $\mathbf{SO}(3)$ *are odd dimensional. There is a unique irreducible representation of* $\mathbf{SO}(3)$ *in space* $\mathbb{R}^{2l+1}$ *for each* $l \in \{0, 1, 2, 3, ...\}$. *The tensor product* $\mathbb{R}^{2l_1+1} \otimes \mathbb{R}^{2l_2+1}$ *decomposes onto the* $\mathbf{SO}(3)$*-irreducible components according to the following Wigner formula*

$$\mathbb{R}^{2l_1+1} \otimes \mathbb{R}^{2l_2+1} = \bigoplus_{l=|l_1-l_2|}^{|l_1+l_2|} \mathbb{R}^{2l+1}. \tag{3.1}$$

The 5-dimensional irreducible representation ρ of $\mathbf{SO}(3)$ with the carrier space

$$\textstyle\bigwedge^1_5 := \mathbb{R}^5$$

was already considered in (2.2). To find the projectors onto the irreducible components of the tensor representations $\bigotimes^2 \mathbb{R}^5$, $\bigwedge^2 \mathbb{R}^5$ and $\bigodot^2 \mathbb{R}^5$ we use the $\mathbf{SO}(3)$ structure (g, Υ). Associated with Υ is the following endomorphism

$$\hat{\Upsilon} : \textstyle\bigotimes^2 \mathbb{R}^5 \longrightarrow \bigotimes^2 \mathbb{R}^5,$$

$$W^{ik} \overset{\hat{\Upsilon}}{\longmapsto} 4\ \Upsilon_{ijm}\Upsilon_{klm}W^{jl},$$

which preserves the decomposition $\bigotimes^2 \mathbb{R}^5 = \bigwedge^2 \mathbb{R}^5 \oplus \bigodot^2 \mathbb{R}^5$. Now, a simple algebra leads to the following proposition.

Proposition 3.3. $\bigotimes^2\mathbb{R}^5 = \bigwedge^2_3 \oplus \bigwedge^2_7 \oplus \bigodot^2_1 \oplus \bigodot^2_5 \oplus \bigodot^2_9$, *where*

$$\begin{aligned}
&\textstyle\bigodot^2_1 = \{\ S \in \bigotimes^2\mathbb{R}^5 \mid \hat{\Upsilon}(S) = 14{\cdot}S\ \} = \{S = \lambda{\cdot}g,\ \lambda \in \mathbb{R}\ \},\\
&\textstyle\bigwedge^2_3 = \{\ F \in \bigotimes^2\mathbb{R}^5 \mid \hat{\Upsilon}(F) = 7{\cdot}F\ \}\ =\ \mathfrak{so}(3)\ =\ \mathrm{Span}(E_1, E_2, E_3),\\
&\textstyle\bigodot^2_5 = \{\ S \in \bigotimes^2\mathbb{R}^5 \mid \hat{\Upsilon}(S) = -3{\cdot}S\ \},\\
&\textstyle\bigwedge^2_7 = \{\ F \in \bigotimes^2\mathbb{R}^5 \mid \hat{\Upsilon}(F) = -8{\cdot}F\ \} =: \mathfrak{n},\\
&\textstyle\bigodot^2_9 = \{\ S \in \bigotimes^2\mathbb{R}^5 \mid \hat{\Upsilon}(S) = 4{\cdot}S\ \}.
\end{aligned}$$

All the representations $\bigwedge^2_j \subset \bigwedge^2\mathbb{R}^5$ *and* $\bigodot^2_k \subset \bigodot^2\mathbb{R}^5$ *are irreducible; the indices* j *and* k *denote their dimensions.*

Remark 3.4. Note that the tensor $\hat{\Upsilon}$ defines a nondegenerate $\mathbf{SO}(3)$ invariant scalar product $(F|F') = *(\hat{\Upsilon}(F) \wedge *F')$ of signature $(3,7)$ on the space of 2-forms

$$\textstyle\bigwedge^2\mathbb{R}^5 = \mathfrak{so}(5) = \mathfrak{so}(3) \oplus \mathfrak{n} = \bigwedge^2_3 \oplus \bigwedge^2_7. \tag{3.2}$$

Although this scalar product differs from the one associated with the Killing form $k(F,F') = -6*(F \wedge *F')$, in both of them we have $\mathfrak{so}(3) \perp \mathfrak{n}$.

Remark 3.5. In agreement with the above notation we will denote the irreducible representation $\mathbb{R}^5$ by

$$\textstyle\bigwedge^1_5 = \mathbb{R}^5 = \bigwedge^1\mathbb{R}^5.$$

Using the $\mathbf{SO}(3)$ structure (g, Υ) we can also build up an endomorphism

$$\check{\Upsilon} : \textstyle\bigodot^2\mathbb{R}^5 \longrightarrow \bigodot^2\mathbb{R}^5$$

given by

$$S^{kl} \xmapsto{\check{\Upsilon}} 4\ \Upsilon_{klm}\Upsilon_{ijm}S^{ij}.$$

It is independent of $\hat{\Upsilon}|_{\odot^2\mathbb{R}^5}$. Note that $\check{\Upsilon}$ is a composition $\check{\Upsilon} = 4\bar{\Upsilon} \circ \grave{\Upsilon}$ of two maps

$$\textstyle\bigodot^2\mathbb{R}^5 \xrightarrow{\grave{\Upsilon}} \bigwedge^1_5 \xrightarrow{\bar{\Upsilon}} \bigodot^2_5$$

given by

$$\grave{\Upsilon}(S)_i = \Upsilon_{ijk}S_{jk}, \qquad \bar{\Upsilon}(v) = \Upsilon_v. \tag{3.3}$$

We have

$$\ker(\grave{\Upsilon}) = \textstyle\bigodot^2_1 \oplus \bigodot^2_9, \qquad \mathrm{im}(\grave{\Upsilon}) = \bigwedge^1_5.$$

Thus $\grave{\Upsilon}$ restricted to $\bigodot^2_5$ is an isomorphic intertwiner between the representations $\bigodot^2_5$ and $\bigwedge^1_5$. Furthermore we have:

$$4\bar{\Upsilon} \circ \grave{\Upsilon}|_{\odot^2_5} = 14{\cdot}\mathrm{id}.$$

Summarising we have the following proposition.

Proposition 3.6. *The eigenvalues of* $\check{\Upsilon}$ *on the representations* $\bigodot^2_1 \oplus \bigodot^2_9$ *and* $\bigodot^2_5$ *are 0 and 14, respectively.*

4. The $\mathbf{SO}(3)$ structure on manifold

Definition 4.1. An $\mathbf{SO}(3)$ structure on a 5-dimensional Riemannian manifold (M,g) is a structure defined by means of a rank 3 tensor field Υ for which the associated linear map

$$\mathrm{T}M \ni v \mapsto \Upsilon_v \in \mathrm{End}(\mathrm{T}M), \qquad (\Upsilon_v)_{ij} = \Upsilon_{ijk}v_k.$$

satisfies the following three conditions

1. it is totally symmetric, i.e. $g(u,\Upsilon_v w) = g(w,\Upsilon_v u) = g(u,\Upsilon_w v)$,
2. it is trace free $\mathrm{tr}(\Upsilon_v)=0$,
3. for any vector field $v\in \mathrm{T}M$
$$\Upsilon_v^2 v = g(v,v)v.$$

Definition 4.2. Two $\mathbf{SO}(3)$ structures (M,g,Υ) and $(\bar{M},\bar{g},\bar{\Upsilon})$ defined on two respective 5-manifolds M and $\bar{M}$ are (locally) *equivalent* iff there exists a (local) diffeomorphism $\phi : M\to \bar{M}$ such that

$$\phi^*(\bar{g}) = g \qquad \text{and} \qquad \phi^*(\bar{\Upsilon}) = \Upsilon.$$

If $\bar{M}=M$, $\bar{g}=g$, $\bar{\Upsilon}=\Upsilon$ the equivalence ϕ is called a (local) *symmetry* of (M,g,Υ). The group of (local) symmetries is called a *symmetry group of* (M,g,Υ).

In view of Corollary 2.10, Theorem 2.6 and Proposition 2.2 tensor field Υ reduces the structure group of the bundle of orthonormal frames over M to the irreducible $\mathbf{SO}(3)$. Thus, locally, we can represent an $\mathbf{SO}(3)$ structure on M by a coframe

$$\theta = (\theta^i) = (\theta^1,\theta^2,\theta^3,\theta^4,\theta^5) \tag{4.1}$$

on M, given up to the $\mathbf{SO}(3)$ transformation

$$\mathrm{T}M\otimes\Omega^1(M)\ni\theta\mapsto\tilde{\theta} = \rho(h)\theta. \tag{4.2}$$

For such a class of coframes the Riemannian metric g is

$$g = \theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_5^2,$$

and the tensor Υ, reducing the structure group from $\mathbf{SO}(5)$ to $\mathbf{SO}(3)$, is

$$\Upsilon = \frac{1}{2}\theta_1\big(\ 6\theta_2^2+6\theta_4^2-2\theta_1^2-3\theta_3^2-3\theta_5^2\ \big)+\frac{3\sqrt{3}}{2}\theta_4(\theta_5^2-\theta_3^2)+3\sqrt{3}\theta_2\theta_3\theta_5. \tag{4.3}$$

Definition 4.3. An orthonormal coframe $(\theta^1,\theta^2,\theta^3,\theta^4,\theta^5)$ in which the tensor Υ of an $\mathbf{SO}(3)$ structure (M,g,Υ) is of the form (4.3) is called a *coframe adapted to* (M,g,Υ), an *adapted coframe*, for short.

4.1. **Topological obstruction.**

The determination of topological obstructions for existence of an irreducible $\mathbf{SO}(3)$ structure on a 5-dimensional manifold is presented in a separate paper of one of us [2]. For the completeness of the present paper we quote the result here. In the theorem below we denote by p_j the jth Pontriagin class.

Theorem 4.4. *Let M be an orientable 5-dimensional manifold. There exists an irreducible $\mathbf{SO}(3)$ structure on M iff M admits the standard $\mathbf{SO}(3)$ structure (i.e. TM splits on the rank 2 trivial bundle and a rank 3 complement) and*

$$p_1(TM) = 5\,\tilde{p}, \quad \textit{where} \quad \tilde{p}\in H^4(M;\mathbb{Z}).$$

Remark 4.5. The irreducible inclusion $\iota(\mathbf{SO}(3)) \subset \mathbf{SO}(5)$ induces the irreducible inclusion of $\tilde{\iota}(\mathbf{Spin}(3)) \subset \mathbf{Spin}(5)$. Assuming that $w_2(TM) = 0$, the $\mathbf{SO}(5)$ structure on M can be lifted to the $\mathbf{Spin}(5)$ structure. It further may be reduced to the $\tilde{\iota}(\mathbf{Spin}(3))$ structure on M, provided that M admits an irreducible $\mathbf{SO}(3)$ structure.

4.2. $\mathfrak{so}(3)$ **connection.** Given an $\mathbf{SO}(3)$ structure as above, we consider an $\mathfrak{so}(3)$ connection on M represented locally by means of an $\mathfrak{so}(3)$-valued 1-form Γ given by

$$\Gamma = (\Gamma^i{}_j) = \gamma^1 E_1 + \gamma^2 E_2 + \gamma^3 E_3, \tag{4.4}$$

where $\gamma^1, \gamma^2, \gamma^3$ are 1-forms on M and E_I with $I = 1, 2, 3$ are given by (2.8). This connection, having values in $\mathfrak{so}(3) \subset \mathfrak{so}(5)$, is necessarily metric. Via the Cartan structure equations,

$$\mathrm{d}\theta^i + \Gamma^i{}_j \wedge \theta^j = T^i \tag{4.5}$$

$$\mathrm{d}\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j = K^i{}_j, \tag{4.6}$$

it defines the torsion 2-form T^i and the $\mathfrak{so}(3)$-curvature 2-form $K^i{}_j$. Using these forms we define the torsion tensor $T^i{}_{jk} \in (\mathbb{R}^5 \otimes \bigwedge^2 \mathbb{R}^5)$ and the $\mathfrak{so}(3)$-curvature tensor $r^I{}_{jk} \in (\mathfrak{so}(3) \otimes \bigwedge^2 \mathbb{R}^5)$, respectively, by

$$T^i = \frac{1}{2} T^i{}_{jk} \theta^j \wedge \theta^k$$

and

$$r^I = \mathrm{d}\gamma^I + \tfrac{1}{2}\epsilon^I{}_{JK}\gamma^J \wedge \gamma^K = \frac{\sqrt{3}}{2} r^I{}_{jk} \theta^j \wedge \theta^k. \tag{4.7}$$

(Note that, $K = (K^i{}_j) = r^1 E_1 + r^2 E_2 + r^3 E_3$.) The connection satisfies the first Bianchi identity

$$K^i{}_j \wedge \theta^j = DT^i \tag{4.8}$$

and the second Bianchi identity

$$DK^i{}_j = 0, \tag{4.9}$$

with the covariant differential defined by

$$DT^i = \mathrm{d}T^i + \Gamma^i{}_j \wedge T^j, \qquad DK^i{}_j = \mathrm{d}K^i{}_j + \Gamma^i{}_k \wedge K^k{}_j - K^i{}_k \wedge \Gamma^k{}_j.$$

Since the irreducible $\mathbf{SO}(3)$ was defined by the demand that it preserves g and Υ we have the following proposition.

