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Twistor bundles, Einstein equations and real structures*

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Abstract. We consider $\mathbf{S}^2$ bundles $\mathcal{P}$ and $\mathcal{P}'$ of totally null planes of maximal dimension and opposite self-duality over a four-dimensional manifold equipped with a Weyl or Riemannian geometry. The fibre product $\mathcal{P}\mathcal{P}'$ of $\mathcal{P}$ and $\mathcal{P}'$ is found to be appropriate for the encoding of both the self-dual and the Einstein–Weyl equations for the 4-metric. This encoding is realized in terms of the properties of certain well defined geometrical objects on $\mathcal{P}\mathcal{P}'$. The formulation is suitable for complex-valued metrics and unifies results for all three possible real signatures. In the purely Riemannian positive-definite case it implies the existence of a natural almost Hermitian structure on $\mathcal{P}\mathcal{P}'$ whose integrability conditions correspond to the self-dual Einstein equations of the 4-metric. All Einstein equations for the 4-metric are also encoded in the properties of this almost Hermitian structure on $\mathcal{P}\mathcal{P}'$.

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1. Introduction

The natural appearance of complex coordinates in the Robinson–Trautman [13] class of metrics was one of the first signs that complex geometric methods may be important in general relativity. The proper understanding of this fact led to the introduction of CR-geometric concepts to the Einstein theory [6, 11, 14, 15, 18]. Penrose's twistor programme was also partially motivated by this result.

In this paper we are concerned with a twistor theory over a four-dimensional manifold. Given a metric on such a manifold the problem of encoding the Einstein equations on the associated twistor bundle arises. Since Penrose's original paper [10] several attempts to construct an encoding have been undertaken [3, 16, 17]. In particular, in the case of a positive-definite metric, Salamon in [16] used well defined differential forms on the twistor bundle and showed that the vanishing of certain differentials corresponded to the anti-self-dual Einstein equations on the base manifold. Our approach in [8] is very much in the spirit of Salamon. There we studied natural forms on the twistor bundle in the Lorentzian case. Our analysis was more complex than in the positive-definite case since we had to deal with directions of forms rather than forms themselves. We showed that if our forms satisfied certain well defined differential conditions on the twistor bundle then the Ricci tensor of the base metric was traceless. Thus, in that case, we succeeded in encoding the full set of Einstein equations, without restricting to anti-self-dual metrics. In this paper we extend results of [8] to 4-manifolds with complex-valued metrics or real metrics of signature $(+,+,+,+)$ or $(+,+,-,-)$. The proposed approach unifies all the signatures and also applies to Weyl geometries.

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Section 2 presents a short reformulation of the main results of [8]. It will be useful for generalizations of the results to other signatures.

Section 3 describes an analogy between Hermitian and optical geometries (see theorem 3.1). We show there that notions such as that of a null direction in four-dimensional Lorentzian geometry and an almost Hermitian structure in the case of a positive-definite metric have a unified description in terms of totally null planes of maximal dimension in the complexification of the tangent space. It turns out that the integrability conditions for both almost Hermitian geometries and optical geometries associated with null rays have a unified description in terms of associated fields of maximal totally null planes. These conditions are given by equation (13).

Section 4 gives necessary information about Weyl geometries. We recall that such geometries are given in terms of a class of pairs (g, A) where g is a metric and A is a 1-form on a manifold. Two pairs (g, A) and (g', A') are in the same class iff $g' = e^{2\varphi}g$ and $A' = A - 2d\varphi$.

In section 5 we study 4-manifolds equipped with Weyl geometries. We consider Weyl geometries in which both g and A may be complex-valued. However, we do not exclude situations in which g and A are real. Purely metric situations $A = 0, \varphi = 0$ are also not excluded in our analysis. Given a 4-manifold $\mathcal{M}$ and a Weyl geometry (g, A) on it we consider a space $\mathcal{P}$ of all self-dual totally null 2-planes in the complexification of $T\mathcal{M}$. This space is an $\mathbf{S}^2$ bundle over $\mathcal{M}$. An analogous bundle $\mathcal{P}'$ of all anti-self-dual maximal totally null spaces is also considered there. Given $\mathcal{P}$ and $\mathcal{P}'$ we also study their fibre product space $\mathcal{PP}'$. This is a bundle over $\mathcal{M}$ with typical fibre $\mathbf{S}^2 \times \mathbf{S}^2$. We call $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{PP}'$ twistor bundles (section 5.1).

The rest of section 5 is devoted to studies of natural geometric structures that exist on twistor bundles. In particular, we find that $\mathcal{P}$ has the following geometrical features. $T\mathcal{P}$ splits naturally into a vertical and horizontal part. One can also naturally define on $\mathcal{P}$ a spin connection 1-form, a class of metrics $\tilde{g}$, a canonical field of horizontal 2-planes and two distinguished fields of 3-planes which are totally null in any metric from the class $\tilde{g}$. We also find a way of writing certain differential equations on $\mathcal{P}$ that have geometrical meaning. Analogous structures are also defined on $\mathcal{P}'$ (section 5.2). Another set of geometrical objects is naturally defined on $\mathcal{PP}'$ (section 5). There we find a natural split of the tangent bundle into vertical and horizontal parts. This enables a canonical field of horizontal 3-planes to be defined on $\mathcal{PP}'$. There is a nice geometry associated with these which, in particular, distinguishes a certain field of (in general complex) directions. This field is null in the naturally defined class of metrics on $\mathcal{PP}'$. It is used to define a canonical 1-form and eight distinguished fields of 4-planes that are totally null in all the natural metrics on $\mathcal{PP}'$.

Section 6 deals with the integrability conditions of the structures defined on twistor bundles. Using them we find a way of encoding (anti-)self-duality equations for the 4-metric on $\mathcal{P}$ and $\mathcal{P}'$ (theorems 6.4 and 6.5). This gives a Weyl-geometric generalization of the Atiyah–Hitchin–Singer [1] and the Penrose [11] theorems. Use of the natural structures on $\mathcal{PP}'$ enables the (anti-)self-dual Einstein–Weyl equations to be encoded there. This encoding is done by means of the integrability conditions of one of the eight naturally defined fields of maximal totally null planes on $\mathcal{PP}'$ (corollary 6.1, theorem 6.3). Other results of this section are included in theorems 6.1 and 6.2. They provide a description of the Einstein–Weyl equations (without restriction to self-dual metrics) on $\mathcal{PP}'$. In the purely metric case $A = 0, \varphi = 0$ they give a signature-independent formulation of the Einstein equations $R_{\mu\nu} = \lambda g_{\mu\nu}$ on $\mathcal{PP}'$.

Section 7 is concerned with the interpretations of the results of section 6 in the case of real Weyl geometries. If the 4-metric has positive-definite or neutral signature (section 7.2), then the main results are included in theorem 7.1. This, in particular, states that there is a preferred almost Hermitian structure on $\mathcal{PP}'$, the integrability conditions of which are equivalent to the self-dual Einstein–Weyl equations for the Weyl geometry. This result, even in the purely metric case, goes a bit beyond the Atiyah–Hitchin–Singer theorem. We are able to encode both the self-duality and Einstein equations in the integrability conditions of natural almost Hermitian structures on $\mathcal{PP}'$. Section 7.2 also includes a geometrical interpretation of the full set of Einstein–Weyl (Einstein, in the pure metric case) equations on $\mathcal{PP}'$. This is given by theorem 7.2. It uses one of the eight distinguished almost Hermitian structures $\mathcal{J}$ on $\mathcal{PP}'$ to analyse the decomposition of the differential of the naturally defined spin connection 1-form on $\mathcal{PP}'$. It turns out that the Einstein–Weyl equations for the Weyl geometry are equivalent to the fact that this differential has zero intersection with the $\mathrm{T}^{*(0,2)}$ space of 2-forms, where $\mathrm{T}^{*(0,2)}$ is defined with respect to $\mathcal{J}$. Section 7.3 deals with the Lorentzian case of the 4-metric. The main result is summarized in theorem 7.3. It, in particular, states that in the purely metric case one can associate a natural seven-dimensional CR structure with the Minkowski 4-metric. The end of section 7 explains why in the Lorentzian case it suffices to work on $\mathcal{P}$ or $\mathcal{P}'$. The results of section 2 are then reobtained in terms of bundles of maximal totally null planes (theorems 7.4 and 7.5).

2. Summary of the Lorentzian case

To make the present paper self-contained we briefly recall our twistor formulation of the Einstein equations in the Lorentzian case [8].

Let $\mathcal{M}$ be a four-dimensional oriented manifold equipped with a Lorentzian metric g. It is convenient to introduce a null tetrad $(m, \bar{m}, l, k)$ on $\mathcal{M}$ with a cotetrad $(\theta^i) = (\theta^1, \theta^2, \theta^3, \theta^4) = (M, \bar{M}, L, K)$ so that

$$g = g_{ij}\theta^i\theta^j = M\bar{M} - LK, \tag{1}$$

where $\theta^i\theta^j = \frac{1}{2}(\theta^i \otimes \theta^j + \theta^j \otimes \theta^i)$. Consider the set $\mathbf{S}_x$ of all null directions outgoing from a given point $x \in \mathcal{M}$. This set is topologically a sphere (the celestial sphere of an observer situated at x). The points of this sphere can be parametrized by a complex number z belonging to the Argand plane $\mathbf{C} \cup \{\infty\}$. A direction associated with $z \neq \infty$ is generated by a vector

$$k(z) = k + z\bar{z}l - zm - \bar{z}\bar{m}. \tag{2}$$

With $z = \infty$ we associate a direction generated by vector l. Conversely, any null direction from x is either parallel to the vector l or can be represented by the unique null vector $k(z)$ such that $g(k(z), l) = -1$. It follows that $k(z)$ has, necessarily, the form (2), and that it defines a certain $z \in \mathbf{C}$. If a direction is parallel to l we associate with it $z = \infty$.

We define a fibre bundle $\mathcal{P} = \bigcup_{x\in\mathcal{M}} \mathbf{S}_x$ over $\mathcal{M}$, so that its fibres are two-dimensional spheres $\mathbf{S}_x$. The anonical projection $\pi : \mathcal{P} \to \mathcal{M}$ is defined by $\pi(\mathbf{S}_x) = x$. We will call the bundle $\mathcal{P}$ 'Penrose's twistor space', or the 'twistor bundle'. This bundle possesses quite a broad family of well defined geometrical objects, which collectively form the so-called 'optical geometry' [19, 21]. Here we recall only those objects that are relevant in the present paper (see [8] for details).

(i) The Levi-Civita connection associated with the metric g on $\mathcal{M}$ distinguishes a horizontal space in $\mathrm{T}\mathcal{P}$. A vertical space consists, by definition, of vectors tangent to the fibres. In this way, at any point $p \in \mathcal{P}$ we have a natural splitting of its tangent space onto a direct sum $\mathrm{T}_p\mathcal{P} = V_p \oplus H_p$, where H_p is a four-dimensional horizontal space and V_p

is a two-dimensional vertical space. The vertical space V_p is identical with a tangent space to a certain point on the two-dimensional sphere. Thus V_p has a natural complex structure related to the complex structure on $\mathbf{S}^2$. Complexification of V_p splits it into eigenspaces V_p^+ and V_p^- with respect to this complex structure. We have a horizontal lift $\tilde{v}$ of any vector v from $\pi(p) \in \mathcal{M}$ to $\mathcal{P}$. This is a vector $\tilde{v}$ such that at p $\tilde{v} \in H_p$ and $\pi_*(\tilde{v}) = v$.

(ii) A Lorentzian metric $\tilde{g}$ can be defined on $\mathcal{P}$ by the requirements that:
   - (a) a scalar product of any two horizontal vectors is defined in $\tilde{g}$ in terms of the scalar product in g of their push forwards to $\mathcal{M}$;
   - (b) a scalar product of any two vertical vectors in $\tilde{g}$ is equal to their scalar product in the natural metric on a two-dimensional sphere (this is consistent since vertical vectors can be considered tangent vectors to $\mathbf{S}^2$);
   - (c) any two vectors such that one is horizontal and the other is vertical are orthogonal in $\tilde{g}$.

(iii) There is a natural congruence of lines on $\mathcal{P}$ which is tangent to the horizontal lifts of null directions from $\mathcal{M}$. It is defined by the following recipe. Take a null vector k at $x \in \mathcal{M}$. This represents a certain null direction $p(k)$ outgoing from x. Correspondingly, this defines a point $p = p(k)$ in the fibre $\pi^{-1}(x)$. Lift k horizontally to p. This defines $\tilde{k}$ which generates a certain direction outgoing from $p \in \mathcal{P}$. Repeating this procedure for all directions outgoing from $x \in \mathcal{M}$ we attach a unique direction to any point of $\pi^{-1}(x)$. If we do it for all points of $\mathcal{M}$, we define a field of directions on $\mathcal{P}$ which, according to its construction and properties of $\tilde{g}$, is null. Integral curves of this field form the desired null congruence. This congruence is called the null spray on $\mathcal{P}$ [17].

Let X be any non-vanishing vector field tangent to the null spray on $\mathcal{P}$. Let Λ_L be a real 1-form on $\mathcal{P}$ defined by $\Lambda_L = \tilde{g}(X)$. Since X is defined up to a multiplication of a non-vanishing real function on $\mathcal{P}$ then Λ_L is also specified up to a multiplication by a real non-vanishing function u on $\mathcal{P}$,

$$\Lambda_L \to \Lambda'_L = u\Lambda_L. \tag{3}$$

One associates another 1-form with the horizontal space in $\mathcal{P}$. This is a complex 1-form E_L on $\mathcal{P}$ that satisfies (i) $E_L(H_p) = E_L(V_p^-) = 0$ and (ii) $E_L \wedge \bar{E}_L \neq 0$ at any point $p \in \mathcal{P}$. E_L is also defined up to a multiplication by a non-vanishing complex function h on $\mathcal{P}$

$$E_L \to E'_L = hE_L. \tag{4}$$

It is easy to see that the metric $\tilde{g}$ on $\mathcal{P}$ can be expressed as

$$\tilde{g} = 2(h\bar{h}E_L\bar{E}_L + \Lambda_L T + F_L\bar{F}_L)$$

with some 1-forms T (real) and F_L (complex) on $\mathcal{P}$. The above expression can be considered a definition of the form F_L. It is given up to transformations

$$F_L \to F'_L = \mathrm{e}^{\mathrm{i}\phi}F_L + p\Lambda_L, \tag{5}$$

where ϕ (real) and p (complex) are some functions on $\mathcal{P}$.

It follows that in the ordered null cotetrad (θ^i) of (1) the forms Λ_L, F_L and E_L can be represented by

$$\Lambda_L = -L - z\bar{z}K - z\bar{M} - \bar{z}M, \tag{6}$$

$$F_L = M + zK, \tag{7}$$

$$E_L = \mathrm{d}z + \gamma^3{}_2 + z(\gamma^1{}_1 - \gamma^4{}_4) + z^2\gamma^2{}_3, \tag{8}$$

where z is the same as in (2) and $\gamma^i{}_j$ are Levi-Civita connection 1-forms associated with the metric g in the cotetrad (θ^i).

Although the above forms are only defined up to transformations (3)–(5) one can use them to write down some well defined equations on $\mathcal{P}$. The following equations, invariant under transformations (3)–(5), are of particular interest.

$$\mathrm{d}\Lambda_L \wedge \Lambda_L \wedge F_L \wedge E_L = 0, \tag{9}$$

$$\mathrm{d}F_L \wedge \Lambda_L \wedge F_L \wedge E_L = 0, \tag{10}$$

$$\mathrm{d}E_L \wedge \Lambda_L \wedge F_L \wedge E_L = 0, \tag{11}$$

$$\mathrm{d}E_L \wedge \Lambda_L \wedge \bar{F}_L \wedge E_L = 0. \tag{12}$$

Note that in equation (12) a form $\bar{F}_L$, which is a complex conjugate of F_L, appears. Since any of the above equations is invariant under (3)–(5) we can use Λ_L, F_L and E_L in a particular representation (6)–(8) to analyse them. It is a matter of a straightforward but lengthy calculation to arrive at the following theorem.

Theorem 2.1.

(i) Equations (9) and (10) are identically satisfied on $\mathcal{P}$.
(ii) Equation (11) is satisfied everywhere on $\mathcal{P}$ if and only if the metric (1) on $\mathcal{M}$ is conformally flat.
(iii) Equation (12) is satisfied everywhere on $\mathcal{P}$ if and only if the traceless part $s_{ij} = r_{ij} - \frac{1}{4}g_{ij}r$ of the Ricci tensor of the metric (1) vanishes on $\mathcal{M}$.

A straightforward corollary from this theorem reads as follows.

Corollary 2.1. Equation (12) is satisfied in $\mathcal{P}$ if and only if the base metric satisfies the Einstein equations $r_{ij} = \kappa g_{ij}$.

To interpret equation (11) geometrically on $\mathcal{P}$ it is convenient to consider it together with equations (9) and (10). It is easily seen then that the system (9)–(11) constitutes the Froebenius condition for the three-dimensional distribution $\mathcal{N}$ which in $\mathcal{P}$ annihilates forms Λ_L, F_L and E_L. It follows that $\mathcal{N}$ is totally null in the metric $\tilde{g}$ and has maximal dimension.

We failed in finding a nice geometrical interpretation for equation (12). Since it is invariant under transformations (3)–(5) such an interpretation should exist.

3. Hermitian and optical geometries

Suppose that we are given a $2m$-dimensional real manifold $\mathcal{R}$ equipped with a real-valued metric g of signature $(2p+\epsilon,\ 2q+\epsilon)$. Here $2m = 2(p+q+\epsilon)$ and $\epsilon = 0$ or 1. Following [5] we call the cases $\epsilon = 0$ and $\epsilon = 1$ pseudo-Euclidean and pseudo-Lorentzian, respectively. We omit the prefix 'pseudo' if $pq = 0$. By complexifying g one endows the complexification $\mathrm{T}\mathcal{R}^{\mathbf{C}}$ of the tangent bundle $\mathrm{T}\mathcal{R}$ with a metric $g^{\mathbf{C}}$. Let $\mathcal{N}$ be a vector sub-bundle of $\mathrm{T}\mathcal{R}^{\mathbf{C}}$ which is totally null with respect to $g^{\mathbf{C}}$ and has m-dimensional fibres. We call such bundles maximal totally null bundles. Given $\mathcal{N}$ we also have its complex-conjugate bundle $\bar{\mathcal{N}}$ as well as bundles $\mathcal{N}\cap\bar{\mathcal{N}}$ and $\mathcal{N}+\bar{\mathcal{N}}$. It is easy to see that $\mathcal{N}\cap\bar{\mathcal{N}}$ and $\mathcal{N}+\bar{\mathcal{N}}$ are, respectively, complexifications of certain vector sub-bundles $\mathcal{K}$ and $\mathcal{L} = \mathcal{K}^{\perp}$ of $\mathrm{T}\mathcal{R}$. The complex fibre dimension r of $\mathcal{N}\cap\bar{\mathcal{N}}$ (or real fibre dimension of $\mathcal{K}$) depends on the signature of g and may take the following values: $r = \epsilon, 2+\epsilon, \ldots, \min(2p+\epsilon, 2q+\epsilon)$ [5]. It is called a real index of $\mathcal{N}$. From now on we only consider such $\mathcal{N}$s for which r is constant over $\mathcal{R}$.

Given $\mathcal{N}$, we have a natural almost complex structure $\mathcal{J}$ in a bundle $\mathcal{H} \stackrel{\text{def}}{=} \mathcal{L}/\mathcal{K}$. To define this we observe that any section l of $\mathcal{L}$ is of the form $l = n + \bar{n}$ where n is some section of $\mathcal{N}$. If $[l]$ denotes the equivalence class associated with l in $\mathcal{H}$ we define $\mathcal{J}$ by

$$\mathcal{J}([l]) = \mathcal{J}([n + \bar{n}]) \stackrel{\text{def}}{=} [-\mathrm{i}(n - \bar{n})].$$

One may prove that $\mathcal{J}$ is well defined. Moreover, since the metric g is degenerate on $\mathcal{K}$ then it descends to a unique metric g' in $\mathcal{H}$. It further follows that $\mathcal{J}$ is an orthogonal transformation for g' (we say that $\mathcal{J}$ is orthogonal with respect to g' or, simply, orthogonal).

If $q = 0$ then the only possible values of the real index r of $\mathcal{N}$ are 0 (Euclidean case) or 1 (Lorentzian case). For such q and $\epsilon = 0$ we see that the corresponding $\mathcal{K} = \{0\}$, $\mathcal{L} = \mathcal{H} = \mathrm{T}\mathcal{R}$, $g' = g$. Thus, in this case, $\mathcal{N}$ defines an almost Hermitian geometry $(g, \mathcal{J})$ in $\mathcal{R}$.

If $q = 0$ and $\epsilon = 1$ then the maximal totally null bundle $\mathcal{N}$ equips $\mathcal{R}$ with the structure of the almost optical geometry of Trautman [19, 21]. This is a sequence

$$\begin{array}{lccccc} & \mathcal{K} & \hookrightarrow & \mathcal{L} & \longrightarrow & \mathcal{H} \\ \text{Fibre dimension} & 1 & & 2m-1 & & 2m-2 \end{array}$$

of real vector sub-bundles $\mathcal{K}$ and $\mathcal{L}$ of $\mathrm{T}\mathcal{R}$ together with an orthogonal almost complex structure $\mathcal{J}$ in the quotient bundle $\mathcal{H}$. Note that in this case the metric g' in $\mathcal{H}$ is purely Euclidean.

To deal with the generic case of $q \neq 0$ we introduce the following definition [9].

Definition 3.1. Consider a real $2m$-dimensional manifold $\mathcal{R}$ equipped with a metric g of signature $(2p+\epsilon, 2q+\epsilon)$. Let $\mathcal{K}$ and $\mathcal{L} = \mathcal{K}^\perp$ $(\mathcal{K} \subset \mathcal{L})$ be vector sub-bundles of $\mathrm{T}\mathcal{R}$ which have respective fibre dimension r and $2m - r$. If the quotient bundle $\mathcal{H} = \mathcal{L}/\mathcal{K}$ is equipped with an almost complex structure $\mathcal{J}$ which is orthogonal with respect to the descended metric g' in $\mathcal{H}$, then $(\mathcal{K}, g, \mathcal{H}, \mathcal{J})$ is called an almost optical geometry with index r (or almost r-optical geometry).

Thus any maximal totally null bundle with real index r gives rise to an almost r-optical geometry. The converse is also true. Given an almost r-optical geometry on $\mathcal{R}$ we define $\mathcal{N}$ as the vector sub-bundle of $\mathrm{T}\mathcal{R}^{\mathbf{C}}$ whose sections are of the form $n = l_1 + \mathrm{i}l_2$, where l_1, l_2 are sections of $\mathcal{L}$ satisfying $\mathcal{J}[l_1] + \mathrm{i}\mathcal{J}[l_2] = -\mathrm{i}[l_1] + [l_2]$. One easily proves that $\mathcal{N}$ thus defined is totally null, has fibre dimension m and real index r.

The above discussion shows a one-to-one correspondence between maximal totally null bundles of a given index r and almost r-optical geometries.

Given a maximal totally null bundle $\mathcal{N}$ we denote the set of all its sections by $\Gamma(\mathcal{N})$. It is natural to consider the following integrability conditions for $\mathcal{N}$:

$$[\Gamma(\mathcal{N}), \Gamma(\mathcal{N})] \subset \Gamma(\mathcal{N}). \tag{13}$$

Here $[\cdot, \cdot]$ denotes a commutator of sections treated as vector fields. We say that an almost r-optical geometry associated with $\mathcal{N}$ is r-optical if and only if the conditions (13) are satisfied.

Definition 3.2. A CR structure [22] is a real $(2m-1)$-dimensional manifold $\mathcal{Q}$ equipped with a sub-bundle $\mathcal{H}$ of the tangent bundle $\mathrm{T}\mathcal{Q}$, which has fibres of dimension $2(m-1)$ and which is endowed with an almost complex structure $\mathcal{J}$.