Proposition 4.6. *Every $\mathfrak{so}(3)$ connection Γ of (4.4) is metric*

$$\overset{\Gamma}{\nabla}_v (g) \equiv 0$$

and preserves tensor Υ

$$\overset{\Gamma}{\nabla}_v (\Upsilon) \equiv 0 \qquad \forall v \in \mathrm{T}M.$$

4.3. **SO(3) structures with vanishing torsion.** In this section we find all $\mathbf{SO}(3)$ structures (M, g, Υ) which admit $\mathfrak{so}(3)$ connections Γ of (4.4) with vanishing torsion

$$T^i \equiv 0. \tag{4.10}$$

Assuming that T^i is identically zero and using the first Bianchi identity (4.8) for Γ we easily find that a lot of components of the $\mathfrak{so}(3)$-curvature r^I vanish. Explicitly, we find that in such a case the curvature forms (r^1, r^2, r^3) are expressible in terms of only one function ${r^1}_{15}$ and read

$$r^1 = {r^1}_{15}\kappa^1, \qquad r^2 = {r^1}_{15}\kappa^2, \qquad r^3 = {r^1}_{15}\kappa^3, \tag{4.11}$$

where

$$\kappa^1 = \sqrt{3}\theta^1 \wedge \theta^5 + \theta^2 \wedge \theta^3 + \theta^4 \wedge \theta^5,$$
$$\kappa^2 = \sqrt{3}\theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^5 + \theta^3 \wedge \theta^4,$$
$$\kappa^3 = 2\theta^2 \wedge \theta^4 + \theta^3 \wedge \theta^5.$$

It further follows, that under the assumption of (4.10), the second Bianchi identity (4.9) implies that

$${r^1}_{15} = \mathrm{const}.$$

This means that ${r^1}_{15}$ is a real parameter and that there is only 1-parameter family of $\mathbf{SO}(3)$ structures with vanishing torsion. This family equips the principal fibre bundle $F(M)$ of $\mathbf{SO}(3)$ frames

$$\mathbf{SO}(3) \to F(M) \xrightarrow{\pi} M$$

over M with an $\mathfrak{so}(3)$-connection

$$\begin{aligned} \tilde{\Gamma} &= \rho(h)\ \Gamma\ \rho(h)^{-1} - \mathrm{d}\rho(h)\ \rho(h)^{-1} \\ &= \tilde{\gamma}^1 E_1 + \tilde{\gamma}^2 E_2 + \tilde{\gamma}^3 E_3. \end{aligned} \tag{4.12}$$

This, together with the lifted coframe

$$\tilde{\theta} = \rho(h)\theta \tag{4.13}$$

of (4.2), satisfies the following differential system

$$\begin{aligned}
\mathrm{d}\tilde{\theta}^1 &= -\sqrt{3}\tilde{\gamma}^1 \wedge \tilde{\theta}^5 - \sqrt{3}\tilde{\gamma}^2 \wedge \tilde{\theta}^3 \\
\mathrm{d}\tilde{\theta}^2 &= -\tilde{\gamma}^1 \wedge \tilde{\theta}^3 - \tilde{\gamma}^2 \wedge \tilde{\theta}^5 - 2\tilde{\gamma}^3 \wedge \tilde{\theta}^4 \\
\mathrm{d}\tilde{\theta}^3 &= \tilde{\gamma}^1 \wedge \tilde{\theta}^2 + \sqrt{3}\tilde{\gamma}^2 \wedge \tilde{\theta}^1 - \tilde{\gamma}^2 \wedge \tilde{\theta}^4 - \tilde{\gamma}^3 \wedge \tilde{\theta}^5 \\
\mathrm{d}\tilde{\theta}^4 &= -\tilde{\gamma}^1 \wedge \tilde{\theta}^5 + \tilde{\gamma}^2 \wedge \tilde{\theta}^3 + 2\tilde{\gamma}^3 \wedge \tilde{\theta}^2 \\
\mathrm{d}\tilde{\theta}^5 &= \sqrt{3}\tilde{\gamma}^1 \wedge \tilde{\theta}^1 + \tilde{\gamma}^1 \wedge \tilde{\theta}^4 + \tilde{\gamma}^2 \wedge \tilde{\theta}^2 + \tilde{\gamma}^3 \wedge \tilde{\theta}^3 \\
\mathrm{d}\tilde{\gamma}^1 &= -\tilde{\gamma}^2 \wedge \tilde{\gamma}^3 + {r^1}_{15}\tilde{\kappa}^1 \\
\mathrm{d}\tilde{\gamma}^2 &= -\tilde{\gamma}^3 \wedge \tilde{\gamma}^1 + {r^1}_{15}\tilde{\kappa}^2 \\
\mathrm{d}\tilde{\gamma}^3 &= -\tilde{\gamma}^1 \wedge \tilde{\gamma}^2 + {r^1}_{15}\tilde{\kappa}^3,
\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}
\tilde{\kappa}^1 &= \sqrt{3}\tilde{\theta}^1 \wedge \tilde{\theta}^5 + \tilde{\theta}^2 \wedge \tilde{\theta}^3 + \tilde{\theta}^4 \wedge \tilde{\theta}^5, \\
\tilde{\kappa}^2 &= \sqrt{3}\tilde{\theta}^1 \wedge \tilde{\theta}^3 + \tilde{\theta}^2 \wedge \tilde{\theta}^5 + \tilde{\theta}^3 \wedge \tilde{\theta}^4, \\
\tilde{\kappa}^3 &= 2\tilde{\theta}^2 \wedge \tilde{\theta}^4 + \tilde{\theta}^3 \wedge \tilde{\theta}^5.
\end{aligned} \tag{4.15}$$

The *eight* linearly independent 1-forms $(\tilde\theta^1,\tilde\theta^2,\tilde\theta^3,\tilde\theta^4,\tilde\theta^5,\tilde\gamma^1,\tilde\gamma^2,\tilde\gamma^3)$ constitute a basis of 1-forms on the *eight* dimensional manifold $F(M)$. Moreover, since equations (4.14) have only constant coefficients on their right hand sides, the basis $(\tilde\theta^1,\tilde\theta^2,\tilde\theta^3,\tilde\theta^4,\tilde\theta^5,\tilde\gamma^1,\tilde\gamma^2,\tilde\gamma^3)$ can be identified with a basis of left invariant forms on a Lie group to which the bundle $F(M)$ is (locally) diffeomorphic. Thus we may identify $F(M)$ with a local Lie group, the structure constants of which can be read off from the system (4.14)-(4.15). We find that, depending on the parameter ${r^1}_{15}$, these structure constants correspond to

i) $\mathbf{SO}(3)\times_\rho\mathbb{R}^5$ group iff ${r^1}_{15}=0$
ii) $\mathbf{SU}(3)$ group iff ${r^1}_{15}>0$
iii) $\mathbf{SL}(3,\mathbb{R})$ group iff ${r^1}_{15}<0$.

It further follows from the system (4.14)-(4.15) that the tensors

$$\tilde g=\tilde\theta_1^2+\tilde\theta_2^2+\tilde\theta_3^2+\tilde\theta_4^2+\tilde\theta_5^2, \tag{4.16}$$

and

$$\tilde\Upsilon=\frac12\tilde\theta_1\big(\,6\tilde\theta_2^2+6\tilde\theta_4^2-2\tilde\theta_1^2-3\tilde\theta_3^2-3\tilde\theta_5^2\,\big)+\frac{3\sqrt3}{2}\tilde\theta_4(\tilde\theta_5^2-\tilde\theta_3^2)+3\sqrt3\tilde\theta_2\tilde\theta_3\tilde\theta_5 \tag{4.17}$$

on $F(M)$ are preserved under the Lie transport along the fibres of $\mathbf{SO}(3)\to F(M)\xrightarrow{\pi}M$. Moreover, these tensors are degenerate in precisely vertical directions. Thus they descend to M defining, respectively, g and Υ, i.e. an $\mathbf{SO}(3)$ structure, there. Locally, depending on the sign of ${r^1}_{15}$, this structure is isomorphic to the homogeneous model M_0 in case i), the homogeneous model M_+ in case ii) and the homogeneous model M_- in case iii).

Theorem 4.7. *All* $\mathbf{SO}(3)$ *structures with vanishing torsion are locally isometric to one of the symmetric spaces*

$$M=G/\mathbf{SO}(3),$$

where

$$G=\mathbf{SO}(3)\times_\rho\mathbb{R}^5,\qquad \mathbf{SU}(3)\qquad\text{or}\qquad \mathbf{SL}(3,\mathbb{R}).$$

The Riemannian metric g and the tensor Υ defining the $\mathbf{SO}(3)$ *structure are obtained via (4.16)-(4.17) by means of the left invariant forms $(\tilde\theta^1,\tilde\theta^2,\tilde\theta^3,\tilde\theta^4,\tilde\theta^5,\tilde\gamma^1,\tilde\gamma^2,\tilde\gamma^3)$ on G, which satisfy (4.14)-(4.15). In all three cases the metric g is Einstein. It is flat in case of* $G=\mathbf{SO}(3)\times_\rho\mathbb{R}^5$*. In the other two cases the metric is not even conformally flat.*

Proof. Only the last three sentences of the theorem remain to be proven. Since there is no torsion, the Levi-Civita connection for g, when written in terms of the coframe $(\tilde\theta^1,\tilde\theta^2,\tilde\theta^3,\tilde\theta^4,\tilde\theta^5,\tilde\gamma^1,\tilde\gamma^2,\tilde\gamma^3)$, is simply $\tilde\Gamma$ of (4.12). Then, the direct calculation shows that the metric is Einstein with both the Ricci scalar and the Weyl tensor being proportional, modulo a constant factor, to ${r^1}_{15}$. □

Remark 4.8. According to the last sentence of the theorem the spaces $M_\pm$ corresponding to nontrivial $\mathbf{SO}(3)$ structures without torsion are not of constant curvature for the Levi-Civita connection of g.

Remark 4.9. Note that

$$-\tilde K_0=\tilde\kappa^I E_I \tag{4.18}$$

is the curvature of the canonical connection [5] on the symmetric space $\mathbf{SU}(3)/\mathbf{SO}(3)$. Moreover, the forms $(\tilde\theta^1,\theta^2,\tilde\theta^3,\tilde\theta^4,\tilde\theta^5,\tilde\gamma^1,\tilde\gamma^2,\tilde\gamma^3)$ define an absolute teleparalelism on $F(M)$. They can be collected to an $\mathfrak{su}(3)$-valued matrix

$$\Gamma_{\rm Cartan}=\begin{pmatrix}0&\tilde\gamma^3&\tilde\gamma^2\\-\tilde\gamma^3&0&\tilde\gamma^1\\-\tilde\gamma^2&-\tilde\gamma^1&0\end{pmatrix}+i\begin{pmatrix}\frac{\tilde\theta^1}{\sqrt3}-\tilde\theta^4&\tilde\theta^2&\tilde\theta^3\\\tilde\theta^2&\frac{\tilde\theta^1}{\sqrt3}+\tilde\theta^4&\tilde\theta^5\\\tilde\theta^3&\tilde\theta^5&-2\frac{\tilde\theta^1}{\sqrt3}\end{pmatrix},\tag{4.19}$$

which defines an $\mathfrak{su}(3)$-valued Cartan connection on the bundle $\mathbf{SO}(3)\to F(M)\to M$. The curvature of this connection

$$\Omega_{\rm Cartan}={\rm d}\Gamma_{\rm Cartan}+\Gamma_{\rm Cartan}\wedge\Gamma_{\rm Cartan}\tag{4.20}$$

is

$$\Omega_{\rm Cartan}=({r^1}_{15}-1)\begin{pmatrix}0&\tilde\kappa^3&\tilde\kappa^2\\-\tilde\kappa^3&0&\tilde\kappa^1\\-\tilde\kappa^2&-\tilde\kappa^1&0\end{pmatrix}$$

and it vanishes iff the corresponding $\mathfrak{so}(3)$ connection Γ has constant positive curvature determined by ${r^1}_{15}=1$.

Remark 4.10. Remark 4.9 can be generalised leading to the description of $\mathbf{SO}(3)$ geometries with arbitrary $\mathfrak{so}(3)$ connection in terms of an $\mathfrak{su}(3)$ Cartan connection on the fibre bundle $\mathbf{SO}(3)\to F(M)\to M$. Indeed, given an $\mathbf{SO}(3)$ geometry with the adapted coframe $(\theta^1,\theta^2,\theta^3,\theta^4,\theta^5)$ and the $\mathfrak{so}(3)$ connection Γ we define the lifted coframe $(\tilde\theta^1,\tilde\theta^2,\tilde\theta^3,\tilde\theta^4,\tilde\theta^5)$ via (4.13) and the 1-forms $(\tilde\gamma^1,\tilde\gamma^2,\tilde\gamma^3)$ via (4.12). Then, the $\mathfrak{su}(3)$-valued Cartan connection on $F(M)$ is given by equation (4.19). The curvature (4.20) of this connection satisfies the Bianchi identity

$${\rm D}\Omega_{\rm Cartan}={\rm d}\Omega_{\rm Cartan}+\Gamma_{\rm Cartan}\wedge\Omega_{\rm Cartan}-\Omega_{\rm Cartan}\wedge\Gamma_{\rm Cartan}\equiv0\tag{4.21}$$

and naturally splits onto the real and imaginary parts

$$\Omega_{\rm Cartan}={\rm Re}(\Omega_{\rm Cartan})+i\sigma(\tilde T)=\begin{pmatrix}0&\tilde r^3&\tilde r^2\\-\tilde r^3&0&\tilde r^1\\-\tilde r^2&-\tilde r^1&0\end{pmatrix}+i\begin{pmatrix}\frac{\tilde T^1}{\sqrt3}-\tilde T^4&\tilde T^2&\tilde T^3\\\tilde T^2&\frac{\tilde T^1}{\sqrt3}+\tilde T^4&\tilde T^5\\\tilde T^3&\tilde T^5&-2\frac{\tilde T^1}{\sqrt3}\end{pmatrix}.$$

The imaginary part is simply the lift of the torsion T of the $\mathfrak{so}(3)$-connection Γ,

$$\tilde T=\rho(h)T.$$

The real part can be collected to a 5×5 matrix

$$\tilde R=\tilde r^1E_1+\tilde r^2E_2+\tilde r^3E_3.$$

This satisfies

$$\tilde R=\tilde K-\tilde K_0,\qquad\qquad \tilde K=\rho(h)K\rho(h)^{-1},\tag{4.22}$$

where K is the $\mathfrak{so}(3)$ curvature of Γ and $\tilde{K}_0$ is given by (4.18). Thus, $\tilde{R}$ is the lift of the $\mathfrak{so}(3)$ curvature K shifted by the curvature $-\tilde{K}_0$ of the canonical connection on the symmetric space $\mathbf{SU}(3)/\mathbf{SO}(3)$.