Given a CR structure $(\mathcal{Q}, \mathcal{H}, \mathcal{J})$ we extend $\mathcal{J}$ to the complexification $\mathcal{H}^{\mathbf{C}}$ by linearity. A CR structure is called an integrable CR structure if for any sections X, Y of $\mathcal{H}$ we have

$$\mathcal{J}[X + \mathrm{i}\mathcal{J}X, Y + \mathrm{i}\mathcal{J}Y] = -\mathrm{i}[X + \mathrm{i}\mathcal{J}X, Y + \mathrm{i}\mathcal{J}'Y]. \tag{14}$$

We say that two CR structures $(\mathcal{Q}, \mathcal{H}, \mathcal{J})$ and $(\mathcal{Q}', \mathcal{H}', \mathcal{J}')$ are (locally) equivalent iff there exists a (local) diffeomorphism $\phi : \mathcal{Q} \to \mathcal{Q}'$ such that

$$\phi_*\mathcal{H} = \mathcal{H}'$$

and

$$\phi^*\mathcal{J} = \mathcal{J}'.$$

In the following we will also need the more general structure.

Definition 3.3. An r-CR structure is a real $(2m - r)$-dimensional manifold $\mathcal{Q}$ equipped with a sub-bundle $\mathcal{H}$ of the tangent bundle $\mathrm{T}\mathcal{Q}$ such that it has fibres of dimension $2m - 2r$ and is endowed with an almost complex structure $\mathcal{J}$.

An r-CR structure is integrable iff any two sections X, Y of the bundle $\mathcal{H}$ satisfy conditions (14).

We note that a 0-CR structure is the same as an almost complex geometry in $\mathcal{Q}$. Its integrability conditions are equivalent to the integrability conditions of this almost complex structure.

Given an almost r-optical geometry $(\mathcal{K}, \mathcal{L}, \mathcal{H}, \mathcal{J})$ on $\mathcal{R}$ we choose a surface $\mathcal{S}$ of dimension $2m - r$ in $\mathcal{R}$ that it is transversal to sections of the bundle $\mathcal{K}$. It is easy to see that any such surface is naturally endowed with an r-CR structure. If it happens that $(\mathcal{K}, \mathcal{L}, \mathcal{H}, \mathcal{J})$ is r-optical then the integrability conditions (13) imply that r-CR structures on any hypersurface $\mathcal{S}$ are integrable and locally equivalent. More formally, given an r-optical geometry satisfying (13) we find that the bundle $\mathcal{K}$ is integrable as a distribution on $\mathcal{R}$. Thus it defines a foliation of $\mathcal{R}$ by r-dimensional real manifolds tangent to $\mathcal{K}$. Consider an equivalence relation $\sim$ in $\mathcal{R}$ which identifies points lying on the same leaf $\mathcal{X}$ of this foliation. We assume that its quotient space $\mathcal{Q} \equiv \mathcal{R}/\sim$ is a manifold. Conditions (13) guarantee that the projection of r-CR structures from any of the surfaces $\mathcal{S}$ to this manifold equip it with the same integrable r-CR structure. Hence, in such a case, the manifold $\mathcal{R}$ is locally equivalent to the Cartesian product $\mathcal{X} \times \mathcal{Q}$. This generalizes the well known fact for almost optical geometries associated with congruences of shear-free and null geodesics in four dimensions [14, 15].

Summing up we have the following theorem.

Theorem 3.1. Let $\mathcal{R}$ be a real $2m$-dimensional manifold equipped with a real metric g of signature $(2p + \epsilon, 2q + \epsilon)$, where $2m = 2(p + q + \epsilon)$ and $\epsilon = 0$ or 1.

(i) There exists a one-to-one correspondence between almost r-optical geometries over $\mathcal{R}$ and maximal totally null bundles $\mathcal{N}$ of constant real index r over $\mathcal{R}$.
(ii) Any integrable $\mathcal{N}$ of index r locally defines an integrable r-CR structure.
(iii) In the case of a Euclidean metric, the bundle $\mathcal{N}$ corresponds to an almost Hermitian structure $(g, \mathcal{J})$ on $\mathcal{R}$. This almost Hermitian structure is integrable iff $\mathcal{N}$ satisfies integrability conditions (13).
(iv) In the case of a Lorentzian metric, the bundle $\mathcal{N}$ corresponds to an almost optical geometry on $\mathcal{R}$. This, when integrable, defines an integrable CR structure.

In the following we will need the interpretation of the integrability conditions (13) in terms of the theory of differential ideals.

Consider a system of complex-valued 1-forms $(A_1, A_2, \ldots, A_s)$ on $\mathcal{R}$. Let $\mathcal{I}$ be an ideal in the exterior algebra of all complex-valued differential forms on $\mathcal{R}$ generated by 1-forms $(A_1, A_2, \ldots, A_s)$. We say that $\mathcal{I}$ is a closed differential ideal iff

$$\begin{aligned}
&\mathrm{d}A_1 \wedge A_1 \wedge A_2 \wedge \ldots \wedge A_s = 0,\\
&\mathrm{d}A_2 \wedge A_1 \wedge A_2 \wedge \ldots \wedge A_s = 0,\\
&\ldots\\
&\ldots\\
&\ldots\\
&\mathrm{d}A_s \wedge A_1 \wedge A_2 \wedge \ldots \wedge A_s = 0.
\end{aligned}$$

Any maximal totally null bundle $\mathcal{N}$ over $\mathcal{R}$ can be defined as the annihilator of m linearly independent, totally null, complex-valued 1-forms, say $(A_1, A_2, \ldots, A_m)$, on $\mathcal{R}$. Given $\mathcal{N}$ defined by such 1-forms we have the following, well known, lemma.

Lemma 3.1. $\mathcal{N}$ satisfies the integrability conditions (13) if and only if the system $(A_1, A_2, \ldots, A_m)$ generates a closed differential ideal on $\mathcal{R}$.

4. Weyl geometry

4.1. Definitions

From now on by a metric on a real manifold we will understand a non-degenerate, bilinear and symmetric, complex-valued form.

Consider a four-dimensional real oriented manifold $\mathcal{M}$ equipped with a metric g. Fixing four complex-valued 1-forms (θ^i) $(i = 1, 2, 3, 4)$ on $\mathcal{M}$ for which $\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \neq 0$ we can represent any metric g by means of its metric coefficients g_{ij}. Thus, given g and (θ^i) we have

$$g = g_{ij}\theta^i\theta^j.$$

The system of forms (θ^i) will be called a cotetrad on $\mathcal{M}$. We equip $\mathcal{M}$ with a Weyl geometry. Such a geometry is defined in terms of a pair (g, A) where g is a metric and $A = A_i\theta^i$ is a complex-valued 1-form on $\mathcal{M}$. The metric and A are related by

$$\mathrm{D}g_{ij} = \mathrm{d}g_{ij} - g_{ik}\Gamma^k{}_j - g_{jk}\Gamma^k{}_i = -Ag_{ij}, \tag{15}$$

where $\Gamma^i{}_j$ are torsion-free connection 1-forms. The torsion-free condition is expressed by

$$\mathrm{d}\theta^i = -\Gamma^i{}_j \wedge \theta^j. \tag{16}$$

Given a Weyl geometry (g, A) on $\mathcal{M}$ the connection 1-forms $\Gamma^i{}_j$ are uniquely determined. They are expressible in terms of A and the Levi-Civita connection 1-forms $\gamma^i{}_j$ of the metric $g = g_{ij}\theta^i\theta^j$. Explicitly we have

$$\Gamma^i{}_j = g^{ik}\Gamma_{kj}, \qquad g^{ik}g_{kj} = \delta^i{}_j \tag{17}$$

where

$$\Gamma_{ij} = \gamma_{ij} + \tfrac{1}{2}g_{ij}A + g_{k[i}A_{j]}\theta^k, \qquad \gamma_{ij} = g_{ik}\gamma^k{}_j \tag{18}$$

and where we have introduced the abbreviation $a_{[i}b_{j]} = \frac{1}{2}(a_ib_j - a_jb_i)$†.

† Round brackets will denote symmetrization of indices, e.g. $a_{(i}b_{j)} = \frac{1}{2}(a_ib_j + a_jb_i)$.

Given connection 1-forms ${\Gamma^i}_j$ one may associate a (Weyl) connection with them and obtain a recipe for parallel transport of vectors on $\mathcal{M}$. It follows that in contrast to the parallel transport of Riemannian geometry, this transport preserves only nullity of vectors (see, for instance, [23] for more information)†.

The curvature of Weyl geometry is defined in terms of curvature 2-forms

$$\Omega_{ij} = \tfrac{1}{2} R_{ijkl}\theta^k \wedge \theta^l = \mathrm{d}\Gamma_{ij} + \Gamma_{ik} \wedge {\Gamma^k}_j. \tag{19}$$

It splits into the curvature ω_{ij} of the Levi-Civita connection, and the remaining A-dependent part. This, in particular, includes the curvature

$$\mathcal{F} = \tfrac{1}{2}\mathcal{F}_{ij}\theta^i \wedge \theta^j = \mathrm{d}A$$

of A‡.

The Ricci tensor R_{jl} and its scalar R are defined, respectively, by $R_{jl} = g^{ik}R_{ijkl}$ and $R = g^{ij}R_{ij}$. Note that R_{ij} is not symmetric in general. The traceless part of $R_{(jl)}$ is defined by

$$S_{ij} = R_{(ij)} - \tfrac{1}{4}Rg_{ij}, \tag{20}$$

which implies

$$S := g^{ij}S_{ij} = 0. \tag{21}$$

We say that Weyl geometry (g, A) satisfies the Einstein–Weyl equations iff

$$S_{ij} = 0. \tag{22}$$

For further use we also define a tensor

$$C_{ijkl} = R_{ijkl} + \tfrac{1}{3}Rg_{i[k}g_{l]j} + R_{j[k}g_{l]i} + R_{i[l}g_{k]j}. \tag{23}$$

This can be decomposed into the Levi-Civita (w_{ijkl}) and A-dependent (f_{ijkl}) parts

$$C_{ijkl} = w_{ijkl} + f_{ijkl} \tag{24}$$

(see appendix A). It turns out that w_{ijkl} are precisely the covariant coefficients of the usual Weyl tensor associated with the metric g. Since w_{ijkl} are antisymmetric in k, l then we may associate with them a collection of 2-forms

$$w_{ij} = \tfrac{1}{2}w_{ijkl}\theta^k \wedge \theta^l, \tag{25}$$

which we call the Weyl-tensor 2-forms.

We close this section with a remark that if $A = 0$ everywhere on $\mathcal{M}$ then such a Weyl geometry reduces to the usual Riemannian geometry associated with metric g. In particular, such objects like Γ_{ij}, R_{ijkl} etc reduce to their respective Levi-Civita parts γ_{ij}, r_{ijkl}, etc (compare with the first footnote below).

† Our point of view on Weyl geometries is non-standard in two respects. First, we admit complex metrics g. Second, we do not stress the conformal invariance. It is easy to check that for a fixed cotetrad (θ^i) equation (15) is invariant under the transformation

$$(g, A) \to (g', A') = (\mathrm{e}^{2\varphi}g, A - 2\mathrm{d}\varphi). \tag{$*$}$$

One can therefore view Weyl geometry as a pair (g, A) given up to transformations $(*)$. In such a formulation only a conformal metric is relevant. We do not refer to this point of view in our discussion since we want to have a nice passage to Riemannian geometries (fixed metrics, not their conformal class) when $A = 0$. However, all relevant formulae such as, for example, (22), (61), (62), (67) and (68) are covariant under $(*)$. See appendix E for a further discussion of this. Thus our results also apply to Weyl geometries viewed in this standard, conformal sense.

‡ Decompositions of various Weyl geometric objects onto the Levi-Civita and A-dependent parts are given in appendix A.