4.4. $\mathfrak{spin}(3)$ **connection.** The even Clifford algebra $Cl_0(5,0)$ has a 4-dimensional faithful representation in which the orthonormal vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5)$ may be represented by

$$\begin{gathered}\mathbf{e}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ \mathbf{e}_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \qquad \mathbf{e}_4 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \qquad \mathbf{e}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\end{gathered} \tag{4.23}$$

One checks, by direct calculations, that

$$\mathbf{e}_i^2 = 1, \qquad \mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0, \qquad j \neq i = 1, 2, 3, 4, 5.$$

Now, the double covering homomorphism $\mathbf{Spin}(5) \to \mathbf{SO}(5)$ induces the isomorphism of the Lie algebras $\mathfrak{spin}(5) \to \mathfrak{so}(5)$. By means of this isomorphism an element $\mathbf{e}_i\mathbf{e}_j \in \mathfrak{spin}(5)$, $i < j$, is mapped to (f_{ij}) - a 5×5 antisymmetric matrix having value 1 at its entry f_{ij}, value -1 at f_{ji} and value 0 in all the remaining entries. This implies that the basis of the Lie algebra $\mathfrak{spin}(3)$ corresponding to the basis (E_1, E_2, E_3) of the irreducible $\mathfrak{so}(3)$ is

$$\mathbf{E}_1 = \tfrac{1}{2}(\sqrt{3}\mathbf{e}_1\mathbf{e}_5 + \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_4\mathbf{e}_5), \qquad \mathbf{E}_2 = \tfrac{1}{2}(\sqrt{3}\mathbf{e}_1\mathbf{e}_3 + \mathbf{e}_2\mathbf{e}_5 + \mathbf{e}_3\mathbf{e}_4),$$

$$\mathbf{E}_3 = \tfrac{1}{2}(2\mathbf{e}_2\mathbf{e}_4 + \mathbf{e}_3\mathbf{e}_5).$$

Explicitly:

$$\mathbf{E}_1 = \tfrac{1}{2}\begin{pmatrix} 0 & i & -\sqrt{3} & i \\ i & 0 & -i & -\sqrt{3} \\ \sqrt{3} & -i & 0 & -i \\ i & \sqrt{3} & -i & 0 \end{pmatrix} \qquad \mathbf{E}_2 = \tfrac{1}{2}\begin{pmatrix} i\sqrt{3} & -1 & 0 & -1 \\ 1 & i\sqrt{3} & -1 & 0 \\ 0 & 1 & -i\sqrt{3} & 1 \\ 1 & 0 & -1 & -i\sqrt{3} \end{pmatrix}$$

$$\mathbf{E}_3 = \tfrac{1}{2}\begin{pmatrix} 2i & 0 & i & 0 \\ 0 & -2i & 0 & i \\ i & 0 & 2i & 0 \\ 0 & i & 0 & -2i \end{pmatrix}. \tag{4.24}$$

Thus we have

$$\mathfrak{spin}(3) = \mathrm{Span}(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3) \quad \subset \quad \mathfrak{spin}(5) = \mathrm{Span}(\tfrac{1}{2}\mathbf{e}_i\mathbf{e}_j, i < j = 1, 2, \ldots, 5).$$

Now, given an $\mathbf{SO}(3)$ structure (M, g, Υ) and an $\mathfrak{so}(3)$ connection $\Gamma = \gamma^1 E_1 + \gamma^2 E_2 + \gamma^3 E_3$, we associate with it a connection

$$\Gamma_{\mathfrak{spin}} = \gamma^1\mathbf{E}_1 + \gamma^2\mathbf{E}_2 + \gamma^3\mathbf{E}_3 \in \mathfrak{spin}(3) \tag{4.25}$$

which we call $\mathfrak{spin}(3)$ connection. This connection will be used in Section 7 to define covariantly constant spinor fields on M.

5. Characteristic connection

Suppose now that we are given an $\mathbf{SO}(3)$ structure (M, g, Υ) on a 5-dimensional manifold M. This defines the Levi-Civita connection $\overset{LC}{\Gamma}$ which, having values in $\mathfrak{so}(5)$, is an element $\overset{LC}{\Gamma}_{ijk}$ of $\mathfrak{so}(5) \otimes \mathbb{R}^5 = \bigwedge^2\mathbb{R}^5 \otimes \mathbb{R}^5$. In the following we will be only interested in a subclass of $\mathbf{SO}(3)$ structures, which we term *nearly integrable*.

Definition 5.1. An $\mathbf{SO}(3)$ structure (M, g, Υ) is called *nearly integrable* iff

$$(\overset{LC}{\nabla}_v \Upsilon)(v, v, v) \equiv 0 \tag{5.1}$$

for the Levi-Civita connection $\overset{LC}{\nabla}$.

The condition (5.1), when written in an adapted coframe (4.1), is

$$\overset{LC}{\Gamma}_{m(ji}\ \Upsilon_{kl)m} \equiv 0. \tag{5.2}$$

This motivates an introduction of the map

$$\Upsilon' : \bigwedge{}^2\mathbb{R}^5 \otimes \mathbb{R}^5 \mapsto \bigodot{}^4\mathbb{R}^5$$

such that

$$\begin{aligned} \Upsilon'(\overset{LC}{\Gamma})_{ijkl} &= 12\overset{LC}{\Gamma}_{m(ji}\ \Upsilon_{kl)m} \\ &= \overset{LC}{\Gamma}_{mji}\ \Upsilon_{mkl} + \overset{LC}{\Gamma}_{mki}\ \Upsilon_{jml} + \overset{LC}{\Gamma}_{mli}\ \Upsilon_{jkm} \\ &+ \overset{LC}{\Gamma}_{mij}\ \Upsilon_{mkl} + \overset{LC}{\Gamma}_{mkj}\ \Upsilon_{iml} + \overset{LC}{\Gamma}_{mlj}\ \Upsilon_{ikm} \\ &+ \overset{LC}{\Gamma}_{mik}\ \Upsilon_{mjl} + \overset{LC}{\Gamma}_{mjk}\ \Upsilon_{iml} + \overset{LC}{\Gamma}_{mlk}\ \Upsilon_{ijm} \\ &+ \overset{LC}{\Gamma}_{mil}\ \Upsilon_{mjk} + \overset{LC}{\Gamma}_{mjl}\ \Upsilon_{imk} + \overset{LC}{\Gamma}_{mkl}\ \Upsilon_{ijm}. \end{aligned} \tag{5.3}$$

We have the following proposition.

Proposition 5.2. *An* $\mathbf{SO}(3)$ *structure* (M, g, Υ) *is nearly integrable if and only if its Levi-Civita connection* $\overset{LC}{\Gamma} \in \ker \Upsilon'$.

It is worthwhile to note that each of the last four raws of (5.3) resembles the l.h.s. of equality (2.7). Thus, $\mathfrak{so}(3) \otimes \mathbb{R}^5 \subset \ker \Upsilon'$. Due to the first equality in (5.3) we also have $\bigwedge^3\mathbb{R}^5 \subset \ker \Upsilon'$. It further follows that $\ker \Upsilon' = [\mathfrak{so}(3) \otimes \mathbb{R}^5] + \bigwedge^3\mathbb{R}^5$. Now, introducing the map

$$\grave{\Upsilon} : \ker \Upsilon' \to \otimes^2\mathbb{R}^5$$

given by

$$\grave{\Upsilon}(\overset{LC}{\Gamma})_{il} = \Upsilon_{ijk}\overset{LC}{\Gamma}_{ljk}$$

and observing that $\ker \grave{\Upsilon} = \bigwedge^3\mathbb{R}^5$ we get the following $\mathbf{SO}(3)$ invariant decomposition

$$\ker \Upsilon' = [\mathfrak{so}(3) \otimes \mathbb{R}^5] \oplus \bigwedge{}^3\mathbb{R}^5.$$

This is the base for the following proposition.

Proposition 5.3. *The Levi-Civita connection* $\overset{LC}{\Gamma}$ *of a nearly integrable* $\mathbf{SO}(3)$ *structure* (M, g, Υ) *uniquely decomposes onto*

$$\overset{LC}{\Gamma} = \Gamma + \tfrac{1}{2}T, \tag{5.4}$$

where

$$\Gamma\in\mathfrak{so}(3)\otimes\mathbb{R}^5\qquad\text{and}\qquad T\in{\textstyle\bigwedge}^3\mathbb{R}^5=\ker\dot{\Upsilon}.$$

The decomposition (5.4) of the Levi-Civita connection $\overset{LC}{\Gamma}$ of a nearly integrable $\mathbf{SO}(3)$ structure defines an $\mathfrak{so}(3)$ connection Γ. Rewriting the Cartan structure equation

$$\mathrm{d}\theta^i+\overset{LC}{\Gamma}{}^i{}_j\wedge\theta^j=0$$

for $\overset{LC}{\Gamma}$ into the form

$$\mathrm{d}\theta^i+\Gamma^i{}_j\wedge\theta^j=\tfrac{1}{2}T^i{}_{jk}\theta^j\wedge\theta^k$$

enables us to interpret T as the *totally skew symmetric torsion* of Γ.

Definition 5.4. An $\mathfrak{so}(3)$ connection Γ of an $\mathbf{SO}(3)$ structure (M,g,Υ) is called a *characteristic connection* if its torsion T_{ijk} is totally skew symmetric.

The consideration of this section can be summarised in the following theorem.

Theorem 5.5. *Among all $\mathbf{SO}(3)$ structures only the nearly integrable ones admit characteristic connection Γ. Every nearly integrable $\mathbf{SO}(3)$ structure defines Γ uniquely.*

Remark 5.6. Note, that out of *a priori* 50 independent components of the Levi-Civita connection $\overset{LC}{\Gamma}$, the nearly integrable condition (5.1) excludes 25. Thus, heuristically, the nearly integrable $\mathbf{SO}(3)$ structures constitute 'a half' of all the possible $\mathbf{SO}(3)$ structures.

Remark 5.7. Note, that given a nearly integrable $\mathbf{SO}(3)$ structure its totally skew symmetric torsion T_{ijk} defines the torsion 3-form

$$T=\tfrac{1}{6}T_{ijk}\theta^i\wedge\theta^j\wedge\theta^k.$$

Since

$${\textstyle\bigwedge}^3\mathbb{R}^5={\textstyle\bigwedge}^2\mathbb{R}^5={\textstyle\bigwedge}^2_3\oplus{\textstyle\bigwedge}^2_7$$

we have two kinds of skew symmetric torsions of 'pure type' - those for which T belongs to the representation $\bigwedge^2_3$ and those whose T is in $\bigwedge^2_7$.
Note that for an $\mathbf{SO}(3)$ structure with arbitrary $\mathfrak{so}(3)$ connection its torsion T_{ijk} belongs to $\bigwedge^2\mathbb{R}^5\otimes\mathbb{R}^5$. Thus, according to the discussion at the beginning of this section, under the action of $\mathbf{SO}(3)$, such T_{ijk} satisfy

$$T_{ijk}\in{\textstyle\bigwedge}^2\mathbb{R}^5\otimes\mathbb{R}^5=\Big([\mathfrak{so}(3)\otimes\mathbb{R}^5]\oplus{\textstyle\bigwedge}^3\mathbb{R}^5\Big)\oplus\mathbb{R}^{25}.$$

Obviously, $\mathbb{R}^{25}$ further decomposes onto irreducibles: $\mathbb{R}^{25}=\mathbb{R}^5\oplus\mathbb{R}^9\oplus\mathbb{R}^{11}$.

We close this section with the analysis of the $\mathbf{SO}(3)$ decomposition of the curvature

$$K^i{}_j=\tfrac{1}{2}K^i{}_{jkl}\theta^k\wedge\theta^l=\mathrm{d}\Gamma^i{}_j+\Gamma^i{}_k\wedge\Gamma^k{}_j$$

of the characteristic connection Γ. Since $K_{ijkl}\in\mathfrak{so}(3)\otimes\bigwedge^2\mathbb{R}^5$, this is given by the following proposition.

Proposition 5.8. *The projectors onto the irreducible components of the decomposition*

$$\mathfrak{so}(3)\otimes{\textstyle\bigwedge}^2\mathbb{R}^5\cong{\textstyle\bigodot}^2_1\oplus{\textstyle\bigwedge}^2_3\oplus{\textstyle\bigwedge}^2_7\oplus{\textstyle\bigodot}^2_5\oplus{\textstyle\bigodot}^2_9\oplus{\textstyle\bigwedge}^1_5,\tag{5.5}$$

are:

$$K_{ijkl} \longmapsto K_{[ijkl]} \in \textstyle\bigwedge^4\mathbb{R}^5 = \bigwedge^1_5$$
$$K_{ijkl} \longmapsto K_{ijil} =: k_{jl} \longmapsto k_{[jl]} \in \textstyle\bigwedge^2_3 \oplus \bigwedge^2_7$$
$$K_{ijkl} \longmapsto K_{ijil} = k_{jl} \longmapsto (k_{(jl)} - \frac{1}{5}k_{ii}\ g_{jl}) \in \bigodot^2_5 \oplus \bigodot^2_9$$
$$K_{ijkl} \longmapsto K_{ijil} = k_{jl} \longmapsto k_{ii} \in \bigodot^2_1.$$

Remark 5.9. Note that the curvature tensor decomposition (5.5) is an analog, but not just the refinement, of the standard Riemann tensor components. The $\mathfrak{so}(3)$-connection, we investigate, is *not* in general (compare Section 4.3 for the exception) the torsion free connection and so the curvature does not have the usual Riemann tensor symmetries.

6. Homogeneous examples

In the present section we look for examples of nearly integrable $\mathbf{SO}(3)$ structures admitting transitive symmetry groups.

Using the fact that the possible subgroups of $\mathbf{SO}(3)$ may have dimensions 0,1,3 we get the following proposition.

Proposition 6.1. *A transitive symmetry group G of an $\mathbf{SO}(3)$ structure may have the following dimension: $5, 6$ or 8.*

6.1. **Examples with 8-dimensional symmetry group.** If the group of transitive symmetries G is 8-dimensional, the $\mathbf{SO}(3)$ frame bundle $F(M)$ may be identified with G. Then, the problem of finding all the examples with such group of symmetries is equivalent to find those Gs for which the basis of left invariant forms $(\tilde{\theta}^1, \tilde{\theta}^2, \tilde{\theta}^3, \tilde{\theta}^4, \tilde{\theta}^5, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3)$ satisfy the pull-backed Cartan equations (4.4)-(4.6) with the torsion coefficients T_{ijk} and the curvature coefficients ${r^I}_{jk}$ constant on G. This is a purely algebraic problem with the following solution.

Proposition 6.2. *There are only three different examples of nearly integrable $\mathbf{SO}(3)$ geometries with 8-dimensional symmetry group. These are the torsion-free models:*

$$M_+ = \mathbf{SU}(3)/\mathbf{SO}(3), \qquad M_0 = (\mathbf{SO}(3)\times_\rho\mathbb{R}^5)/\mathbf{SO}(3), \qquad M_- = \mathbf{SL}(3,\mathbb{R})/\mathbf{SO}(3).$$

6.2. **Examples with 6-dimensional symmetry group.** To obtain all the examples with 6-dimensional transitive symmetry groups we do as follows. We further reduce the lifted system (4.4)-(4.6) from the $\mathbf{SO}(3)$ frame bundle $F(M)$ to a 6-dimensional group G fibred over M. We will identify G with the transitive symmetry group of the considered structure. Thus, M will be a homogeneous space

$$M = G/H$$

with H - a 1-dimensional subgroup of G.