4.2. Weyl geometries in null tetrads

Of particular interest are null cotetrads on $\mathcal{M}$. These are cotetrads

$$(\theta^1, \theta^2, \theta^3, \theta^4) = (M, P, N, K) \tag{26}$$

related to the metric g by

$$g = g_{ij}\theta^i\theta^j = MP - NK. \tag{27}$$

A tetrad dual to (26) will be denoted by

$$(e_1, e_2, e_3, e_4) = (m, p, n, k). \tag{28}$$

Given a null cotetrad (M, P, N, K) it is convenient to introduce the form

$$\eta = M \wedge P \wedge N \wedge K. \tag{29}$$

Using it one splits the set of all null cotetrads into two classes. Cotetrads (M', P', N', K') from the first class satisfy $M' \wedge P' \wedge N' \wedge K' = \eta$ and cotetrads (M'', P'', N'', K'') from the second class satisfy $M'' \wedge P'' \wedge N'' \wedge K'' = -\eta$. From now on we restrict our attention only to null cotetrads from the first class.

Given a null cotetrad and A we find Weyl connection 1-forms $\Gamma^i{}_j$, and calculate curvature 2-forms. Their convenient decomposition relates to the notion of self-duality.

Given a p-form ω on $\mathcal{M}$ we define its Hodge dualization $*\omega$ by

$$(*\omega)(X_1, \ldots, X_{4-p})\eta = \omega \wedge g(X_1) \wedge \ldots \wedge g(X_{4-p}), \tag{30}$$

where $g(X_i)$ is a 1-form associated with a vector field X_i $(i = 1, 2, \ldots, (4 - p))$ by $\langle g(X_i), X_j\rangle = g(X_i, X_j)$. Since the metric g induces an isomorphism between forms and vectors on $\mathcal{M}$ then, in an obvious way, we also have a Hodge dualization of p-vectors.

Hodge dualization is an involutive ($*^2 = \mathrm{id}$) automorphism of the complexified space $\bigwedge^2$ of 2-forms on $\mathcal{M}$. Its $\pm$ eigenspaces $\bigwedge^2_+$ and $\bigwedge^2_-$ consist of self-dual and anti-self-dual forms, respectively. A convenient basis for $\bigwedge^2_+$ is

$$P \wedge K, \qquad N \wedge K - M \wedge P, \qquad N \wedge M \tag{31}$$

and for $\bigwedge^2_-$

$$M \wedge K, \qquad N \wedge K + M \wedge P, \qquad N \wedge P. \tag{32}$$

Any 2-form can be decomposed onto these bases. Decompositions of the curvature $\mathcal{F}$ and the Weyl tensor 2-forms w_{ij} onto these bases define coefficients ϕ_0, ϕ_1, ϕ_2, ϕ'_0, ϕ'_1, ϕ'_2 and Ψ_0, Ψ_1, Ψ_2, Ψ_3, Ψ_4, Ψ'_0, Ψ'_1, Ψ'_2, Ψ'_3, Ψ'_4 by

$$\begin{aligned}\mathcal{F} = {}&\phi'_0 N \wedge P + \phi'_1(N \wedge K + M \wedge P) + \phi'_2 M \wedge K + \phi_0 N \wedge M\\ &+\phi_1(N \wedge K - M \wedge P) + \phi_2 P \wedge K,\end{aligned} \tag{33}$$

$$\begin{aligned}&w_{14} = \Psi'_0 M \wedge K + \Psi'_1(N \wedge K + M \wedge P) + \Psi'_2 N \wedge P\\ &w_{23} = -\Psi'_2 M \wedge K - \Psi'_3(N \wedge K + M \wedge P) - \Psi'_4 N \wedge P\\ &\tfrac{1}{2}(w_{34} + w_{12}) = \Psi'_1 M \wedge K + \Psi'_2(N \wedge K + M \wedge P) + \Psi'_3 N \wedge P\end{aligned} \tag{34}$$

$$\begin{aligned}&w_{24} = \Psi_0 P \wedge K + \Psi_1(N \wedge K - M \wedge P) + \Psi_2 N \wedge M\\ &w_{13} = -\Psi_2 P \wedge K - \Psi_3(N \wedge K - M \wedge P) - \Psi_4 N \wedge M\\ &\tfrac{1}{2}(w_{34} - w_{12}) = \Psi_1 P \wedge K + \Psi_2(N \wedge K - M \wedge P) + \Psi_3 N \wedge M.\end{aligned} \tag{35}$$

Decompositions of $\Omega_{[ij]}$ can be found in appendix B.

It follows from the above decompositions that the Weyl tensor 2-forms w_{ij} are anti-self-dual (respectively, self-dual) if and only if all the coefficients Ψ_0, Ψ_1, Ψ_2, Ψ_3,

Ψ_4 (respectively, Ψ'_0, Ψ'_1, Ψ'_2, Ψ'_3, Ψ'_4) vanish† on $\mathcal{M}$. This allows for the following terminology. Weyl geometries (g, A) on $\mathcal{M}$ are called anti-self-dual (respectively, self-dual) if and only if all coefficients Ψ_μ (respectively Ψ'_μ), $\mu = 0, 1, 2, 3, 4$ vanish on $\mathcal{M}$. It follows that this definition does not depend on the choice of the null cotetrad.

5. Geometry of twistor bundles

5.1. Twistor bundles

Let $\mathcal{M}$ be a real oriented 4-manifold equipped with a Weyl geometry (g, A). At any point x of $\mathcal{M}$ we consider vector subspaces of the complexification of the tangent space $\mathrm{T}_x\mathcal{M}$ which

(i) are totally null with respect to g and
(ii) have maximal dimension.

Such spaces are necessarily two dimensional and can be represented by a complex bivector. It turns out that bivectors associated with spaces satisfying (i) and (ii) are either self-dual or anti-self-dual in the Hodge dualization associated with g and η. This shows that the set of all spaces that at x satisfy (i) and (ii) consists of two disjoint parts S_x and S'_x. We call S_x (respectively, S'_x) a set of all self-dual (anti-self-dual) maximal totally null spaces at x. A pair of spaces (s, s') such that $s \in S_x$ and $s' \in S'_x$ is called a pair of maximal totally null spaces of opposite self-duality. It is easy to see that both S_x and S'_x are diffeomorphic to a two-dimensional sphere $\mathbf{S}_2$. A stereographic projection gives a convenient parametrization of these spheres in terms of points of the Argand plane $\mathbf{C} \cup \{\infty\}$. Using the null tetrad (28) for g we find that elements $s \in S_x$ and $s' \in S'_x$ can be represented, respectively, by

$$s = \mathrm{Span}\{zm - k, zn - p\} \qquad s' = \mathrm{Span}\{z'p - k, z'n - m\}, \tag{36}$$

where $z, z' \in \mathbf{C} \cup \{\infty\}$ and thus we identified points of $\mathbf{S}_2$ with the points of $\mathbf{C} \cup \{\infty\}$. For further use we also note that any two spaces s and s' have nonzero intersection at x. It is easy to see that this intersection is one dimensional and is spanned by a null vector X which, if s and s' are represented by (36), has the form

$$X = k + zz'n - zm - z'p. \tag{37}$$

Collecting the sets S_x, S'_x point by point we have two fibre bundles $\mathcal{P} = \bigcup_{x\in\mathcal{M}} S_x$ and $\mathcal{P}' = \bigcup_{x\in\mathcal{M}} S'_x$ over $\mathcal{M}$. Any of these bundles has $\mathbf{S}_2$ as its typical fibre and is equipped with respective projections $\pi : \mathcal{P} \to \mathcal{M}$ and $\pi' : \mathcal{P}' \to \mathcal{M}$. Any point p of $\mathcal{P}$ is a certain totally null (necessarily self-dual) space of maximal dimension at the corresponding point x of $\mathcal{M}$ (there is an analogous statement for points of $\mathcal{P}'$). It can therefore be parametrized by $(x, z, \bar{z})$, where z is as in (36). A point $p' \in \mathcal{P}'$ is parametrized by $(x, z', \bar{z}')$, where $\pi'(p') = x$ and z' is as in (36). Using $\mathcal{P}$ and $\mathcal{P}'$ one defines their fibre product $\mathcal{P}\mathcal{P}' = \bigcup_{x\in\mathcal{M}}(S_x \times S'_x)$ which is:

(i) a fibre bundle over $\mathcal{M}$ with a natural projection $\Pi : \mathcal{P}\mathcal{P}' \to \mathcal{M}$ and typical fibre diffeomorphic to $\mathbf{S}_2 \times \mathbf{S}_2$;
(ii) a fibre bundle over $\mathcal{P}$ with a natural projection $\mathrm{pr} : \mathcal{P}\mathcal{P}' \to \mathcal{P}$ and typical fibre diffeomorphic to $\mathbf{S}_2$;

† It is known that conditions $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$ or $\Psi'_0 = \Psi'_1 = \Psi'_2 = \Psi'_3 = \Psi'_4 = 0$ are invariant under the conformal transformations of the metric. A less well known fact states that they are also invariant under transformations $(*)$ of the second footnote of section 4.1. This is related to the fact that the conditions $C_{ijkl} = 0$ and $w_{ijkl} = 0$ (hence also $f_{ijkl} = 0$) are invariant under $(*)$.

(iii) a fibre bundle over $\mathcal{P}'$ with a natural projection $\mathrm{pr}' : \mathcal{PP}' \to \mathcal{P}'$ and typical fibre diffeomorphic to $\mathbf{S}_2$.

In particular, any point of $\mathcal{PP}'$ may be understood as a pair of maximal totally null spaces of opposite self-duality at the corresponding point of $\mathcal{M}$. A convenient parametrization of $\mathcal{PP}'$ is $(x, z, \bar{z}, z', \bar{z}')$. The projections associated with these bundles satisfy

$$\Pi = \pi \circ \mathrm{pr} = \pi' \circ \mathrm{pr}', \tag{38}$$

and in the above coordinates on $\mathcal{PP}'$, $\mathcal{P}$ and $\mathcal{P}'$ are given by

$$\Pi(x, z, \bar{z}, z', \bar{z}') = x, \qquad \mathrm{pr}(x, z, \bar{z}, z', \bar{z}') = (x, z, \bar{z}), \qquad \text{etc.}$$

Using the projections we can pull back forms. For example, using π we pull back forms from $\mathcal{M}$ to $\mathcal{P}$, using Π we pull back forms from $\mathcal{M}$ to $\mathcal{PP}'$ and using pr' we pull back forms from $\mathcal{P}'$ to $\mathcal{PP}'$†. In this way we can, for example, pull back tetrad 1-forms θ^i and Weyl connection 1-forms $\Gamma^i{}_j$ from $\mathcal{M}$ to $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{PP}'$. Since it follows from the context on which manifold a given form is placed, we use in the following the same letters to denote forms and their pullbacks.

Weyl geometry (g, A) induces interesting geometrical structures on bundles $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{PP}'$. We only outline constructions for $\mathcal{P}$ and $\mathcal{PP}'$.

5.2. Natural structures on $\mathcal{P}$ and $\mathcal{P}'$

(i) *The tangent bundle to $\mathcal{P}$ and its complexification split naturally into a vertical and horizontal part.* To show this we give a recipe for the lifting of a given vector v from $x \in \mathcal{M}$ to a chosen point $p \in \mathcal{P}$ in the fibre over x. Recall that a point p can be considered a maximal totally null self-dual space at x. Take any curve $x(t)$ that is tangent to v at x. Use the Weyl connection associated with (g, A) to propagate the maximal totally null space represented by p parallelly along $x(t)$. Since Weyl-geometric parallel propagation preserves the nullity of vectors, then at any point of our curve we get a certain totally null space. Due to the continuity of $x(t)$ any such space is maximal and self-dual. Thus given a curve $x(t)$ tangent to v at x we have a corresponding curve $p(t)$ in $\mathcal{P}$ which starts at p. It follows that a direction of a tangent vector to $p(t)$ in p does not depend on the choice of $x(t)$. This is the direction of the desired horizontal lift $\tilde{v}$ of v to p. The lift is determined completely by the additional demand that $\pi_*(\tilde{v}) = v$. Thus we are able to lift any vector from $x \in \mathcal{M}$ to a chosen point p from the fibre over x. Moreover, it is true that horizontal lifts of four linearly independent vectors from x constitute four linearly independent vectors in p. This means that we have a well defined lift of the tangent space $\mathrm{T}_x\mathcal{M}$ to a four-dimensional subspace H_p of $\mathrm{T}_p\mathcal{P}$. This subspace is called a horizontal space at p. The vertical space V_p consists of vectors at p that are tangent to the fibres. This space is two dimensional and may be identified with the tangent space to a certain point of $\mathbf{S}^2$. A direct sum $H_p \oplus V_p$ equals $\mathrm{T}_p\mathcal{P}$.

The horizontal lift that we described above can be also used to lift horizontally vectors w from the complexification of the tangent space at x to $p \in \pi^{-1}(x) \subset \mathcal{P}$. This is achieved by extending the horizontal lift map $\tilde{\ } : v \to \tilde{v}$ by linearity to the complexification of $\mathrm{T}_x\mathcal{M}$. Thus $\tilde{w} = \tilde{v}_1 + \mathrm{i}\tilde{v}_2$, where v_1 and v_2 are, respectively, real and imaginary parts of w. This leads to a split of the complexification of the tangent bundle to $\mathcal{P}$ into vertical and horizontal parts $(\mathrm{T}\mathcal{P})^{\mathbf{C}} = \mathcal{H}^{\mathbf{C}} \oplus \mathcal{V}^{\mathbf{C}}$.

† Note that due to property (38) the direct pull back of a form from $\mathcal{M}$ to $\mathcal{PP}'$ is the same as a pullback of this form via $\mathcal{P}$ (first using π and then pr) or $\mathcal{P}'$.

There is a natural complex structure I_p in V_p that comes from the natural complex structure on $\mathbf{S}^2$. This, when prolonged to the complexification $V_p^{\mathbf{C}}$, gives a split $V_p^{\mathbf{C}} = V_p^+ \oplus V_p^-$, where $I_p V_p^{\pm} = \pm \mathrm{i} V_p^{\pm}$.

(ii) *Spin connection 1-form.* We look for a complex-valued 1-form E such that in some neighbourhood $\mathcal{U}$ of $\mathcal{P}$ it satisfies (a) $E(\mathcal{H}) = 0$, (b) $E(\bigcup_{p\in\mathcal{U}} V_p^-) = 0$ and (c) $E \wedge \bar{E} \neq 0$. In general, starting from a given point $x \in \mathcal{M}$, we can solve these conditions only in a cylindrical $\mathcal{U}$ over a sufficiently small neighbourhood of x. Outside $\mathcal{U}$ conditions (a)–(c) may be contradictory. From now on we restrict our considerations to such $\mathcal{M}$s for which the corresponding E is defined globally. This can be achieved either by assuming some additional structure on $\mathcal{M}$ (e.g. spin manifold structure) or restricting ourselves to $\mathcal{M}$s to be open subsets of $\mathbf{R}^4$. With such an assumption, conditions (a)–(c) define E on $\mathcal{P}$ up to a multiplication by a non-vanishing complex-valued function h on $\mathcal{P}$

$$E \to hE. \tag{39}$$

E is called a spin connection 1-form on $\mathcal{P}$.

Using the null cotetrad (26) and the coordinates $(x, z, \bar{z})$ on $\mathcal{P}$ one easily finds that the form E may be represented by

$$E = \mathrm{d}z - \Gamma^3{}_2 + z(\Gamma^1{}_1 - \Gamma^4{}_4) + z^2\Gamma^2{}_3. \tag{40}$$

(iii) *Metrics.* Pullback the metric g from $\mathcal{M}$ to $\mathcal{P}$ and add to it a tensor $h\bar{h}E\bar{E}$ with h being a non-vanishing function on $\mathcal{P}$. This defines a class of metrics $\tilde{g}$ on $\mathcal{P}$, which can be represented by

$$\tilde{g} = \pi^*(g) + h\bar{h}E\bar{E}. \tag{41}$$

(iv) *Canonical field of 2-planes.* Take a point p of $\mathcal{P}$. It corresponds to a certain self-dual maximal totally null plane at $x = \pi(p) \in \mathcal{M}$. Lift this plane horizontally to p. This associates a horizontal 2-plane to any point p of $\mathcal{P}$. Thus on $\mathcal{P}$ we have a distinguished field of 2-planes, which we call the canonical field of 2-planes. Note that any 2-plane in this field is totally null in any metric from the class (41).

Dually, the canonical field of 2-planes defines a pair of 1-forms (F, B) on $\mathcal{P}$ which, by definition, annihilate the vertical space and the canonical field of 2-planes. F and B are given up to transformations

$$(F, B) \to (\alpha F + \beta B, \gamma F + \delta B), \qquad \alpha\delta - \beta\gamma \neq 0. \tag{42}$$

This shows that a *direction* of a 2-form $\Sigma = F \wedge B$ on $\mathcal{P}$ is well defined.

It is easy to see that in the null cotetrad (26) and in the coordinates $(x, z, \bar{z})$ the forms F and B may be represented by

$$F = M + zK, \qquad B = N + zP. \tag{43}$$

(v) *Distinguished totally null planes of maximal dimension.* Given a point $p \in \mathcal{P}$ consider a canonical 2-plane σ passing through this point. There are only two three-dimensional planes at p that are totally null in any metric $\tilde{g}$ and that contain σ as a vector subspace. These may be defined as vector spaces n_E and $n_{\bar{E}}$ annihilating (F, B, E) and $(F, B, \bar{E})$, respectively. Point by point they define two bundles of maximal totally null planes $\mathcal{N}_E$ and $\mathcal{N}_{\bar{E}}$ over $\mathcal{P}$. According to section 2, in the case of real geometries, they will define a pair of distinguished complex or optical structures on $\mathcal{P}$.

(vi) *Invariant equations.* Although forms E, F, B and Σ are only given up to certain transformations, one can use them to write down several geometric equations on $\mathcal{P}$. In particular, note that the equation

$$\mathrm{d}E \wedge F \wedge B \wedge E = 0 \tag{44}$$

as well as the system of equations

$$\mathrm{d}F \wedge F \wedge B \wedge E = 0 \qquad \mathrm{d}B \wedge F \wedge B \wedge E = 0 \tag{45}$$

are invariant under the transformations (39), (42).

(i′), (ii′), (iii′), (iv′), (v′), (vi′). Analogous constructions as in (i)–(vi) can be performed for $\mathcal{P}'$. In this way on $\mathcal{P}'$ we have a split of $\mathrm{T}\mathcal{P}'$ into a vertical and a horizontal part. Also the spin connection 1-form E', metrics, canonical field of 2-planes, classes of forms (F', B'), Σ', distinguished maximal totally null planes and invariant equations are defined there.

5.3. Natural structures on $\mathcal{PP}'$

(i) *The tangent bundle to $\mathcal{PP}'$ and its complexification have a natural split into vertical and horizontal parts.* The recipe for having this split is almost as in the case of $\mathcal{P}$ with the exception that now a point $p \in \mathcal{PP}'$ corresponds to a pair (s, s') of maximal totally null spaces of opposite self-duality at $x = \Pi(p) \in \mathcal{M}$. Thus if we want to lift a vector v from $x \in \mathcal{M}$ to a point $p \in \Pi^{-1}(x) \subset \mathcal{PP}'$ we take a curve tangent to v at x and propagate parallelly spaces s and s' along this curve. This produces a pair of maximal totally null spaces of opposite self-duality at any point along the curve. Correspondingly, we get a curve in $\mathcal{PP}'$ starting at p which defines the direction of the lift $\tilde{\tilde{v}}$ of v. As before the lift is specified uniquely by the demand that $\Pi_*(\tilde{\tilde{v}}) = v$. Lifting $\mathrm{T}_{\Pi(p)}\mathcal{M}$ horizontally we get a horizontal space H_p in p. The vertical space V_p is defined as the vector space tangent at p to the fibre of $\mathcal{PP}'$ over $\Pi(p)$. Note that now V_p is four-dimensional and is isomorphic to the tangent space of $\mathbf{S}^2 \times \mathbf{S}^2$ at the point corresponding to p.

(ii) *Connection 1-forms.* These are the complex-valued 1-forms on $\mathcal{PP}'$ that annihilate the horizontal space in $\mathrm{T}(\mathcal{PP}')$. It follows that the basis of such forms on $\mathcal{PP}'$ is given by the four pullbacks $\mathrm{pr}^*(E)$, $\mathrm{pr}^*(\bar{E})$, $\mathrm{pr}'^*(E')$ and $\mathrm{pr}'^*(\bar{E}')$. Here we wrote pullback signs explicitly. They will be omitted in the following.

Since the form E (respectively E') was defined on $\mathcal{P}$ (respectively on $\mathcal{P}'$) up to a scaling by a function, the four above-mentioned forms are given up to a scaling by a non-vanishing complex function on $\mathcal{PP}'$.

Local representations of E and E' may be given in terms of the coordinates $(x, z, \bar{z}, z', \bar{z}')$ introduced on $\mathcal{PP}'$ in section 4.1. Since they were chosen in such a way that by projections we were getting corresponding coordinates $(x, z, \bar{z})$ on $\mathcal{P}$ and $(x, z', \bar{z}')$ on $\mathcal{P}'$ then we easily find that

$$E = \mathrm{d}z - \Gamma^3{}_2 + z(\Gamma^1{}_1 - \Gamma^4{}_4) + z^2\Gamma^2{}_3, \tag{46}$$

$$E' = \mathrm{d}z' - \Gamma^3{}_1 + z'(\Gamma^2{}_2 - \Gamma^4{}_4) + z'^2\Gamma^1{}_3. \tag{47}$$

Here, as usual, connection 1-forms are expressed with respect to the cotetrad (26).

(iii) *Metrics.* The following metrics are of particular interest on $\mathcal{PP}'$:

$$\tilde{\tilde{g}} = \Pi^*(g) + h\bar{h}E\bar{E} + h'\bar{h}'E\bar{E}', \tag{48}$$

where h and h' are non-vanishing complex-valued functions on $\mathcal{PP}'$.

(iv) *The canonical field of 3-planes and associated bundles.* At every point p of $\mathcal{PP}'$ there is a natural 3-plane which is obtained as follows. Take a pair (s, s') of maximal totally null spaces of opposite self-duality which at $x = \Pi(p)$ correspond to p. Lift spaces s and s' horizontally to a point $p \in \Pi^{-1}(x)$ corresponding to (s, s'). This gives a pair of vector spaces $\tilde{\tilde{s}}$ and $\tilde{\tilde{s}}'$ at p. But as we noticed in section 4.1, s and s' have a one-dimensional intersection. Hence the vector space $\tilde{\tilde{s}} + \tilde{\tilde{s}}'$ has complex dimension equal to three. Thus at every point $p \in \mathcal{PP}'$ we have a three-dimensional space $\tilde{\tilde{s}} + \tilde{\tilde{s}}'$, which we call a canonical field of 3-planes.

Actually the above considerations show that we have a list of vector sub-bundles $\mathcal{S} = \bigcup_{p\in\mathcal{PP}'} \tilde{\tilde{s}}$, $\mathcal{S}' = \bigcup_{p\in\mathcal{PP}'} \tilde{\tilde{s}}'$, $\mathcal{L} = \bigcup_{p\in\mathcal{PP}'} (\tilde{\tilde{s}} + \tilde{\tilde{s}}')$, $\mathcal{K} = \bigcup_{p\in\mathcal{PP}'} (\tilde{\tilde{s}} \cap \tilde{\tilde{s}}')$ of the complexification of the tangent bundle to $\mathcal{PP}'$ which give rise to the following sequence:

$$\begin{array}{lccccc} & \mathcal{K} & \hookrightarrow & \mathcal{L} & \longrightarrow & \mathcal{L}/\mathcal{K} \\ \text{fibre dimension} & 1 & & 3 & & 2 \end{array}.$$

Note that by definition bundles $\mathcal{S}$, $\mathcal{S}'$ are sub-bundles of $\mathcal{L}$, and that $\mathcal{S}$, $\mathcal{S}'$, and $\mathcal{K}$ are null bundles with respect to any metric from the class $\tilde{\tilde{g}}$. This indicates parallels between the structures defined here and the optical geometries of Trautman [19, 21].