The reduction of the lifted system (4.4)-(4.6) from $F(M)$ to G implies that on G, the two of the connection 1-forms $(\tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3)$, say $\tilde{\gamma}^1$ and $\tilde{\gamma}^2$, must be $\mathbb{R}$-linearly dependent on the lift of the adapted coframe $(\tilde{\theta}^1, \tilde{\theta}^2, \tilde{\theta}^3, \tilde{\theta}^4, \tilde{\theta}^5)$. Thus, in such case, the basis for 1-forms on G is $(\tilde{\theta}^1, \tilde{\theta}^2, \tilde{\theta}^3, \tilde{\theta}^4, \tilde{\theta}^5, \tilde{\gamma}^3)$. It is subject to the lift of the structure equations (4.4)-(4.6). One of the integrability conditions for these equations require that $\tilde\gamma^1$ and $\tilde\gamma^2$ must be of the form

$$\begin{aligned}\tilde\gamma^1 &= -b\,\tilde\theta^3 + a\,\tilde\theta^5,\\ \tilde\gamma^2 &= a\,\tilde\theta^3 + b\,\tilde\theta^5,\end{aligned} \tag{6.1}$$

where $a, b \in \mathbb{R}$.

Due to the fact that all the coefficients in the pullback of the Cartan structure equations (4.4)-(4.6) are constant on G, the closure of these equations implies the following proposition.

Proposition 6.3. *All* $\mathbf{SO}(3)$ *nearly integrable structures with 6-dimensional symmetry group have (skew symmetric) torsion of the form*

$$T = t_1\theta^1\wedge\theta^2\wedge\theta^4 + t_2\theta^1\wedge\theta^3\wedge\theta^5.$$

There are three families of such geometries

1. $b = t_1 = t_2 = 0$, *and* a *arbitrary;*
2. $a = b = 0$ *and* t_1, t_2 *arbitrary;*
3. $a = 0$, $b = \frac{t_1 - 2t_2}{2\sqrt{3}}$ *and* t_1, t_2 *arbitrary.*

Below we discuss all possibilities.

The point 1 of Proposition 6.3. In this case the torsion is obviously zero and the $\mathfrak{so}(3)$ curvature form is

$$K = -a^2\Big[\kappa^1\cdot E_1 + \kappa^2\cdot E_2 + \kappa^3\cdot E_3\Big],$$

where $\kappa^1, \kappa^2, \kappa^3$ are given by (4.11). Thus, in this case, we reconstruct two of the three torsion-free examples. For $a = 0$ the respective $\mathbf{SO}(3)$ structure is equivalent to M_0. For $a \neq 0$ we reconstruct the structure $M_- = \mathbf{SL}(3,\mathbb{R})/\mathbf{SO}(3)$. The latter case corresponds to the following 6-dimensional subgroup of $\mathbf{SL}(3,\mathbb{R})$

$$G = \left\{ M = \begin{pmatrix} d & e & f\\ g & h & k\\ 0 & 0 & m\end{pmatrix} : \quad \det M = 1\right\}, \qquad H = \mathbf{SO}(2) = \left\{\begin{pmatrix} \cos t & \sin t & 0\\ -\sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}\right\}.$$

The point 2 of Proposition 6.3. In this case an invariant coframe $(\tilde\theta^1, \dots, \tilde\theta^5, \tilde\gamma^3)$ on G satisfies the following differential system:

$$\begin{aligned}
\mathrm{d}\tilde\theta^1 &= t_1\tilde\theta^2\wedge\tilde\theta^4 + t_2\tilde\theta^3\wedge\tilde\theta^5\\
\mathrm{d}\tilde\theta^2 &= -t_1\tilde\theta^1\wedge\tilde\theta^4 + 2\tilde\theta^4\wedge\tilde\gamma^3\\
\mathrm{d}\tilde\theta^3 &= -t_2\tilde\theta^1\wedge\tilde\theta^5 + \tilde\theta^5\wedge\tilde\gamma^3\\
\mathrm{d}\tilde\theta^4 &= t_1\tilde\theta^1\wedge\tilde\theta^2 - 2\tilde\theta^2\wedge\tilde\gamma^3\\
\mathrm{d}\tilde\theta^5 &= t_2\tilde\theta^1\wedge\tilde\theta^3 - \tilde\theta^3\wedge\tilde\gamma^3\\
\mathrm{d}\tilde\gamma^3 &= -\frac{t_1t_2}{2}(\tilde\theta^3\wedge\tilde\theta^5 + 2\tilde\theta^2\wedge\tilde\theta^4).
\end{aligned}$$

The symmetry group $G = G_{(t_1,t_2)}$ depends on the torsion parameters (t_1, t_2). We depict the possible Gs on the (t_1, t_2)-plane in Figure 1.

Below we discuss each $G_{(t_1,t_2)}$ separately.

(i) $t_1\,t_2\,(t_1 - 2t_2) \neq 0$. In this case G is always of the form

$$G = G_1 \times G_2, \tag{6.2}$$

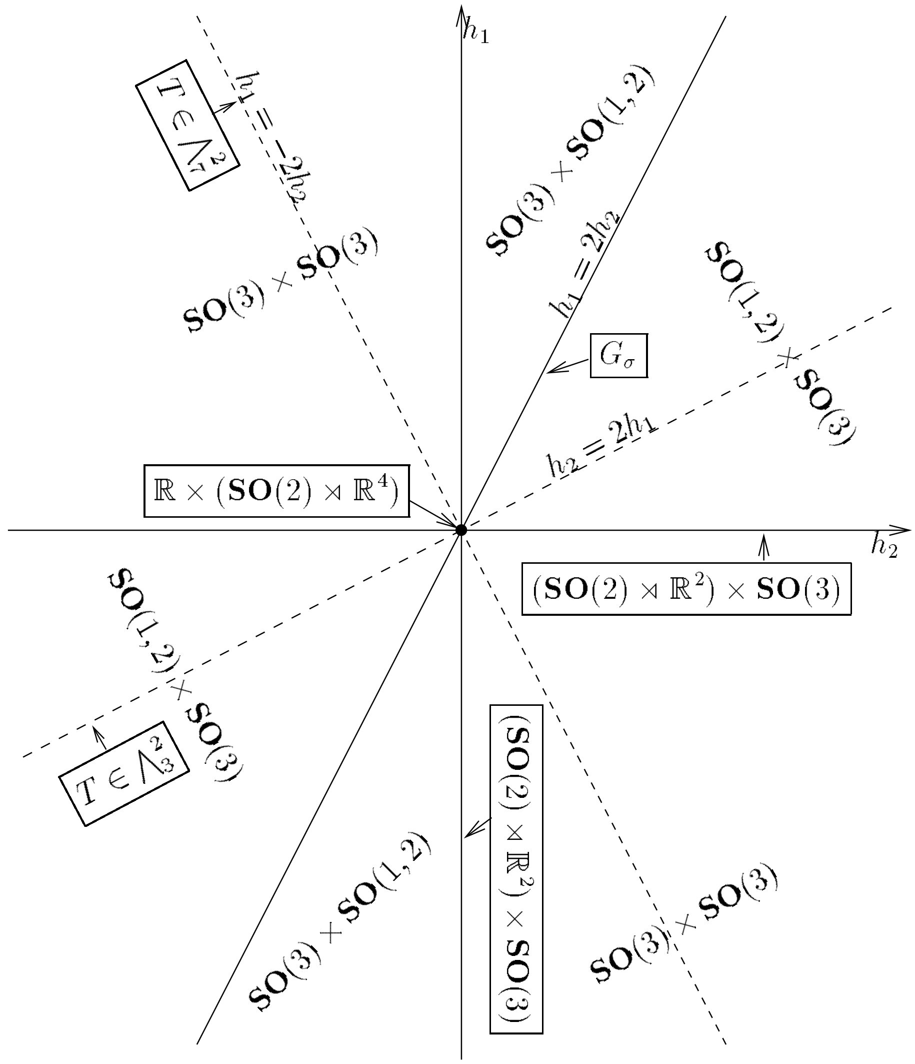

FIGURE 1. Groups $G_{(t_1,t_2)}$ of $\mathbf{SO}(3)$ structures of Proposition 6.3 (2)

where G_j is either $\mathbf{SO}(3)$ or $\mathbf{SO}(1,2)$ – see Figure 1. There is a standard inclusion of $\mathbf{SO}(2)$ in both of the above groups. The inclusion of $H = \mathbf{SO}(2)$ in the product G is given by $\mathbf{SO}(2) \ni h \longmapsto (h^2, h) \in G_1 \times G_2$. We consider the standard 3-dimensional representations of $\mathfrak{so}(1,2)$ and $\mathfrak{so}(3)$ so that the Maurer-Cartan form $\tilde{\theta}_{MC}$ on G is given by

$$\tilde{\theta}_{MC} = \begin{pmatrix} 0 & c\tilde{\alpha}^1 + 2\tilde{\eta} & \tilde{\alpha}^2 & 0 & 0 & 0 \\ -(c\tilde{\alpha}^1 + 2\tilde{\eta}) & 0 & \tilde{\alpha}^4 & 0 & 0 & 0 \\ \epsilon_1\tilde{\alpha}^2 & \epsilon_1\tilde{\alpha}^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2c\tilde{\alpha}^1 + \tilde{\eta} & \tilde{\alpha}^3 \\ 0 & 0 & 0 & -(-2c\tilde{\alpha}^1 + \tilde{\eta}) & 0 & \tilde{\alpha}^5 \\ 0 & 0 & 0 & \epsilon_2\tilde{\alpha}^3 & \epsilon_2\tilde{\alpha}^5 & 0 \end{pmatrix}, \tag{6.3}$$

where

$$\begin{aligned} c &= \tfrac{1}{\sqrt{5}},\\ \epsilon_1 &= -\mathrm{sgn}[t_1(t_1-2t_2)],\\ \epsilon_2 &= \mathrm{sgn}[t_2(t_1-2t_2)] \end{aligned}$$

and $(\tilde\alpha^i,\tilde\eta)$ is a left invariant coframe on G. We have the following relations between $(\tilde\alpha^i,\tilde\eta)$ and the canonical coframe $(\tilde\theta^i,\tilde\gamma^3)$:

$$\begin{aligned} &\tilde\gamma^3 = \tilde\eta - \tfrac{2t_1+t_2}{\sqrt{5}(t_1-2t_2)}\cdot\tilde\alpha^1 && \tilde\theta^1 = \tfrac{\sqrt{5}}{t_1-2t_2}\cdot\tilde\alpha^1\\ &\tilde\theta^2 = \sqrt{\tfrac{\epsilon_1}{-t_1(t_1-2t_2)}}\cdot\tilde\alpha^2 && \tilde\theta^4 = \sqrt{\tfrac{\epsilon_1}{-t_1(t_1-2t_2)}}\cdot\tilde\alpha^4\\ &\tilde\theta^3 = \sqrt{\tfrac{2\epsilon_2}{t_2(t_1-2t_2)}}\cdot\tilde\alpha^3 && \tilde\theta^5 = \sqrt{\tfrac{2\epsilon_2}{t_2(t_1-2t_2)}}\cdot\tilde\alpha^5. \end{aligned} \tag{6.4}$$

Now, we take $(\tilde g,\tilde\Upsilon)$ in the canonical form (4.16), (4.17). These descend to the $\mathbf{SO}(3)$ structure (g,Υ) on $M=G/H$ due to the isotropy invariance of $(\tilde g,\tilde\Upsilon)$. The G-invariant $\mathfrak{so}(3)$ connection Γ on M has the following form

$$\Gamma = \Gamma_0 - \tfrac{1}{5}(2t_1+t_2)\theta^1\cdot E_3,$$

where Γ_0 is the canonical connection on the reductive homogeneous space G/H – see [5].

Remark 6.4. It is worth to notice that on the line $t_2=-2t_1$ the connection Γ coincides with the canonical connection Γ_0. The example from this line corresponding to $(t_1,t_2)=(\frac{1}{5},-\frac{2}{5})$ is due to Th. Friedrich [4].

In general, the torsion T has components in the both possible irreducible $\mathbf{SO}(3)$ representations $\bigwedge^2_3$ and $\bigwedge^2_7$ (see Remark 5.7). On the line $t_2=2t_1$ the torsion is of pure type $\bigwedge^2_3$; on the line $t_1=-2t_2$ it is of pure type $\bigwedge^2_7$ – see the Figure 1.

The $\mathfrak{so}(3)$ curvature is of the form

$$K = -t_1t_2\kappa^3\cdot E_3.$$

It belongs to $\mathfrak{so}(3)\otimes\mathfrak{so}(3)$. If $t_1t_2\neq 0$ the curvature has non-zero values in all of the components $\bigodot^2_1\oplus\bigodot^2_5\oplus\bigwedge^1_5$ of the irreducible decomposition (5.5).

(ii) $t_1=0,\ t_2\neq 0$. The group G_1 of the previous case contracts and the symmetry group becomes

$$G = (\mathbf{SO}(2)\rtimes\mathbb{R}^2)\times\mathbf{SO}(3).$$

The inclusion of $H=\mathbf{SO}(2)$ in the product G is given by

$$\mathbf{SO}(2)\ni h\longmapsto (h^2,h).$$

The Maurer-Cartan form on G has the form (6.3) with $\epsilon_1=0$. The relations (6.4) remain valid after passing to the limit $\frac{\epsilon_1}{t_1}\to -\mathrm{sgn}[(t_1-2t_2)]=\mathrm{sgn}t_2$

$$\begin{aligned} &\tilde\gamma^3 = \tilde\eta + \tfrac{1}{2\sqrt{5}}\cdot\tilde\alpha^1 && \tilde\theta^1 = -\tfrac{\sqrt{5}}{2t_2}\cdot\tilde\alpha^1\\ &\tilde\theta^2 = \tfrac{1}{\sqrt{2|t_2|}}\cdot\tilde\alpha^2 && \tilde\theta^4 = \tfrac{1}{\sqrt{2|t_2|}}\cdot\tilde\alpha^4\\ &\tilde\theta^3 = \tfrac{1}{|t_2|}\cdot\tilde\alpha^3 && \tilde\theta^5 = \tfrac{1}{|t_2|}\cdot\tilde\alpha^5. \end{aligned}$$

These define an $\mathbf{SO}(3)$ structure on $M = G/H$ in an analogous way as in the previous case. The torsion $T \neq 0$ is never of a pure type and the $\mathfrak{so}(3)$ curvature $K \equiv 0$.

(iii) $t_2 = 0,\ t_1 \neq 0$. This case is the same as the previous one. One has to put $\epsilon_2 = 0$ in (6.3) and $\frac{\epsilon_2}{t_2} \to \mathrm{sgn} t_1$ in (6.4). The statements about curvature and torsion are the same as in the previous point.