Given the above bundles on $\mathcal{PP}'$ it is interesting to ask whether such geometric conditions as $[\mathcal{S},\mathcal{S}] \subset \mathcal{S}$, $[\mathcal{S}',\mathcal{S}'] \subset \mathcal{S}'$, $[\mathcal{S},\mathcal{S}'] \subset \mathcal{L}$, $[\mathcal{K},\mathcal{L}] \subset \mathcal{K}$ etc, have some interpretation in terms of the Weyl geometry on $\mathcal{M}$.

(v) *Canonical 1-form on* $\mathcal{PP}'$. The bundle $\mathcal{K}$ has one-complex-dimensional fibres. It can be used to define a direction of 1-form Λ on $\mathcal{PP}'$. Indeed, if X is a section of $\mathcal{K}$, then we define Λ by

$$\Lambda = \tilde{\tilde{g}}(X). \tag{49}$$

Taking another section of $\mathcal{K}$ we see that

$$\Lambda \to u\Lambda, \tag{50}$$

which shows that a direction of Λ is well defined. We call Λ a canonical 1-form. We notice that Λ, together with E, $\bar{E}$, E', $\bar{E}'$, can be used to define a convenient basis of 1-forms on $\mathcal{PP}'$. Indeed, one easily finds that

$$\tilde{\tilde{g}} = h\bar{h}E\bar{E} + h'\bar{h}'E'\bar{E}^{\ prime} + \Lambda T + FF', \tag{51}$$

with some complex-valued 1-forms F, F' and T on $\mathcal{PP}'$. These forms are defined up to the following transformations:

$$F \to \alpha F + \beta\Lambda, \tag{52}$$

$$F' \to \frac{1}{\alpha}F' + \gamma\Lambda, \tag{53}$$

$$T \to \frac{1}{u}\left(T - \alpha\gamma F - \frac{\beta}{\alpha}F' - \beta\gamma\Lambda\right), \tag{54}$$

where $\alpha \neq 0$, β, γ are some functions on $\mathcal{PP}'$. It follows that the forms F and F' are in the class of forms obtained by taking pullbacks of the forms F and F' of section 4.2. The set of forms $(E, \bar{E}, E', \bar{E}', F, F', \Lambda, T)$ constitutes a convenient basis of 1-forms on $\mathcal{PP}'$. In the coordinates $(x, z, \bar{z}, z', \bar{z}')$ and in the cotetrad (26) they may be represented as

$$\begin{aligned} E &= \mathrm{d}z - \Gamma^3{}_2 + z(\Gamma^1{}_1 - \Gamma^4{}_4) + z^2\Gamma^2{}_3, \\ E' &= \mathrm{d}z' - \Gamma^3{}_1 + z'(\Gamma^2{}_2 - \Gamma^4{}_4) + z'^2\Gamma^1{}_3, \\ \Lambda &= -N - zz'K - zP - z'M, \\ F &= M + zK, \\ F' &= P + z'K, \\ T &= K. \end{aligned} \tag{55}$$

It is useful to consider transformations

$$\begin{aligned} &z \leftrightarrow z', \\ &1 \leftrightarrow 2, \qquad 3 \leftrightarrow 3, \qquad 4 \leftrightarrow 4 \end{aligned} \tag{56}$$

where the last transformation means that for any object on $\mathcal{M}$, $\mathcal{P}$, $\mathcal{P}'$ or $\mathcal{P}\mathcal{P}'$, with tetrad indices 1, 2, 3, 4 interchange indices 1 and 2 and do not change indices 3 and 4.

Examples:

$$\theta^1 = M \leftrightarrow \theta^2 = P, \qquad \theta^3 = N \leftrightarrow \theta^3 = N, \qquad \theta^4 = K \leftrightarrow \theta^4 = K, \tag{57}$$

$$e_1 = m \leftrightarrow e_2 = p, \qquad e_3 = n \leftrightarrow e_3 = n, \qquad e_4 = k \leftrightarrow e_4 = k, \tag{58}$$

$$\begin{aligned} &S_{13} \leftrightarrow S_{23}, \qquad S_{34} \leftrightarrow S_{34}, \qquad S_{12} \leftrightarrow S_{21}, \qquad \text{etc}, \\ &\Gamma^1{}_4 \leftrightarrow \Gamma^2{}_4, \qquad \Gamma^2{}_2 \leftrightarrow \Gamma^1{}_1, \qquad \text{etc}, \\ &\Lambda \leftrightarrow \Lambda, \qquad F \leftrightarrow F', \qquad E \leftrightarrow E', \end{aligned} \tag{59}$$

$$\Psi_\mu \leftrightarrow \Psi'_\mu, \mu = 0, 1, 2, 3, 4, \qquad \phi_a \leftrightarrow \phi'_a, \qquad a = 0, 1, 2. \tag{60}$$

It will be important that Λ is invariant under the transformations (56).

(vi) *Distinguished totally null planes of maximal dimension.* Given a point $p \in \mathcal{P}\mathcal{P}'$, consider its corresponding pair (s, s') of maximal totally null planes of opposite self-duality in $\Pi(p) \in \mathcal{M}$. Lift s horizontally to $\tilde{\tilde{s}}$ in p. It follows that $\tilde{\tilde{s}}$ is a totally null 2-plane in any metric $\tilde{\tilde{g}}$ in $\mathcal{P}\mathcal{P}'$. There are only four four-dimensional planes at p which are totally null with respect to *any* metric $\tilde{\tilde{g}}$ and which contain $\tilde{\tilde{s}}$ as a subspace†. These may be defined as vector spaces $n_{F'EE'}$, $n_{F'E\bar{E}'}$, $n_{F'\bar{E}E'}$ and $n_{F'\bar{E}\bar{E}'}$ annihilating, respectively, (Λ, F', E, E'), $(\Lambda, F', E, \bar{E}')$, $(\Lambda, F', \bar{E}, E')$ and $(\Lambda, F', \bar{E}, \bar{E}')$. Point by point they define four maximal totally null bundles $\mathcal{N}_{F'EE'}$, $\mathcal{N}_{F'E\bar{E}'}$, $\mathcal{N}_{F'\bar{E}E'}$ and $\mathcal{N}_{F'\bar{E}\bar{E}'}$ over $\mathcal{P}\mathcal{P}'$.

Similarly, considering extensions of $\tilde{\tilde{s}}'$ we find four other maximal totally null bundles $\mathcal{N}_{FEE'}$, $\mathcal{N}_{FE\bar{E}'}$, $\mathcal{N}_{F\bar{E}E'}$ and $\mathcal{N}_{F\bar{E}\bar{E}'}$. Thus, in real cases, we will have eight different distinguished r-optical structures on $\mathcal{P}\mathcal{P}'$.

(vii) *Invariant equations.* One set of geometrical equations on $\mathcal{P}\mathcal{P}'$ was already mentioned at the end of section 5.3 iv. By using forms E, E', Λ, F and F' we can write further equations and systems of equations. Only those which are invariant under the allowed transformations of the forms have geometrical meaning. Among them there are the following:

$$\begin{aligned} &\mathrm{d}\Lambda \wedge E \wedge \Lambda \wedge F = 0 \qquad &&\mathrm{d}F \wedge E \wedge \Lambda \wedge F = 0 \\ &\mathrm{d}E \wedge E \wedge \Lambda \wedge F = 0 \qquad &&\mathrm{d}\Lambda \wedge E' \wedge \Lambda \wedge F' = 0 \\ &\mathrm{d}F' \wedge E' \wedge \Lambda \wedge F' = 0 \qquad &&\mathrm{d}E' \wedge E' \wedge \Lambda \wedge F' = 0 \\ &\mathrm{d}E \wedge E \wedge \Lambda \wedge F' = 0 \qquad &&\mathrm{d}E' \wedge E' \wedge \Lambda \wedge F = 0 \\ &\mathrm{d}E \wedge E' \wedge \Lambda \wedge F' = 0 \qquad &&\mathrm{d}E \wedge E' \wedge \Lambda \wedge F = 0. \end{aligned}$$

One can continue this list. We discuss some of these equations in section 6.

6. Self-duality and the Einstein–Weyl equations

Let $(\mathcal{M}, g, A)$ be a four-dimensional Weyl geometry. In this section we study relations between curvature properties of $(\mathcal{M}, g, A)$ and the integrability conditions for the natural objects on the corresponding twistor bundles $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{P}\mathcal{P}'$. We start our analysis by

† The requirement that these planes must be null in any of metrics $\tilde{\tilde{g}}$ is crucial to obtain a discrete number of them!

giving a geometrical interpretation of the invariant equations on $\mathcal{PP}'$ (see section 5.3 vii). We use the local representation (55) of the natural forms on $\mathcal{PP}'$ to prove the following lemma.

Lemma 6.1. The equations

$$\mathrm{d}\Lambda \wedge \Lambda \wedge F \wedge E = 0 \tag{61}$$

$$\mathrm{d}F \wedge \Lambda \wedge F \wedge E = 0 \tag{62}$$

are automatically satisfied everywhere on $\mathcal{PP}'$.

Proof. By using definitions (55) and (16) we easily compute that

$$\begin{aligned}
\mathrm{d}\Lambda \wedge \Lambda \wedge F \wedge E = {} & [\Gamma^3{}_4 + z(\Gamma^2{}_4 - \Gamma^3{}_1) + z'(\Gamma^1{}_4 - \Gamma^3{}_2) \\
& + zz'(\Gamma^3{}_3 + \Gamma^4{}_4 - \Gamma^2{}_2 - \Gamma^1{}_1) - z^2\Gamma^2{}_1 - z'^2\Gamma^1_2 + zz'^2(\Gamma^1_3 - \Gamma^4_2) \\
& + z'z^2(\Gamma^2_3 - \Gamma^4_1) + z^2z'^2\Gamma^4_3]K \wedge \Lambda \wedge F \wedge E \\
& + [z(\Gamma^2{}_2 + \Gamma^1{}_1 - \Gamma^3{}_3 - \Gamma^4{}_4) + z'\Gamma^1{}_2 \\
& + zz'(\Gamma^4{}_2 - \Gamma^1{}_3) - z^2z'\Gamma^4{}_3]F' \wedge \Lambda \wedge F \wedge E
\end{aligned} \tag{63}$$

and

$$\begin{aligned}
\mathrm{d}F \wedge \Lambda \wedge F \wedge E = {} & [z^2\Gamma^4{}_3 + z(\Gamma^1{}_3 - \Gamma^4{}_2) - \Gamma^1{}_2]F' \wedge \Lambda \wedge F \wedge E \\
& + [\Gamma^3{}_2 - \Gamma^1{}_4 + z'\Gamma^1{}_2 + z^2(\Gamma^4{}_1 - \Gamma^2{}_3) + zz'(\Gamma^4{}_2 - \Gamma^1{}_3) - z'z^2\Gamma^4{}_3)] \\
& \times K \wedge \Lambda \wedge F \wedge E.
\end{aligned} \tag{64}$$

Now, the right-hand sides of the above expressions are actually equal to zero due to the Weyl geometry relations (15), which in the tetrad (26) read

$$\begin{gathered}
\Gamma^1{}_2 = \Gamma^2{}_1 = \Gamma^3{}_4 = \Gamma^4{}_3 = 0, \\
\Gamma^1{}_3 = \Gamma^4{}_2, \qquad \Gamma^2{}_3 = \Gamma^4{}_1, \qquad \Gamma^1{}_4 = \Gamma^3{}_2, \qquad \Gamma^2{}_4 = \Gamma^3{}_1, \\
\Gamma^1{}_1 + \Gamma^2{}_2 = \Gamma^3{}_3 + \Gamma^4{}_4 = A.
\end{gathered}$$

This concludes the proof of the lemma†. □

Since equations (61), (62) are satisfied on $\mathcal{PP}'$ it is natural to ask when the forms Λ, F and E form a closed differential ideal. Given the equations (61), (62), this question is equivalent to a question as to when an object $\mathrm{d}E \wedge \Lambda \wedge F \wedge E$ vanishes identically on $\mathcal{PP}'$. Also the related question of the vanishing of $\mathrm{d}E \wedge \Lambda \wedge F' \wedge E$ may be interesting. A long but straightforward calculation leads to the following expressions:

$$\mathrm{d}E \wedge \Lambda \wedge F \wedge E = [-\Psi_0 + 4\Psi_1 z - 6\Psi_2 z^2 + 4\Psi_3 z^3 - \Psi_4 z^4]F' \wedge K \wedge \Lambda \wedge F \wedge E \tag{65}$$

$$\begin{aligned}
\mathrm{d}E \wedge \Lambda \wedge F' \wedge E = {} & [-\tfrac{1}{2}S_{44} + z'S_{24} + zS_{14} - zz'(S_{12} + S_{34}) - \tfrac{1}{2}z'^2S_{22} \\
& - \tfrac{1}{2}z^2S_{11} + z'^2zS_{23} + z^2z'S_{13} - \tfrac{1}{2}z^2z'^2S_{33}]F \wedge K \wedge \Lambda \wedge F' \wedge E.
\end{aligned} \tag{66}$$

Here we have used the notation of section 3 applied to null tetrad (26). The above formulae are implied by the general expressions for differentials of Λ, F, F', E, E' and K, which can be found in appendix C.

† Note that the Weyl connections (17), (18) are not the only connections that imply equations (61), (62). For our purposes, however, it is enough to restrict ourselves to Weyl connections.

Theorem 6.1.

(i) The equation

$$\mathrm{d}E \wedge \Lambda \wedge F \wedge E = 0 \tag{67}$$

is satisfied everywhere on $\mathcal{P}\mathcal{P}'$ if and only if the underlying Weyl geometry (g, A) on $\mathcal{M}$ is anti-self-dual.

(ii) The equation

$$\mathrm{d}E \wedge \Lambda \wedge F' \wedge E = 0 \tag{68}$$

is satisfied everywhere on $\mathcal{P}\mathcal{P}'$ if and only if the underlying Weyl geometry (g, A) satisfies the Einstein–Weyl equations on $\mathcal{M}$.

Proof. In the null tetrad (26), (27) condition (21) expressing the zero trace of S_{ij} reads $S_{12} - S_{34} = 0$. Then the theorem follows immediately from expressions (65), (66) and the requirement that their right-hand sides vanish for any z and z'. □

Applying the transformations (56) we also have analogous facts for primed objects.

Lemma 6.2. The equations

$$\mathrm{d}\Lambda \wedge \Lambda \wedge F' \wedge E' = 0 \tag{69}$$

$$\mathrm{d}F' \wedge \Lambda \wedge F' \wedge E' = 0 \tag{70}$$

are automatically satisfied everywhere on $\mathcal{P}\mathcal{P}'$.

Theorem 6.2.

(i) The equation

$$\mathrm{d}E' \wedge \Lambda \wedge F' \wedge E' = 0 \tag{71}$$

is satisfied everywhere on $\mathcal{P}\mathcal{P}'$ if and only if the underlying Weyl geometry (g, A) on $\mathcal{M}$ is self-dual.

(ii) The equation

$$\mathrm{d}E' \wedge \Lambda \wedge F \wedge E' = 0 \tag{72}$$

is satisfied everywhere on $\mathcal{P}\mathcal{P}'$ if and only if the underlying Weyl geometry (g, A) satisfies the Einstein–Weyl equations on $\mathcal{M}$.

Transformations (56) show (Λ is invariant!) that $\mathrm{d}E \wedge \Lambda \wedge F' \wedge E = 0$ if and only if $\mathrm{d}E' \wedge \Lambda \wedge F \wedge E' = 0$†. This observation, together with the above theorems, leads to the following, interesting corollary.

† Both equations here are equivalent to the Einstein–Weyl equations for $(\mathcal{M}, g, A)$. This is due to the fact that the symmetric Ricci tensor $R_{(ij)}$ (or, equivalently S_{ij} and R) is fully encoded in the differential of the spin connection E. The same information about the symmetric Ricci tensor is also encoded in $\mathrm{d}E'$. To see this, it is enough to note that R as well as the quantity

$$\left[-\tfrac{1}{2}S_{44} + z'S_{24} + zS_{14} - zz'(S_{12} + S_{34}) - \tfrac{1}{2}z'^2S_{22} - \tfrac{1}{2}z^2S_{11} + z'^2zS_{23} + z^2z'S_{13} - \tfrac{1}{2}z^2z'^2S_{33}\right]$$

of equation (66) are invariant under the transformations (56). See also appendix C for explicit forms of $\mathrm{d}E$ and $\mathrm{d}E'$.

Corollary 6.1. A Weyl geometry $(\mathcal{M}, g, A)$ is anti-self-dual and satisfies the Einstein–Weyl equations if and only if the forms (Λ, F, E, E') form a closed differential ideal on $\mathcal{PP}'$, i.e. iff on $\mathcal{PP}'$ we have

$$\begin{aligned}
&\mathrm{d}\Lambda \wedge E \wedge \Lambda \wedge F = 0\\
&\mathrm{d}F \wedge E \wedge \Lambda \wedge F = 0\\
&\mathrm{d}E \wedge E \wedge \Lambda \wedge F = 0\\
&\mathrm{d}E' \wedge E' \wedge \Lambda \wedge F = 0.
\end{aligned}$$

$(\mathcal{M}, g, A)$ is self-dual and satisfies the Einstein–Weyl equations if and only if on $\mathcal{PP}'$ we have

$$\begin{aligned}
&\mathrm{d}\Lambda \wedge E' \wedge \Lambda \wedge F' = 0\\
&\mathrm{d}F' \wedge E' \wedge \Lambda \wedge F' = 0\\
&\mathrm{d}E' \wedge E' \wedge \Lambda \wedge F' = 0\\
&\mathrm{d}E \wedge E \wedge \Lambda \wedge F' = 0,
\end{aligned}$$

i.e. iff (Λ, F', E, E') form a closed differential ideal on $\mathcal{PP}'$.

The second part of the corollary follows from the first by applying (56).

According to section 5.3 vi the forms (Λ, F, E, E') and (Λ, F', E, E') define natural maximal totally null bundles $\mathcal{N}_{FEE'}$ and $\mathcal{N}_{F'EE'}$ on $\mathcal{PP}'$. Using lemma 3.1 we find that the above corollary has the following geometrical interpretation.

Theorem 6.3.

(i) The natural totally null bundle $\mathcal{N}_{FEE'}$ of maximal dimension over $\mathcal{PP}'$ satisfies the integrability conditions (13) if and only if the corresponding Weyl geometry $(\mathcal{M}, g, A)$ is anti-self-dual and satisfies the Einstein–Weyl equations.

(ii) The natural totally null bundle $\mathcal{N}_{F'EE'}$ of maximal dimension on $\mathcal{PP}'$ satisfies the integrability conditions (13) if and only if the corresponding Weyl geometry $(\mathcal{M}, g, A)$ is self-dual and satisfies the Einstein–Weyl equations.

Looking at the formula (C3) (see appendix C), which gives a differential $\mathrm{d}\Lambda$, we see that $\mathrm{d}\Lambda \wedge \Lambda \wedge F \wedge \bar{E} \wedge E' = F' \wedge E \wedge \Lambda \wedge F \wedge \bar{E} \wedge E' \neq 0$. This means that the system $(\Lambda, F, \bar{E}, E')$ never forms a closed differential ideal. Thus the totally null bundle $\mathcal{N}_{F\bar{E}E'}$ has no chance of being integrable.

Similarly, since $\mathrm{d}\Lambda \wedge \Lambda \wedge F \wedge \bar{E} \wedge \bar{E}' = F' \wedge E \wedge \Lambda \wedge F \wedge \bar{E} \wedge \bar{E}' \neq 0$, $\mathrm{d}\Lambda \wedge \Lambda \wedge F' \wedge E \wedge \bar{E}' = F \wedge E' \wedge \Lambda \wedge F' \wedge E \wedge \bar{E}' \neq 0$, $\mathrm{d}\Lambda \wedge \Lambda \wedge F' \wedge \bar{E} \wedge \bar{E}' = F \wedge E' \wedge \Lambda \wedge F' \wedge \bar{E} \wedge \bar{E}' \neq 0$ then also neither of the bundles $\mathcal{N}_{F\bar{E}\bar{E}'}$, $\mathcal{N}_{F'E\bar{E}'}$, $\mathcal{N}_{F'\bar{E}\bar{E}'}$ is integrable in the sense of definition (13). To study the integrability conditions (13) for bundles $\mathcal{N}_{FE\bar{E}'}$ and $\mathcal{N}_{F'\bar{E}E'}$ we need to specify the real structure on $\mathcal{M}$. We postpone discussion of this case to the next section.

Analogous results about geometrical objects on $\mathcal{P}$ and $\mathcal{P}'$ (see appendix D for differentials of the basis 1-forms) are summarized below.

Theorem 6.4. A Weyl geometry $(\mathcal{M}, g, A)$ is anti-self-dual if and only if the forms (F, B, E) form a closed differential ideal on $\mathcal{P}$, i.e. iff

$$\begin{aligned}
&\mathrm{d}F \wedge E \wedge F \wedge B = 0\\
&\mathrm{d}B \wedge E \wedge F \wedge B = 0\\
&\mathrm{d}E \wedge E \wedge F \wedge B = 0.
\end{aligned} \tag{73}$$

Similarly, a Weyl geometry $(\mathcal{M}, g, A)$ is self-dual iff on $\mathcal{P}'$ we have

$$\begin{aligned} &\mathrm{d}F' \wedge E' \wedge F' \wedge B' = 0 \\ &\mathrm{d}B' \wedge E' \wedge F' \wedge B' = 0 \\ &\mathrm{d}E' \wedge E' \wedge F' \wedge B' = 0. \end{aligned} \tag{74}$$

An obvious reinterpretation of this theorem in terms of the natural maximal totally null bundles on $\mathcal{P}$ and $\mathcal{P}'$ reads as follows.

Theorem 6.5. A Weyl geometry $(\mathcal{M}, g, A)$ is anti-self-dual if and only if the natural totally null bundle $\mathcal{N}_E$ on $\mathcal{P}$ is integrable.

Similarly, a Weyl geometry $(\mathcal{M}, g, A)$ is self-dual iff on $\mathcal{P}'$ the natural totally null bundle $\mathcal{N}_{E'}$ is integrable.

Looking at the differentials of F on $\mathcal{P}$ and F' on $\mathcal{P}'$ (see appendix D) we find that $\mathrm{d}F \wedge F \wedge B \wedge \bar{E} = E \wedge K \wedge F \wedge B \wedge \bar{E} \neq 0$ and $\mathrm{d}F' \wedge F' \wedge B' \wedge \bar{E}' = E' \wedge K \wedge F' \wedge B' \wedge \bar{E}' \neq 0$, respectively, on $\mathcal{P}$ and $\mathcal{P}'$. This proves the following statement.

Theorem 6.6. Neither of the natural totally null bundles $\mathcal{N}_{\bar{E}}$ on $\mathcal{P}$ and $\mathcal{N}_{\bar{E}'}$ on $\mathcal{P}'$ is integrable.

The above two theorems are the Weyl-geometric counterparts of the Atiyah–Hitchin–Penrose–Singer theorems [1, 11] for Lorentzian and Euclidean Riemannian 4-manifolds. It is interesting that the integrability conditions of $\mathcal{N}_E$ and $\mathcal{N}_{E'}$ say nothing about A. They only restrict the possible metrics on $\mathcal{M}$.