(iv) $t_1 = 0,\ t_2 = 0$. In this case both the torsion and the $\mathfrak{so}(3)$ curvature vanish. Thus, this case corresponds to the flat model M_0. Hence the symmetry group G is extendable to $\mathbf{SO}(3) \times_\rho \mathbb{R}^5$. For the purpose of the next point it is useful to analyse G more carefully. Let τ be the standard representation of $\mathbf{SO}(2)$ in $\mathbb{R}^2$. In conform with the Figure 1 we observe that

$$G = \mathbb{R} \times (\mathbf{SO}(2) \rtimes \mathbb{R}^4), \qquad H = \mathbf{SO}(2),$$

where the semi-direct product is taken with respect to the representation $\tau^2 \oplus \tau$ of $\mathbf{SO}(2)$ on $\mathbb{R}^4$. The Maurer-Cartan form $\tilde{\theta}_{MC}$ on G is

$$\tilde{\theta}_{MC} = \begin{pmatrix} 0 & 2\tilde{\eta} & \tilde{\alpha}^2 & 0 & 0 & 0 & 0 \\ -2\tilde{\eta} & 0 & \tilde{\alpha}^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{\eta} & \tilde{\alpha}^3 & 0 \\ 0 & 0 & 0 & -\tilde{\eta} & 0 & \tilde{\alpha}^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\alpha}^1 \end{pmatrix}. \tag{6.5}$$

The relation between $(\tilde{\alpha}^j, \tilde{\eta})$ and $(\tilde{\theta}^j, \tilde{\gamma}^3)$ is $\tilde{\theta}^j = \tilde{\alpha}^j$ and $\tilde{\gamma}^3 = \tilde{\eta}$.

(v) $t_1 = 2t_2,\ t_2 \neq 0$. In this case the group $G = G_\sigma$ has the following abstract description. We present the Lie algebra of G as a central extension by $\mathbb{R}$ of a 5-dimensional algebra $\mathfrak{l}$. Let us recall (see [13]) that such extensions are classified by closed 2-forms $\sigma \in \bigwedge^2 \mathfrak{l}^*$.

Let $L = \mathbf{SO}(2) \rtimes \mathbb{R}^4$ with the representation $\tau^2 \oplus \tau$ of $\mathbf{SO}(2)$ as in the previous point; $\mathfrak{l}$ is the Lie algebra of L. We take the Maurer-Cartan forms $(\tilde{\alpha}^2, \tilde{\alpha}^3, \tilde{\alpha}^4, \tilde{\alpha}^5, \tilde{\eta})$, defined in (6.5), as the basis of the left invariant forms on L. One can check that the following 2-form on L

$$\tilde{\sigma} = \tilde{\alpha}^3 \wedge \tilde{\alpha}^5 + 2\tilde{\alpha}^2 \wedge \tilde{\alpha}^4, \qquad \sigma \colon = \tilde{\sigma}_e \in \textstyle\bigwedge^2 \mathfrak{l}^* \tag{6.6}$$

is closed.

We define the Lie algebra $\mathfrak{g} = \mathfrak{g}_\sigma$ as a central extension of $\mathfrak{l}$ by $\mathbb{R}$

$$0 \to \mathbb{R} \longrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{l} \to 0 \tag{6.7}$$

characterised by the element σ. Let $G = G_\sigma$ be a Lie group with Lie algebra $\mathfrak{g}_\sigma$. We extend the basis of left invariant forms on L to the (left invariant) basis $(\tilde{\alpha}^1, \tilde{\alpha}^2, \tilde{\alpha}^3, \tilde{\alpha}^4, \tilde{\alpha}^5, \tilde{\eta})$ on G. The differential $\mathrm{d}\tilde{\alpha}^1$ is (see [13])

$$\mathrm{d}\tilde{\alpha}^1 = \tilde{\sigma}.$$

The exact sequence of Lie algebras (6.7) has a partial splitting $s : \mathfrak{so}(2) \hookrightarrow \mathfrak{g}$ (i.e. the composition $\pi \circ s$ is the inclusion of $\mathfrak{so}(2)$ into $\mathfrak{l}$) which defines the inclusion $H = \mathbf{SO}(2) \subset G$.

Finally, the relation between this basis $(\tilde{\alpha}^j, \tilde{\eta})$ and the canonical coframe $(\tilde{\theta}^j, \tilde{\gamma}^3)$ is as follows

$$\tilde{\gamma}^3 = \tilde{\eta} - t_2^2{\cdot}\tilde{\alpha}^1, \quad \tilde{\theta}^1 = t_2{\cdot}\tilde{\alpha}^1, \quad \tilde{\theta}^2 = \tilde{\alpha}^2, \quad \tilde{\theta}^3 = \tilde{\alpha}^3, \quad \tilde{\theta}^4 = \tilde{\alpha}^4, \quad \tilde{\theta}^5 = \tilde{\alpha}^5.$$

These define a nearly integrable $\mathbf{SO}(3)$ structure on $M = G/H$ as in each of the previous cases. The torsion $T \neq 0$ is never of a pure type and the $\mathfrak{so}(3)$-curvature has the form

$$K = -2(t_2)^2\kappa^3 \cdot E_3.$$

The point 3 of Proposition 6.3. We start with the observation that the line $t_1 = 2t_2$ on the (t_1, t_2)-plane in the present case and the line $t_1 = 2t_2$ of the previous case coincide (see Proposition 6.3). Thus, in the entire analysis of this case, we assume that $t_1 \neq 2t_2$.

We have the following differential system on G

$$\begin{aligned}
{\rm d}\tilde\theta^1 &= t_1\tilde\theta^2\wedge\tilde\theta^4 + (t_1-t_2)\tilde\theta^3\wedge\tilde\theta^5\\
{\rm d}\tilde\theta^2 &= -t_1\tilde\theta^1\wedge\tilde\theta^4 + 2\tilde\theta^4\wedge\tilde\gamma^3\\
{\rm d}\tilde\theta^3 &= -\tfrac12 t_1\tilde\theta^1\wedge\tilde\theta^5 + \tilde\theta^5\wedge\tilde\gamma^3 + \tfrac{t_1-2t_2}{2\sqrt3}\tilde\theta^2\wedge\tilde\theta^3 + \tfrac{t_1-2t_2}{2\sqrt3}\tilde\theta^4\wedge\tilde\theta^5\\
{\rm d}\tilde\theta^4 &= t_1\tilde\theta^1\wedge\tilde\theta^2 - 2\tilde\theta^2\wedge\tilde\gamma^3\\
{\rm d}\tilde\theta^5 &= \tfrac12 t_1\tilde\theta^1\wedge\tilde\theta^3 - \tilde\theta^3\wedge\tilde\gamma^3 - \tfrac{t_1-2t_2}{2\sqrt3}\tilde\theta^2\wedge\tilde\theta^5 - \tfrac{t_1-2t_2}{2\sqrt3}\tilde\theta^3\wedge\tilde\theta^4\\
{\rm d}\tilde\gamma^3 &= -\tfrac23(t_1^2 - t_1t_2 + t_2^2)\tilde\theta^2\wedge\tilde\theta^4 - \tfrac12 t_1(t_1-t_2)\tilde\theta^3\wedge\tilde\theta^5.
\end{aligned}\tag{6.8}$$

It follows, that off the line $t_1 = 2t_2$, independently of (t_1, t_2), the symmetry group $G = G_{\epsilon\sigma}$ is a central extension of the group

$$L = \mathbf{SL}(2,\mathbb{R}) \rtimes \mathbb{R}^2$$

by a 1-dimensional Lie group. $G_{\epsilon\sigma}$ is characterised by a closed 2-form $\epsilon\sigma \in \bigwedge^2\mathfrak{l}^*$, $\epsilon = {\rm sgn}|t_1 - t_2|$.

It is convenient to choose the basis of left invariant forms $(\tilde\alpha^2, \tilde\alpha^3, \tilde\alpha^4, \tilde\alpha^5, \tilde\eta)$ on L so that the Maurer-Cartan form $\tilde\theta_{MC}$ on L reads

$$\tilde\theta_{MC} = \begin{pmatrix} -\tilde\alpha^4 & \tilde\alpha^2+\tilde\eta & \tilde\alpha^3\\ \tilde\alpha^2-\tilde\eta & \tilde\alpha^4 & \tilde\alpha^5\\ 0 & 0 & 0\end{pmatrix}.$$

Obviously, we have $\mathbf{SO}(2) \subset \mathbf{SL}(2,\mathbb{R}) \subset L$.

Now, the possible symmetry groups $G = G_{\epsilon\sigma}$, $\epsilon = 0, 1$, are presented on Figure 2. Below, we discuss cases $\epsilon = 1$ and $\epsilon = 0$ separately.

(i) $\epsilon = 1$. This case corresponds to $t_1 \neq t_2$. Here, we observe that

$$\tilde\sigma = \tilde\alpha^3\wedge\tilde\alpha^5, \qquad \sigma :\, = \tilde\sigma_e \in \textstyle\bigwedge^2\mathfrak{l}^*$$

is closed on L. It is this form that defines the desired central extension of the Lie algebra $\mathfrak{l}$ to the Lie algebra $\mathfrak{g} = \mathfrak{g}_\sigma$ of the symmetry group G_σ. Now, the forms $(\tilde\alpha^2, \tilde\alpha^3, \tilde\alpha^4, \tilde\alpha^5, \tilde\eta)$ extend to the left invariant forms on G_σ. Together with the form $\tilde\alpha^1$ such that ${\rm d}\tilde\alpha^1 = \tilde\sigma$ they define the left invariant coframe on G_σ. This coframe is related to the canonical coframe $(\tilde\theta^i, \tilde\gamma^3)$ of

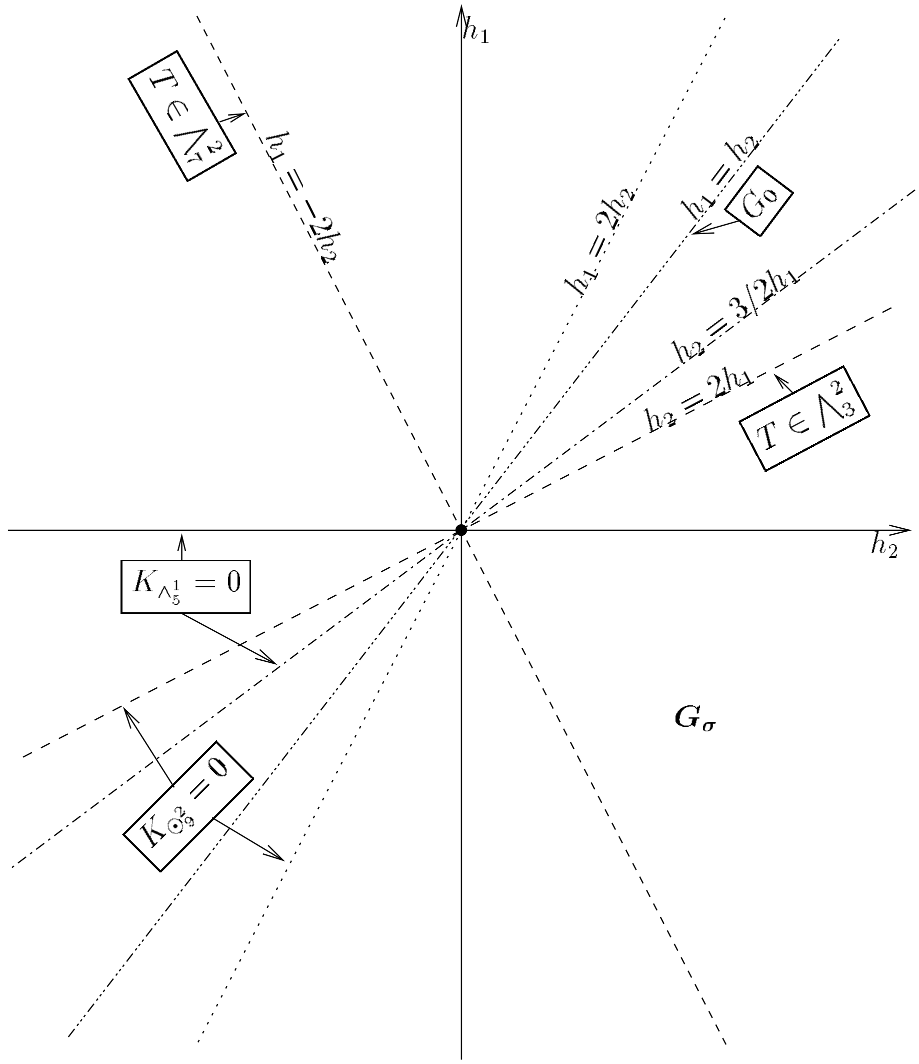

FIGURE 2. Groups $G = G_{\epsilon\sigma}$ of $\mathbf{SO}(3)$ structures of Proposition 6.3 (3)

(6.8) via

$$\begin{aligned}
&\tilde\theta^1 = -\tfrac{6t_1}{(t_1-2t_2)^2}\cdot\tilde\eta + \tfrac{2(t_1-t_2)}{\epsilon}\cdot\tilde\alpha^1 \\
&\tilde\theta^2 = \tfrac{2\sqrt{3}}{t_1-2t_2}\cdot\tilde\alpha^2 && \tilde\theta^4 = \tfrac{2\sqrt{3}}{t_1-2t_2}\cdot\tilde\alpha^4 \\
&\tilde\theta^3 = \tilde\alpha^3 - \tilde\alpha^5 && \tilde\theta^5 = \tilde\alpha^3 + \tilde\alpha^5 \\
&\tilde\gamma^3 = \tfrac{(t_1-2t_2)^2+3t_1^2}{(t_1-2t_2)^2}\cdot\tilde\eta - \tfrac{t_1(t_1-t_2)}{\epsilon}\cdot\tilde\alpha^1.
\end{aligned} \tag{6.9}$$

Now, in analogy to the case (v) of page 22, we use the partial splitting $s : \mathfrak{so}(2) \to \mathfrak{l}$, to recover the inclusion $H = \mathbf{SO}(2) \subset G_\sigma$. Then the $\mathbf{SO}(3)$ structure on $M = G/H$ is obtained via the standard procedure of taking

$(\tilde{g}, \tilde{\Upsilon})$ in the form (4.16), (4.17) and passing to the quotient structure (g, Υ). The G-invariant $\mathfrak{so}(3)$ connection on M is given by

$$\Gamma = \Gamma_0 - \tfrac{t_1-2t_2}{2\sqrt{3}}\theta^3{\cdot}E_1 + \tfrac{t_1-2t_2}{2\sqrt{3}}\theta^5{\cdot}E_2 - \tfrac{t_1}{2}\theta^1{\cdot}E_3,$$

where Γ_0 is the canonical connection on G/H.

As in the entire point 2 of the present Proposition, the torsion T has the pure type $\bigwedge^2_3$ iff $t_2 = 2t_1$; it is of the pure type $\bigwedge^2_7$ iff $t_1 = -2t_2$; in all other cases it is not of a pure type (see Figure 2).

In contrast to the point 2 of the present Proposition, the $\mathfrak{so}(3)$ curvature has the form

$$\begin{aligned} K = \frac{1}{12}\Big[&\Big(\sqrt{3}t_1(t_1-2t_2)\,\theta^1\wedge\theta^5 - (t_1-2t_2)^2(\theta^2\wedge\theta^3+\theta^4\wedge\theta^5)\Big){\cdot}E_1 \\ &+\Big(\sqrt{3}t_1(t_1-2t_2)\,\theta^1\wedge\theta^3 - (t_1-2t_2)^2(\theta^2\wedge\theta^5+\theta^3\wedge\theta^4)\Big){\cdot}E_2 \\ &+\Big(-8(t_1^2-t_1t_2+t_2^2)\,\theta^2\wedge\theta^4 + (-7t_1^2+10t_1t_2-4t_2^2)\,\theta^3\wedge\theta^5\Big){\cdot}E_3\Big], \end{aligned}$$

and (off the line $t_1 = 2t_2$) it is never of type $\mathfrak{so}(3)\otimes\mathfrak{so}(3)$. In general, the curvature can assume values in all of the components of the decomposition (5.5), but $\bigwedge^2_3$ and $\bigwedge^2_7$:

$$K \in \textstyle\bigodot^2_1 \oplus \bigodot^2_5 \oplus \bigodot^2_9 \oplus \bigwedge^1_5.$$

Independently of (t_1, t_2) the curvature has always the $\bigodot^2_1$ and $\bigodot^2_5$ part; it is without the $\bigodot^2_9$ component on the line $t_2 = 2t_1$ and without the $\bigwedge^1_5$ component on lines $t_1 = 0$ and $3t_1 = 2t_2$ – see Figure 2.

(ii) $\epsilon = 0$. This corresponds to the line $t_1 = t_2$. Now, all the formulas of the previous case remain valid, but the formulas for $\tilde{\theta}^1$ and $\tilde{\gamma}^3$. To get correct expressions for them, one has to pass to the limit $\frac{t_1-t_2}{\epsilon} \to 1$ in (6.9).

It is worthwhile to note that the central extension G_0 is, in this case, trivial. Hence, the symmetry group is simply a product

$$G_0 = \mathbb{R}\times(\mathbf{SL}(2,\mathbb{R})\rtimes\mathbb{R}^2) \qquad \text{with} \qquad H = \mathbf{SO}(2)\subset\mathbf{SL}(2,\mathbb{R}).$$

6.3. **Examples with 5-dimensional symmetry group.** The first set of examples in this section is characterised by the requirement that a nearly integrable $\mathbf{SO}(3)$ geometry has flat characteristic connection. The full list of such geometries is given in Section 6.3.1. In Theorem 6.5 we prove that flatness of the characteristic connection implies that the corresponding nearly integrable $\mathbf{SO}(3)$ geometry has at least 5-dimensional transitive symmetry group. Inspection of the examples of Section 6.3.1 shows that, in generic cases, their symmetry groups are strictly 5-dimensional.

Of course, examples with flat characteristic connections do not exhaust the list of all nearly integrable $\mathbf{SO}(3)$ structures with strictly 5-dimensional transitive symmetry group. We obtained another two classes of examples assuming that, in addition to the action of a 5-dimensional transitive symmetry group, the torsion of characteristic connection is of pure type. The results are given in respective Sections 6.3.2 and 6.3.3. It is worth noticing that it was possible to find *all* structures with 5-dimensional transitive symmetry group and torsion in $\bigwedge^2_3$ (see Theorem 6.7). In case of $\bigwedge^2_7$ type torsion we were only able to find a 2-parameter family of examples.

6.3.1. *Vanishing curvature.*

Theorem 6.5. *Let (M, g, Υ) be a nearly integrable* $\mathbf{SO}(3)$ *structure with vanishing curvature of its characteristic connection. Then M has a structure of a 5-dimensional Lie group G and the* $\mathbf{SO}(3)$*-structure is G-invariant.*

Proof. Since the characteristic connection of an $\mathbf{SO}(3)$ structure is flat, one can assume that the connection (locally) vanishes. Thus, in a suitably chosen local coframe (4.1) the first Cartan structure equations are

$$\begin{aligned}
\mathrm{d}\theta^1 &= t_1\theta^2\wedge\theta^3 + t_2\theta^2\wedge\theta^4 + t_3\theta^2\wedge\theta^5 + t_4\theta^3\wedge\theta^4 + t_5\theta^3\wedge\theta^5 + t_6\theta^4\wedge\theta^5\\
\mathrm{d}\theta^2 &= -t_1\theta^1\wedge\theta^3 - t_2\theta^1\wedge\theta^4 - t_3\theta^1\wedge\theta^5 + t_7\theta^3\wedge\theta^4 + t_8\theta^3\wedge\theta^5 + t_9\theta^4\wedge\theta^5\\
\mathrm{d}\theta^3 &= t_1\theta^1\wedge\theta^2 - t_4\theta^1\wedge\theta^4 - t_5\theta^1\wedge\theta^5 - t_7\theta^2\wedge\theta^4 - t_8\theta^2\wedge\theta^5 + t_{10}\theta^4\wedge\theta^5\\
\mathrm{d}\theta^4 &= t_2\theta^1\wedge\theta^2 + t_4\theta^1\wedge\theta^3 - t_6\theta^1\wedge\theta^5 + t_7\theta^2\wedge\theta^3 - t_9\theta^2\wedge\theta^5 - t_{10}\theta^3\wedge\theta^5\\
\mathrm{d}\theta^5 &= t_3\theta^1\wedge\theta^2 + t_5\theta^1\wedge\theta^3 + t_6\theta^1\wedge\theta^4 + t_8\theta^2\wedge\theta^3 + t_9\theta^2\wedge\theta^4 + t_{10}\theta^3\wedge\theta^4.
\end{aligned}\tag{6.10}$$

Here the functional coefficients $t_i, i = 1, 2, \dots, 10$ are related to the torsion 3-form T via:

$$(6.11)\quad T = t_1\theta^1\wedge\theta^2\wedge\theta^3 + t_2\theta^1\wedge\theta^2\wedge\theta^4 + t_3\theta^1\wedge\theta^2\wedge\theta^5 + t_4\theta^1\wedge\theta^3\wedge\theta^4 +$$
$$t_5\theta^1\wedge\theta^3\wedge\theta^5 + t_6\theta^1\wedge\theta^4\wedge\theta^5 + t_7\theta^2\wedge\theta^3\wedge\theta^4 + t_8\theta^2\wedge\theta^3\wedge\theta^5 + t_9\theta^2\wedge\theta^4\wedge\theta^5 + t_{10}\theta^3\wedge\theta^4\wedge\theta^5.$$

Now, the Bianchi identities are equivalent to the following integrability conditions of the system (6.10):

(a) all the functions $t_i, i = 1, 2, \dots, 10$ are *constants*
(b) they are subject to the following constraints

$$\begin{aligned}
t_3t_{10} + t_6t_8 - t_5t_9 &= 0\\
t_1t_{10} + t_5t_7 - t_4t_8 &= 0\\
t_3t_7 - t_2t_8 + t_1t_9 &= 0\\
t_2t_{10} + t_6t_7 - t_4t_9 &= 0\\
t_3t_4 - t_2t_5 + t_1t_6 &= 0.
\end{aligned}\tag{6.12}$$

The point (a) above proves the theorem, showing that M can be identified with the symmetry group G which has t_i as its structure constants. □

Below we solve conditions (6.12) to fully characterise G under the genericity assumption

$$t_{10} \neq 0.$$

If this is assumed the general solution of system (6.12) is

$$t_1 = \tfrac{1}{t_{10}}(t_4t_8 - t_5t_7),\quad t_2 = \tfrac{1}{t_{10}}(t_4t_9 - t_6t_7),\quad t_3 = \tfrac{1}{t_{10}}(t_5t_9 - t_6t_8).$$

Now it is easy to see that the following linearly independent ($t_{10} \neq 0$!) 1-forms

$$\begin{aligned}
\alpha^4 &= t_{10}\theta^1 - t_6\theta^3 + t_5\theta^4 - t_4\theta^5\\
\alpha^5 &= t_{10}\theta^2 - t_9\theta^3 + t_8\theta^4 - t_7\theta^5
\end{aligned}$$

are closed. They can be further supplemented to a coframe $(\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5)$ on M such that

$$\begin{aligned}
\mathrm{d}\alpha^1 &= \alpha^2 \wedge \alpha^3 \\
\mathrm{d}\alpha^2 &= \alpha^3 \wedge \alpha^1 \\
\mathrm{d}\alpha^3 &= \alpha^1 \wedge \alpha^2 \\
\mathrm{d}\alpha^4 &= 0 \\
\mathrm{d}\alpha^5 &= 0.
\end{aligned}$$

This proves the following proposition.

Proposition 6.6. *If the torsion coefficient $t_{10} \neq 0$, the symmetry group G of a nearly integrable $\mathbf{SO}(3)$-structure with flat characteristic connection is isomorphic to $\mathbf{SO}(3) \times \mathbb{R}^2$.*

6.3.2. *Torsion in $\bigwedge^2_3$.*

In the following a parameter $\delta = 0, 1$ labels 5-dimensional Lie groups G_δ. By definition $G_0 = \mathbf{SO}(3) \times Aff(1)$, the direct product of $\mathbf{SO}(3)$ and the affine group $Aff(1)$ in dimension 1. We characterise the group G_1 by specifying the structure equations for a left invariant coframe on G_1. Thus, G_1 is such a 5-dimensional Lie group for which there exists a coframe $(\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5)$ which satisfies the following equations:

$$\begin{aligned}
\mathrm{d}\alpha^1 &= 0 \\
\mathrm{d}\alpha^2 &= \alpha^1 \wedge \alpha^2 \\
\mathrm{d}\alpha^3 &= -2\alpha^1 \wedge \alpha^3 \\
\mathrm{d}\alpha^4 &= -\alpha^1 \wedge \alpha^4 + \alpha^2 \wedge \alpha^3 \\
\mathrm{d}\alpha^5 &= \alpha^2 \wedge \alpha^4.
\end{aligned}$$

This group has the Lie algebra $\mathfrak{g}_1$ which is a central extension $0 \to \mathbb{R} \to \mathfrak{g}_1 \to \mathfrak{h} \to 0$ of the 4-dimensional Lie algebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} x^1 & x^3 & x^4 \\ 0 & -x^1 & x^2 \\ 0 & 0 & 0 \end{pmatrix},\ x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}$$

by a real line $\mathbb{R}$. The extension is given by means of a closed 2-form (see [13]) $\sigma = \alpha^2 \wedge \alpha^4$.

The following theorem is obtained by a successive application of the Bianchi identities on the system (4.5)-(4.6) in which the characteristic connection Γ is supposed to have torsion in $\bigwedge^2_3$ and for which all the connection coefficients, the curvature coefficients and the torsion coefficients are constants.

Theorem 6.7. *Let (M, g, Υ) be a nearly integrable $\mathbf{SO}(3)$ geometry admitting a 5-dimensional transitive symmetry group G. Assume, in addition, that the torsion of its characteristic connection is of pure type $\bigwedge^2_3$. Then:*

- *Modulo a constant* $\mathbf{SO}(3)$ *gauge transformation, it is defined by means of the adapted coframe* $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ *satisfying the following differential system:*

$$\begin{aligned}
d\theta^1 &= -\tfrac{2}{3}\sqrt{3}\varrho\epsilon\Big(\theta^2\wedge\theta^4 + (2-3\delta)\theta^3\wedge\theta^5\Big)\\
d\theta^2 &= -2\varrho\cos\varphi\ \theta^2\wedge\theta^4\\
d\theta^3 &= -\varrho\cos\varphi\ \theta^2\wedge\theta^5 + \sqrt{3}\varrho\epsilon(1-\delta)\theta^1\wedge\theta^5 + \varrho\epsilon\delta\theta^2\wedge\theta^3 + \varrho(\delta\epsilon-\sin\varphi)\theta^4\wedge\theta^5\\
d\theta^4 &= -2\varrho\sin\varphi\ \theta^2\wedge\theta^4\\
d\theta^5 &= \varrho\cos\varphi\ \theta^2\wedge\theta^3 + \sqrt{3}\varrho\epsilon(\delta-1)\theta^1\wedge\theta^3 - \varrho\epsilon\delta\theta^2\wedge\theta^5 - \varrho(\delta\epsilon+\sin\varphi)\theta^3\wedge\theta^4,
\end{aligned}$$

  with constant parameters $\varrho > 0, \varphi \in [0, 2\pi[, \epsilon = \pm 1, \delta = 0, 1.$
- $G \cong G_\delta$.
- *For all values of the parameters* $\epsilon, \delta, \varrho, \varphi$ *the curvature of the characteristic connection is of type* $\bigodot^2_1 \oplus \bigodot^2_5 \oplus \bigwedge^1_5$ *with all the irreducible components non-zero.*

6.3.3. *Torsion in* $\bigwedge^2_7$.

It is easy to check that an adapted coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ with differentials given by:

$$\begin{aligned}
d\theta^1 &= 0\\
d\theta^2 &= -\varrho\cos\varphi\theta^2\wedge\theta^4\\
d\theta^3 &= \tfrac{1}{2}\sqrt{3}\varrho\cos\varphi\ \theta^1\wedge\theta^3 - \tfrac{1}{2}\sqrt{3}\varrho\sin\varphi\ \theta^1\wedge\theta^5 - \tfrac{1}{2}\varrho\sin\varphi\ \theta^2\wedge\theta^3 + \tfrac{1}{2}\varrho\cos\varphi\ \theta^3\wedge\theta^4\\
d\theta^4 &= \varrho\sin\varphi\ \theta^2\wedge\theta^4\\
d\theta^5 &= -\tfrac{1}{2}\sqrt{3}\varrho\sin\varphi\ \theta^1\wedge\theta^3 - \tfrac{1}{2}\sqrt{3}\varrho\cos\varphi\ \theta^1\wedge\theta^5 - \tfrac{1}{2}\varrho\sin\varphi\ \theta^2\wedge\theta^5 - \tfrac{1}{2}\varrho\cos\varphi\ \theta^4\wedge\theta^5,
\end{aligned}$$

where the parameters $\varrho > 0,\ \varphi \in [0, 2\pi[$ are constants, defines a nearly integrable $\mathbf{SO}(3)$ geometry whose characteristic torsion has pure type $\bigwedge^2_7$. Its symmetry group is transitive, strictly 5-dimensional and has the following Maurer-Cartan form

$$\theta_{MC} = \begin{pmatrix} \alpha^4 & 0 & 0 & \alpha^1\\ 0 & \alpha^5 & 0 & \alpha^2\\ 0 & 0 & -(\alpha^4+\alpha^5) & \alpha^3\\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the forms $(\alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5)$ are related to the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ via an appropriate ϱ-dependent $\mathbf{GL}(5, \mathbb{R})$ transformation.

It is worth noting that the curvature of the characteristic connection in this 2-parameter family of examples is always of the type $\bigodot^2_1 \oplus \bigodot^2_9$ with both the irreducible components non-zero.

7. Ricci tensor and covariantly constant spinors

7.1. **Ricci tensor.** We have the following proposition

Proposition 7.1. *For every nearly integrable* $\mathbf{SO}(3)$ *structure* (M, g, Υ) *the Ricci tensor* Ric^{LC} *of the Levi-Civita connection* $\overset{LC}{\Gamma}$ *is related to the Ricci tensor* Ric^{Γ} *of* *the characteristic* $\mathfrak{so}(3)$ *connection* Γ *via*

$$Ric^{LC}_{ij} = Ric^{\Gamma}_{ij} + \tfrac{1}{4}T_{ikl}T_{jkl} + \tfrac{1}{2}(*\mathrm{d}*T)_{ij}.$$

Corollary 7.2. *Given a nearly integrable* $\mathbf{SO}(3)$ *structure* (M, g, Υ) *the following two conditions are equivalent.*

- *The codifferential of the torsion 3-form* T *vanishes.*
- *The Ricci tensor* Ric^{Γ} *of the characteristic connection* Γ *is symmetric.*

Thus, for nearly integrable $\mathbf{SO}(3)$ structures we have

$$*\mathrm{d}*T \equiv 0 \qquad \Longleftrightarrow \qquad Ric^{\Gamma}_{ij} \equiv Ric^{\Gamma}_{ji}.$$

In the rest of this section we discuss the torsion/curvature properties of the homogeneous examples of Section 6. It is interesting to note that all these examples satisfy

$$*\mathrm{d}*T \equiv 0.$$

Thus, the Ricci tensor Ric^{Γ} is symmetric for them. In many cases both the Ricci tensors Ric^{Γ} and Ric^{LC} are diagonal[1]. More explicitly,

- in case (1) of Proposition 6.3 we have:
$$Ric^{LC} = Ric^{\Gamma} = -6a^2 g, \qquad T \equiv 0$$
- in case (2) of Proposition 6.3 we have:
$$Ric^{LC} = \tfrac{1}{2}(t_1^2 + t_2^2)g + \tfrac{1}{24}(16t_1^2 + 12t_1t_2 - t_2^2)E_3^2 + \tfrac{1}{24}(4t_1^2 - t_2^2)E_3^4,$$
$$Ric^{\Gamma} = \tfrac{1}{2}t_1t_2E_3^2,$$
$$\mathrm{d}T = -2t_1t_2\theta^2\wedge\theta^3\wedge\theta^4\wedge\theta^5$$
- in case (3) of Proposition 6.3 we have:
$$Ric^{LC} = (t_1^2 - t_1t_2 + \tfrac{1}{2}t_2^2)g + \tfrac{1}{24}(44t_1^2 - 58t_1t_2 + 27t_2^2)E_3^2 + \tfrac{1}{24}(8t_1^2 - 10t_1t_2 + 3t_2^2)E_3^4,$$
$$Ric^{\Gamma} = \tfrac{1}{2}t_1(t_1 - 2t_2)g + \tfrac{1}{12}(14t_1^2 - 29t_1t_2 + 14t_2^2)E_3^2 + \tfrac{1}{12}(t_1 - 2t_2)(2t_1 - t_2)E_3^4,$$
$$\mathrm{d}T = -t_1^2\theta^2\wedge\theta^3\wedge\theta^4\wedge\theta^5$$
- for the examples of Theorem 6.5 we have:
$$Ric^{\Gamma} \equiv 0,$$
$$\mathrm{d}T \equiv 0$$
and Ric^{LC} has a rather complicated form depending on the torsion parameters t_a, $a = 1, 2, \ldots 10$; for some values of the parameters the Levi-Civita Ricci tensor Ric^{LC} may be diagonal, e.g.: if $t_a = 0, \forall a \neq 1$ then
$$Ric^{LC} = \tfrac{1}{2}t_1^2 \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$
- for the examples of Theorem 6.7 we have:
$$Ric^{LC} = \varrho^2(\tfrac{10}{3} - 2\delta)g + 2\varrho^2E_3^2,$$
$$Ric^{\Gamma} = -2\varrho^2\delta g + \tfrac{4}{3}\varrho^2E_3^2,$$
$$\mathrm{d}T = \tfrac{4}{3}\varrho^2(3\delta - 4)\theta^2\wedge\theta^3\wedge\theta^4\wedge\theta^5$$

[1]Note that the square of the matrix E_3 and its fourth power are diagonal matrices.

- for the examples of Section 6.3.3 we have:

$$Ric^{LC}=-\tfrac{3\varrho^2}{2}\begin{pmatrix}1&0&0&0&0\\0&\sin^2(\varphi)&0&\frac12\sin(2\varphi)&0\\0&0&0&0&0\\0&\frac12\sin(2\varphi)&0&\cos^2(\varphi)&0\\0&0&0&0&0\end{pmatrix},\quad Ric^{\Gamma}=-\tfrac{\varrho^2}{2}\begin{pmatrix}3&0&0&0&0\\0&2-\cos(2\varphi)&0&\sin(2\varphi)&0\\0&0&1&0&0\\0&\sin(2\varphi)&0&2+\cos(2\varphi)&0\\0&0&0&0&1\end{pmatrix},$$

$$\mathrm{d}T\equiv 0.$$

7.2. **Absence of covariantly constant spinors.** We now pass to the question if a manifold with an $\mathbf{SO}(3)$ structure (M,g,Υ) and an $\mathfrak{so}(3)$ connection Γ may admit a covariantly constant spinor field. We look for $\Psi: M\to\mathbb{C}^4$ such that

$$\mathrm{d}\Psi+\Gamma_{\mathfrak{spin}}\Psi=0, \tag{7.1}$$

where $\Gamma_{\mathfrak{spin}}$ is a spin connection (4.25) corresponding to Γ.

We use the curvature

$$\Omega_{\mathfrak{spin}}=\mathrm{d}\Gamma_{\mathfrak{spin}}+\Gamma_{\mathfrak{spin}}\wedge\Gamma_{\mathfrak{spin}}$$

of $\Gamma_{\mathfrak{spin}}$. This curvature is expressible in terms of the curvature $K=\frac{\sqrt3}{2}r^I{}_{jk}\theta^j\theta^k E_I$ of Γ and the (Dirac) matrices $\mathbf{E}_I$ of (4.24). We have

$$\Omega_{\mathfrak{spin}}=\tfrac{\sqrt3}{2}r^I{}_{jk}\theta^j\wedge\theta^k\mathbf{E}_I.$$

It is easy to see that the integrability conditions for the equations (7.1) are

$$\Omega_{\mathfrak{spin}}\Psi=0.$$

These equations should be satisfied for each element of the basis of 2-forms $\theta^i\wedge\theta^k$. Thus, they are equivalent to

$$W_{ij}\Psi=0,\qquad \forall i<j=1,2,3,4,5$$

where W_{ij} is a 4x4 matrix

$$W_{ij}=r^I{}_{ij}\mathbf{E}_I.$$

This shows that an existence of a non-zero solution for Ψ gives a severe restrictions on the curvature $\Omega_{\mathfrak{spin}}$. In particular, this implies that

$$\det(W_{ij})=\det(r^I{}_{ij}\mathbf{E}_I)=0\qquad\forall i<j=1,2,3,4,5. \tag{7.2}$$

But

$$\det(W_{ij})=\tfrac{9}{16}\Big((r^1{}_{ij})^2+(r^2{}_{ij})^2+(r^3{}_{ij})^2\Big)^2.$$

Thus, equations (7.2) are satisfied only if *all* the curvature coefficients $r^I{}_{ij}$ are zero. In such case $\Omega_{\mathfrak{spin}}=0$, which means that the corresponding $\mathfrak{so}(3)$ connection Γ is *flat.* This proves the following proposition.

Proposition 7.3. *Let (M,g,t) be a 5-dimensional $\mathbf{SO}(3)$ geometry equipped with an $\mathfrak{so}(3)$ connection Γ. Then (M,g,Υ) admits a covariantly constant spinor field with respect to the corresponding $\mathfrak{spin}(3)$ connection $\Gamma_{\mathfrak{spin}}$ if the connection Γ is flat. If this condition is satisfied then, locally, one has a 4-parameter family of constant spinors.*

8. The twistor bundle $\mathbb{T}$

It is remarkable that each 5 dimensional manifold M with an $\mathbf{SO}(3)$ structure (g,Υ) on it defines a natural 2-sphere bundle $\mathbb{S}^2 \to \mathbb{T} \to M$. This bundle, which via analogy with the twistor theory, we call the *twistor bundle*, will be defined by recalling that at every point x of M we have a distinguished subspace $(\bigwedge^2_3)_x$ of those 2-forms that span the irreducible $\mathfrak{so}(3)$. Considered point by point, spaces $(\bigwedge^2_3)_x$ form a rank 3 vector bundle $\bigwedge^2_3 M$ over M with the following basis of sections

$$\begin{aligned}
\kappa^1 &= \sqrt{3}\theta^1\wedge\theta^5+\theta^2\wedge\theta^3+\theta^4\wedge\theta^5,\\
\kappa^2 &= \sqrt{3}\theta^1\wedge\theta^3+\theta^2\wedge\theta^5+\theta^3\wedge\theta^4,\\
\kappa^3 &= 2\theta^2\wedge\theta^4+\theta^3\wedge\theta^5.
\end{aligned}$$

Here we have used the adapted coframe $(\theta^1,\theta^2\theta^3,\theta^4,\theta^5)$ for (M,g,Υ). It is also convenient to note that the forms $(\kappa^1,\kappa^2,\kappa^3)$ are related to the basis (E_1,E_2,E_3) of the irreducible $\mathfrak{so}(3)\subset\mathfrak{so}(5)$ via $\kappa^I=\frac{1}{2}E_{Iij}\theta^i\wedge\theta^j$, $I=1,2,3$, see (2.8).

Definition 8.1. The *twistor bundle* over a 5-dimensional manifold M equipped with an $\mathbf{SO}(3)$ structure (g,Υ) is the 2-sphere bundle $\mathbb{S}^2\to\mathbb{T}\xrightarrow{\pi}M$ defined by

$$\mathbb{T}=\Big\{\omega\in{\textstyle\bigwedge^2_3}M\ :\ *(\omega\wedge *\omega)=5\ \Big\}. \tag{8.1}$$

Remark 8.2. The constant 5 in the above normalisation means that $\omega\in\bigwedge^2_3 M$ iff $\omega=b_1\kappa^1+b_2\,\kappa^2+b_3\,\kappa^3$ where $b_1^2+b_2^2+b_3^2=1$.

Consider the complexification $\mathrm{T}^{\mathbb{C}}M$ of the tangent bundle of (M,g,Υ) and denote by the same letters the complexifications of the tensors g and Υ. At every point $x\in M$ consider the space

$$N_x=\{n\in\mathrm{T}^{\mathbb{C}}_xM\ :\ \Upsilon(n,n,\cdot)\equiv 0\}$$

of vectors, which are *null* with respect to the complexified Υ. Given any complexified vector $0\neq v\in\mathrm{T}^{\mathbb{C}}_xM$ we define

$$\mathrm{dir}(v)=\{\lambda v\in\mathrm{T}^{\mathbb{C}}_xM\ :\ \lambda\in\mathbb{C}\}.$$

We have the following proposition

Proposition 8.3. *The space of null directions*

$$\mathbb{P}N_x=\{\mathrm{dir}(n)\ :\ n\in N_x\}$$

is a disjoint sum of two connected components

$$\mathbb{P}N_x=\mathbb{P}N_x^+\sqcup\mathbb{P}N_x^-,\qquad\qquad \mathbb{P}N_x^-=\overline{\mathbb{P}N_x^+}.$$

Each of them is naturally diffeomorphic to the fibre $\mathbb{T}_x=\pi^{-1}(x)=\mathbb{S}^2$ of the twistor bundle $\mathbb{T}$.

Proof. Consider a 2-form $\omega\in\mathbb{T}_x$. In the adapted coframe $(\theta^1,\theta^2,\theta^3,\theta^4,\theta^5)$ it reads $\omega=\frac{1}{2}\omega_{ij}\theta^i\wedge\theta^j$. It defines a linear map

$$\mathrm{T}^{\mathbb{C}}_xM\ni v_i\longmapsto(\omega v)_j=\omega_{ji}v_i\in\mathrm{T}^{\mathbb{C}}_xM.$$

It is easy to see that the eigenvalues of this endomorphism are $\{0,\pm i,\pm 2i\}$. The corresponding $\pm 2i$ eigenspaces are *null* with respect to Υ due to the following argument. The $\mathbf{SO}(3)$ invariance of the tensor Υ, see (2.7), when applied to form ω and a vector n belonging to the $\pm 2i$ eigenspaces of ω_{ij} reads

$$0 = \Upsilon(\omega n, n, \cdot) + \Upsilon(n, \omega n, \cdot) + \Upsilon(n, n, \omega \cdot) = 4i\Upsilon(n, n, \cdot) + \Upsilon(n, n, \omega \cdot).$$

Now, if v belongs to any eigenspace of ω_{ij} the implication of this equality is $\Upsilon(n, n, v) = 0$, which means that $\Upsilon(n, n, \cdot) \equiv 0$.

Thus, the following map

$$\mathbb{T}_x \ni \omega \longmapsto \ker(\omega \mp 2i) \in \mathbb{P}N_x^{\pm}.$$

is well defined. It further follows that it provides the desired diffeomorphism between $\mathbb{T}_x$ and $\mathbb{P}N_x^{\pm}$.

□

Now we define the 2-sphere *bundle of null directions* for Υ to be

$$\mathbb{P}N = \bigcup_{x \in M} \mathbb{P}N_x^{+}$$

and, as a corollary to the above proposition, we get:

Proposition 8.4. *There exists a natural bundle isomorphism between the bundle $\mathbb{P}N$ of null directions for Υ and the twistor bundle $\mathbb{T}$.*

Remark 8.5. The above proposition enables one to view the twistor bundle $\mathbb{T}$ as an analog of the twistor bundles of 4-dimensional (pseudo)Riemannian geometries (see e.g. [8]). Historically, the first such bundle - Penrose's bundle of light rays over the Minkowski space-time [10] - is a 2-sphere bundle of null directions. It proved to be very useful in General Relativity Theory, especially in the case of complexified Minkowski space-time and its curved generalisations. Motivated by the utility of Penrose's bundle of light rays Atiyah, Hitchin and Singer [1] considered the 2-sphere bundle of complexified null 2-planes over a 4-dimensional Riemannian manifold. This bundle, which they identified with the bundle of almost hermitian structures over the 4-manifold, they termed the *twistor bundle*. Later, mathematicians generalised the notion of *twistor bundle* in many directions, so that the relation between null directions and todays twistors is weaker and weaker. We find a particularly remarkable the fact that the 5-dimensional geometries considered in the present paper lead to twistor bundle $\mathbb{T}$ whose relation to null directions is very apparent.

8.1. **Elements of geometry of $\mathbb{T}$.** Now, we consider an arbitrary $\mathbf{SO}(3)$ structure (M, g, Υ) equipped with an $\mathfrak{so}(3)$ connection Γ (we do not assume that Γ is the characteristic connection). These data induce interesting geometrical structures on the twistor bundle $\mathbb{T}$. The rest of this section is devoted to their brief description.

(1) The connection Γ splits the tangent space $\mathrm{T}\mathbb{T}$ into horizontal and vertical parts:

$$\mathrm{T}\mathbb{T} = \mathcal{H} \oplus \mathcal{V}.$$

The fibre of the twistor bundle $\mathbb{T}_x$ is naturally embedded in the vector space $(\bigwedge^2_3)_x$. It is a unit sphere $\mathbb{S}^2$ with respect to the natural scalar product Σ on two-forms, which explicitly reads

$$\Sigma(\sigma_1, \sigma_2) = \tfrac{1}{5} * (\sigma_1 \wedge * \sigma_2), \qquad \forall \sigma_1, \sigma_2 \in (\textstyle\bigwedge^2_3)_x.$$

Thus the vertical tangent space $\mathcal{V}_\omega$ at a point $\omega \in \mathbb{T}_x$ may be identified with the orthogonal complement of ω with respect to Σ. Hence

$$\mathcal{V}_\omega = \{\sigma \in (\textstyle\bigwedge^2_3)_x \quad : \quad \Sigma(\sigma, \omega) = 0\}.$$

(2) There is a natural Riemannian metric $\tilde{g}$ on $\mathbb{T}$. This metric is given by $\tilde{g} = \Sigma_2 \oplus \pi^* g$, where Σ_2 is the natural scalar product induced on the fibre by Σ.

(3) There is a natural complex structure J on the fibre $\mathbb{T}_x$, given by

$$J_\omega(\sigma) = [\omega, \sigma] \qquad \sigma \in \mathcal{V}_\omega \subset (\textstyle\bigwedge^2_3)_x.$$

Here, we view the forms ω and σ as elements of the Lie algebra $\mathfrak{so}(3) \cong (\bigwedge^2_3)_x$, so that $[\cdot,\cdot]$ is the Lie bracket in $\mathfrak{so}(3)$. Obviously, J is compatible with the metric Σ_2. Now, the metric Σ_2 together with orientation given by J determine the volume 2-form η_2 on the fibre.

(4) There is a tautological horizontal 2-form ω on $\mathbb{T}$.

(5) $\mathbb{T}$ is equipped with the horizontal vector field u given by

$$\tilde{g}(u) = \tfrac{1}{4}\tilde{*}(\eta_2 \wedge \omega \wedge \omega)$$

where $\tilde{*}$ is the Hodge star operation on $(\mathbb{T}, \tilde{g})$. The vector field u is unital: $\tilde{g}(u,u) = 1$. We denote the $\tilde{g}$-orthogonal complement of u in $\mathcal{H}$ by $\mathcal{H}^u$.

(6) At every point $x \in \mathbb{T}$ the metric $\tilde{g}$ descends to the 4-dimensional, naturally oriented, vector space $\mathcal{H}^u_x$. Thus, in $\mathcal{H}^u_x$, the Hodge star operator is well defined. By using it we decompose the restriction of the tautological 2-form $\omega_{|\mathcal{H}^u}$ into the self-daul and anti-self-dual parts

$$\omega_{|\mathcal{H}^u} = \omega_+ + \omega_-.$$

The forms $\omega_\pm$ define the pair of $\pi^* g$-compatible complex structures $J_\pm$ on $\mathcal{H}^u$

$$\pi^* g(J_\pm v_1, v_2) \;=\; \frac{2}{2 \pm 1}\,\omega_\pm(v_1, v_2), \qquad v_1, v_2 \in \Gamma(\mathcal{H}^u).$$

These two complex structures commute:

$$[J_+, J_-] = 0.$$

8.2. **Almost CR-structures on $\mathbb{T}$ and their integrability conditions.** We recall that an odd-dimensional real manifold P is equipped with an *almost CR-structure* if there exists on P a distinguished codimension one distribution $\mathcal{N}$ endowed with an almost complex structure $\mathcal{J}$ (see e.g. [9]). The $\pm i$ eigenspaces of $\mathcal{J}$ define the split

$$\mathbb{C} \otimes \mathcal{N} = \mathcal{N}^{(1,0)} \oplus \mathcal{N}^{(0,1)}.$$

An almost CR-structure $(\mathcal{N}, \mathcal{J})$ on P is called an *integrable* CR-structure iff the following integrability conditions are satisfied

$$[\mathcal{N}^{(1,0)}, \mathcal{N}^{(1,0)}] \subset \mathcal{N}^{(1,0)}.$$

The twistor bundle $\mathbb{T}$ is naturally equipped with *four* almost CR-structures. They are genuinely distinct i.e. not related by the conjugacy operation. One obtains these structures by defining the distribution $\mathcal{N}$ to be $\mathcal{N}_\mathbb{T} = u^\perp$, the orthogonal complement of the unit vector u with respect to the metric $\tilde{g}$ on $\mathbb{T}$. Since $\mathcal{N}_\mathbb{T} = \mathcal{V} \oplus \mathcal{H}^u$, then the four almost complex structures on $\mathcal{N}_\mathbb{T}$ may be defined by

$$\mathcal{J} = J \oplus \epsilon J_\pm, \qquad \epsilon = 1 \text{ or } -1.$$

Thus we have four natural almost CR-structures on $\mathbb{T}$ defined by means of four $\mathcal{J}$s on $\mathcal{N}_{\mathbb{T}}$. Among them the most interesting is

$$(\mathcal{N}_{\mathbb{T}},\mathcal{J}_0),\quad \text{where}\quad \mathcal{J}_0=J\oplus J_+.$$

This structure is the only one among $(\mathcal{N}_{\mathbb{T}},\mathcal{J})$ that may be integrable. More specifically, we have the following theorem.

Theorem 8.6.

(1) *Among the four natural almost CR-structures $(\mathcal{N}_{\mathbb{T}}, J\oplus\epsilon J_\pm)$ on $\mathbb{T}$, the only one that may be integrable is $(\mathcal{N}_{\mathbb{T}},\mathcal{J}_0)$.*

(2) *Let (M,g,Υ) be a nearly integrable $\mathbf{SO}(3)$ structure and let $(\mathcal{N}_{\mathbb{T}},\mathcal{J}_0)$ be the almost CR-structure on $\mathbb{T}$ induced by the characteristic connection of (M,g,Υ). This CR-structure is integrable if and only if*

$$K_{\odot^2_9}\equiv 0,\quad \textit{and}\quad T\in \textstyle\bigwedge^2_3.$$

Sketch of the proof. We start by choosing an $\mathbf{SO}(3)$ adapted coframe $(\theta^1,\theta^2,\theta^3,\theta^4,\theta^5)$ on $U\subset M$. We parametrise $\mathbb{T}|_U$ by $U\times\overline{\mathbb{C}}$, so that the tautological 2-form ω reads

$$\omega=\tfrac{z+\overline{z}}{1+|z|^2}\kappa_1+\tfrac{i(\overline{z}-z)}{1+|z|^2}\kappa_2+\tfrac{1-|z|^2}{1+|z|^2}\kappa_3,\qquad z\in\overline{\mathbb{C}}. \tag{8.2}$$

The horizontal-vertical splitting of the tangent bundle $T\mathbb{T}$ with respect to an $\mathfrak{so}(3)$-connection $\Gamma=\gamma_1E_1+\gamma_2E_2+\gamma_3E_3$ is given by the following complex valued 1-form

$$\tilde{h}=\tfrac{1}{1+|z|^2}\Big(\mathrm{d}z+\tfrac{1-z^2}{2i}\gamma_1+\tfrac{1+z^2}{2}\gamma_2+iz\,\gamma_3\Big).$$

The horizontal subspace $\mathcal{H}\subset T\mathbb{T}$ is the kernel of $\tilde{h}$.

The 1-form $\tilde{u}=\tilde{g}(u)$ - the $\tilde{g}$-dual to the unit horizontal vector field u - is given by

$$\tilde{u}=-\tfrac{1-4|z|^2+|z|^4}{(1+|z|^2)^2}\theta^1+\tfrac{i\sqrt{3}(z-\overline{z})(z+\overline{z})}{(1+|z|^2)^2}\theta^2-\tfrac{\sqrt{3}(z+\overline{z})(|z|^2-1)}{(1+|z|^2)^2}\theta^3-\tfrac{\sqrt{3}(z^2+\overline{z}^2)}{(1+|z|^2)^2}\theta^4-\tfrac{i\sqrt{3}(z-\overline{z})(|z|^2-1)}{(1+|z|^2)^2}\theta^5.$$

Since there exist two *commuting* complex structures $J_\pm$ on every 4-dimensional horizontal subspace $\mathcal{H}^u_x$, the complexification of this subspace decomposes onto the common eigenspaces of $J_\pm$. Explicitly we have

$$(\mathcal{H}^u_x)^{\mathbb{C}}=N_1\oplus N_2\oplus\overline{N_1}\oplus\overline{N_2},$$

where the spaces N_1 and N_2 are defined by

$$J_\pm N_1=iN_1,\qquad J_\pm N_2=\pm iN_2,$$

and $\overline{N_1}$, $\overline{N_2}$ denote their respective complex conjugates. The explicit formulae for the $\tilde{g}$-duals $\tilde{n}_1$ and $\tilde{n}_2$ of the vectors n_1 and n_2 generating the subspaces N_1 and N_2 are the following

$$\tilde{n}_1=\tfrac{i2\sqrt{3}z(|z|^2-1)}{(1+|z|^2)^2}\theta^1-\tfrac{2(z^3+\overline{z})}{(1+|z|^2)^2}\theta^2-\tfrac{i(1-3z^2-3z\overline{z}+z^3\overline{z})}{(1+|z|^2)^2}\theta^3-\tfrac{2i(z^3-\overline{z})}{(1+|z|^2)^2}\theta^4-\tfrac{1+3z^2-3z\overline{z}-z^3\overline{z}}{(1+|z|^2)^2}\theta^5,$$
$$\tilde{n}_2=\tfrac{i2\sqrt{3}z^2}{(1+|z|^2)^2}\theta^1+\tfrac{z^4-1}{(1+|z|^2)^2}\theta^2-\tfrac{2iz(z^2-1)}{(1+|z|^2)^2}\theta^3+\tfrac{i(z^4+1)}{(1+|z|^2)^2}\theta^4+\tfrac{2z(z^2+1)}{(1+|z|^2)^2}\theta^5.$$

The space $\mathcal{N}^{(1,0)}_{\mathbb{T}}$ of $(1,0)$-forms with respect to the almost complex structure $\mathcal{J}_0$ is spanned by

$$\mathcal{N}^{(1,0)}_{\mathbb{T}}=\mathrm{Span}_{\mathbb{C}}(\tilde{h},\tilde{n}_1,\tilde{n}_2).$$

Thus the integrability conditions for the CR structure $(\mathcal{N}_{\mathbb{T}}, \mathcal{J}_0)$ have the form

$$\begin{aligned}
\mathrm{d}\tilde{u} \wedge \tilde{u} \wedge \tilde{h} \wedge \tilde{n}_1 \wedge \tilde{n}_2 &\equiv 0\\
\mathrm{d}\tilde{h} \wedge \tilde{u} \wedge \tilde{h} \wedge \tilde{n}_1 \wedge \tilde{n}_2 &\equiv 0\\
\mathrm{d}\tilde{n}_1 \wedge \tilde{u} \wedge \tilde{h} \wedge \tilde{n}_1 \wedge \tilde{n}_2 &\equiv 0\\
\mathrm{d}\tilde{n}_2 \wedge \tilde{u} \wedge \tilde{h} \wedge \tilde{n}_1 \wedge \tilde{n}_2 &\equiv 0,
\end{aligned}$$

The expression for the other almost CR structures are analogous.

The remaining part of proof of the theorem is skipped due to its purely computational character.

□

Remark 8.7. We close this section with a remark that on $\mathbb{T}$ there exist also other natural geometries whose integrability conditions may encode the torsion/curvature properties of $\mathbf{SO}(3)$ structures. Let us define the following real 1-forms

$$\vartheta^1 = \mathrm{Re}(\tilde{n}_1), \quad \vartheta^2 = \mathrm{Im}(\tilde{n}_1), \quad \vartheta^3 = \mathrm{Re}(\tilde{n}_2), \quad \vartheta^4 = \mathrm{Im}(\tilde{n}_1),$$
$$\vartheta^5 = \tilde{u}, \quad \vartheta^6 = -\mathrm{Im}(\tilde{h}), \quad \vartheta^7 = \mathrm{Re}(\tilde{h}).$$

They define the $\tilde{g}$-orthonormal (local) coframe on $\mathbb{T}$. The following 3-forms

$$\begin{aligned}
\phi_1 &= \tfrac{i}{2}(\tilde{n}_1 \wedge \overline{\tilde{n}}_1 - \tilde{n}_2 \wedge \overline{\tilde{n}}_2) \wedge \tilde{u}\\
\phi_2 &= \tfrac{i}{2}(\tilde{n}_1 \wedge \overline{\tilde{n}}_2 \wedge \tilde{h} - \overline{\tilde{n}}_1 \wedge \tilde{n}_2 \wedge \overline{\tilde{h}})\\
\phi_3 &= \tfrac{i}{2}\tilde{u} \wedge \tilde{h} \wedge \overline{\tilde{h}}
\end{aligned}$$

are well defined on $\mathbb{T}$. They may be collected to a single well defined 3-form

$$\phi = \phi_1 + \phi_2 + \phi_3.$$

This, when expressed in terms of the orthonormal coframe $(\vartheta^1, \vartheta^2, \vartheta^3, \vartheta^4, \vartheta^5, \vartheta^6, \vartheta^7)$, reads

$$\phi = (\vartheta^1\wedge\vartheta^2 - \vartheta^3\wedge\vartheta^4)\wedge\vartheta^5 + (\vartheta^1\wedge\vartheta^3 - \vartheta^4\wedge\vartheta^2)\wedge\vartheta^6 + (\vartheta^1\wedge\vartheta^4 - \vartheta^2\wedge\vartheta^3)\wedge\vartheta^7 + \vartheta^5\wedge\vartheta^6\wedge\vartheta^7.$$

It equips $\mathbb{T}$ with a $G_2 \subset \mathbf{SO}(\tilde{g})$ structure (see [11]).

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