7. Real structures

7.1. Reality conditions for the natural structures on twistor bundles

In this section we consider real Weyl geometries $(\mathcal{M}, g, A)$. This means that the metric g and the 1-form A are real-valued. Such Weyl geometries and their twistor bundles are particular cases of the Weyl geometries considered in previous sections. Hence, our results of the previous sections are also valid here. In particular, a null cotetrad (M, P, N, K) for g may be chosen in such a way that

$$\bar{M} = P, \qquad \bar{N} = (1-|\varepsilon|)N + \varepsilon K, \qquad \bar{K} = \varepsilon N + (1-|\varepsilon|)K, \tag{75}$$

where $\varepsilon = 0, 1, -1$ for Lorentzian, neutral and Euclidean signature, respectively. Equations (75) imply the following reality conditions for the Weyl connection 1-forms:

$$\begin{aligned} &\bar{\Gamma}^2_2 = \Gamma^1{}_1 \\ &\bar{\Gamma}^2_3 = (1-|\varepsilon|)\Gamma^1{}_3 + \varepsilon\Gamma^1{}_4 \\ &\bar{\Gamma}^2_4 = (1-|\varepsilon|)\Gamma^1{}_4 + \varepsilon\Gamma^1{}_3 \\ &\bar{\Gamma}^1_3 = (1-|\varepsilon|)\Gamma^2{}_3 + \varepsilon\Gamma^2{}_4 \\ &\bar{\Gamma}^1_4 = (1-|\varepsilon|)\Gamma^2{}_4 + \varepsilon\Gamma^2{}_3 \\ &\bar{\Gamma}^4_4 = (1-2|\varepsilon|)\Gamma^4{}_4 + |\varepsilon|A \\ &\bar{\Gamma}^3_3 = (1-2|\varepsilon|)\Gamma^3{}_3 + |\varepsilon|A. \end{aligned} \tag{76}$$

Reality conditions for the curvature coefficients can be obtained from these equations.

The above reality conditions imply the following properties of the natural maximal totally null bundles on $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{PP}'$.

(i) *Reality conditions for the maximal totally null bundles on $\mathcal{P}$ and $\mathcal{P}'$.* On $\mathcal{P}$ we have two natural maximal totally null bundles $\mathcal{N}_E$ and $\mathcal{N}_{\bar{E}}$. It is easy to see that their real indices r_E and $r_{\bar{E}}$ are equal. They depend on the signature of g according to $r_E = r_{\bar{E}} = 1 - |\varepsilon|$. In the local representation $(x, z, \bar{z})$ of $\mathcal{P}$ one finds that

$$\mathcal{N}_E \cap \bar{\mathcal{N}}_E = \mathcal{N}_{\bar{E}} \cap \bar{\mathcal{N}}_{\bar{E}} = (1 - |\varepsilon|)(\tilde{k} - z\tilde{m} - \bar{z}\tilde{p} + z\bar{z}\tilde{n}), \tag{77}$$

where $(\tilde{m}, \tilde{p}, \tilde{n}, \tilde{k})$ are the horizontal lifts of the null tetrad (m, p, n, k) from $\mathcal{M}$ to $\mathcal{P}$. An analogous formula for $\mathcal{N}_{E'}$ and $\mathcal{N}_{\bar{E}'}$ reads

$$\mathcal{N}_{E'} \cap \bar{\mathcal{N}}_{E'} = \mathcal{N}_{\bar{E}'} \cap \bar{\mathcal{N}}_{\bar{E}'} = (1 - |\varepsilon|)(\tilde{k} - z'\tilde{p} - \bar{z}'\tilde{m} + z'\bar{z}'\tilde{n}). \tag{78}$$

(ii) *Reality conditions for the maximal totally null bundles on $\mathcal{PP}'$.* We have eight natural totally null bundles on $\mathcal{PP}'$. Their reality conditions are given below in the local representation $(x, z, \bar{z}, z', \bar{z}')$ of $\mathcal{PP}'$.

$$\begin{aligned}
\mathcal{N}_{FEE'} \cap \bar{\mathcal{N}}_{FEE'} &= \mathcal{N}_{FE\bar{E}'} \cap \bar{\mathcal{N}}_{FE\bar{E}'} = \mathcal{N}_{F\bar{E}E'} \cap \bar{\mathcal{N}}_{F\bar{E}E'} = \mathcal{N}_{F\bar{E}\bar{E}'} \cap \bar{\mathcal{N}}_{F\bar{E}\bar{E}'} \\
&= (1 - |\varepsilon|)(\tilde{\tilde{k}} - z\tilde{\tilde{m}} - \bar{z}\tilde{\tilde{p}} + z\bar{z}\tilde{\tilde{n}}), \\
\mathcal{N}_{F'EE'} \cap \bar{\mathcal{N}}_{F'EE'} &= \mathcal{N}_{F'E\bar{E}'} \cap \bar{\mathcal{N}}_{F'E\bar{E}'} = \mathcal{N}_{F'\bar{E}E'} \cap \bar{\mathcal{N}}_{F'\bar{E}E'} = \mathcal{N}_{F'\bar{E}\bar{E}'} \cap \bar{\mathcal{N}}_{F'\bar{E}\bar{E}'} \\
&= (1 - |\varepsilon|)(\tilde{\tilde{k}} - z'\tilde{\tilde{p}} - \bar{z}'\tilde{\tilde{m}} + z'\bar{z}'\tilde{\tilde{n}}).
\end{aligned}$$

Here $(\tilde{\tilde{m}}, \tilde{\tilde{p}}, \tilde{\tilde{n}}, \tilde{\tilde{k}})$ are horizontal lifts of null tetrad (m, p, n, k) to $\mathcal{PP}'$.

Note that the real indices of all of the natural maximal totally null bundles on $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{PP}'$ are either 0 or 1. Thus, the twistor bundles get naturally equipped with either Hermitian or optical geometries. We do not see possibilities for distinguishing r-optical geometries with $r > 1$ on $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{PP}'$.

7.2. Euclidean and neutral signature

These cases are characterized by $|\varepsilon| = 1$. It follows from section 7.1 that the real indices of all the natural maximal totally null bundles are equal to zero. This, in particular, means that $\mathcal{N}_E$ and $\mathcal{N}_{\bar{E}}$ define almost complex structures $\mathcal{J}_E$ and $\mathcal{J}_{\bar{E}}$ on $\mathcal{P}$. These structures are almost Hermitian in any metric $\tilde{g}$. Similarly, we have two natural almost Hermitian structures $(\mathcal{J}_{E'}, \tilde{g}')$ and $(\mathcal{J}_{\bar{E}'}, \tilde{g}')$ on $\mathcal{P}'$, and eight almost Hermitian structures $(\mathcal{J}_{FEE'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{FE\bar{E}'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F\bar{E}E'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F\bar{E}\bar{E}'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F'EE'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F'E\bar{E}'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F'\bar{E}E'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F'\bar{E}\bar{E}'}, \tilde{\tilde{g}})$ on $\mathcal{PP}'$. Integrability conditions of these almost Hermitian structures are equivalent to conditions (13) for the corresponding $\mathcal{N}$s. Most of them have already been studied in section 6. The integrability conditions for the remaining two structures $(\mathcal{J}_{FE\bar{E}'}, \tilde{\tilde{g}})$ and $(\mathcal{J}_{F'\bar{E}E'}, \tilde{\tilde{g}})$ follow from the expressions of appendix C. In particular, looking at (C3) and the primed counterpart of (C2) we find that

$$\begin{aligned}
&\mathrm{d}\Lambda \wedge \Lambda \wedge F' \wedge E' \wedge \bar{E} \equiv 0, \\
&\mathrm{d}F' \wedge \Lambda \wedge F' \wedge E' \wedge \bar{E} \equiv 0.
\end{aligned}$$

On the other hand (C8) and the primed counterpart of (C1) show that

$$\begin{aligned}
&\mathrm{d}E' \wedge \Lambda \wedge F' \wedge E' \wedge \bar{E} \equiv 0, \\
&\mathrm{d}\bar{E} \wedge \Lambda \wedge F' \wedge E' \wedge \bar{E} \equiv 0
\end{aligned}$$

if and only if the Weyl geometry is self-dual and satisfies the Einstein–Weyl equations. Thus we find that $(\mathcal{J}_{F'\bar{E}E'}, \tilde{\tilde{g}})$ is integrable only for such Weyl geometries. An analogous result also holds for $(\mathcal{J}_{FE\bar{E}'}, \tilde{\tilde{g}})$. This leads to the following theorem.

Theorem 7.1. Let $\mathcal{M}$ be a four-dimensional real manifold equipped with a Euclidean or a neutral signature Weyl geometry (g, A). Let $\mathcal{P}$ (respectively, $\mathcal{P}'$) be a corresponding twistor bundle of all self-dual (respectively, anti-self-dual) maximal totally null spaces over $\mathcal{M}$. Let $\mathcal{P}\mathcal{P}'$ be a fibre product of the bundles $\mathcal{P}$ and $\mathcal{P}'$.

(i) There are two natural almost Hermitian structures $\mathcal{J}_E$ and $\mathcal{J}_{\bar{E}}$ on $\mathcal{P}$. $\mathcal{J}_E$ is integrable if and only if the Weyl geometry is anti-self-dual. $\mathcal{J}_{\bar{E}}$ is never integrable.

(ii) There are two natural almost Hermitian structures $\mathcal{J}_{E'}$ and $\mathcal{J}_{\bar{E}'}$ on $\mathcal{P}'$. $\mathcal{J}_E'$ is integrable if and only if the Weyl geometry is self-dual. $\mathcal{J}_{\bar{E}'}$ is never integrable.

(iii) There are eight natural almost Hermitian structures $(\mathcal{J}_{FEE'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{FE\bar{E}'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F\bar{E}E'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F\bar{E}\bar{E}'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F'EE'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F'E\bar{E}'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F'\bar{E}E'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F'\bar{E}\bar{E}'}, \tilde{\tilde{g}})$ on $\mathcal{P}\mathcal{P}'$.

   (a) $(\mathcal{J}_{FEE'}, \tilde{\tilde{g}})$ is integrable if and only if the Weyl geometry is anti-self-dual and satisfies the Einstein–Weyl equations. These integrability conditions are also equivalent to the integrability of $(\mathcal{J}_{FE\bar{E}'}, \tilde{\tilde{g}})$.

   (b) $(\mathcal{J}_{F'EE'}, \tilde{\tilde{g}})$ is integrable if and only if the Weyl geometry is self-dual and satisfies the Einstein–Weyl equations. These integrability conditions are also equivalent to the integrability of $(\mathcal{J}_{F\bar{E}E'}, \tilde{\tilde{g}})$.

   (c) $(\mathcal{J}_{F\bar{E}E'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F\bar{E}\bar{E}'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F'E\bar{E}'}, \tilde{\tilde{g}})$, $(\mathcal{J}_{F'\bar{E}\bar{E}'}, \tilde{\tilde{g}})$ are never integrable.

Note that there are no restrictions on the potential A in this theorem. This is a generalization of the classical Atiyah–Hitchin–Singer theorem [1]. It is interesting that in a purely Euclidean case ($\varepsilon = -1$, $A = 0$) we have a holomorphic interpretation of the self-dual Einstein equations.

In section 6, theorem 6.1, we interpreted an invariant equation

$$\mathrm{d}E \wedge \Lambda \wedge F' \wedge E = 0 \tag{79}$$

on $\mathcal{P}\mathcal{P}'$ as a necessary and sufficient condition for the Weyl geometry to satisfy the Einstein–Weyl equations. In the present real case we can reformulate this fact in holomorphic language.

Given an almost complex structure $\mathcal{J}$ on $\mathcal{P}\mathcal{P}'$ we can decompose the complexification $\mathrm{T}(\mathcal{P}\mathcal{P}')^{\mathbf{C}}$ of its tangent bundle onto the eigenspaces of $\mathcal{J}$. The $+\mathrm{i}$, $-\mathrm{i}$ eigenspaces are denoted, respectively, by $\mathrm{T}^{(1,0)}$ and $\mathrm{T}^{(0,1)}$. One easily finds that $\mathrm{T}^{(0,1)}$ is the same as the maximal totally null bundle $\mathcal{N}$ representing $\mathcal{J}$. The above decomposition of $\mathrm{T}(\mathcal{P}\mathcal{P}')^{\mathbf{C}}$ induces an analogous decomposition of the complexification $\mathrm{T}^*(\mathcal{P}\mathcal{P}')^{\mathbf{C}}$ of the cotangent space. We denote by $\mathrm{T}^{*(1,0)}$ the annihilator of $\mathrm{T}^{(0,1)}$ and by $\mathrm{T}^{*(0,1)}$ the annihilator of $\mathrm{T}^{(1,0)}$. In an analogous way $\mathrm{T}^{*(u,w)}$ denotes an exterior product of u copies of $\mathrm{T}^{*(1,0)}$ and w copies of $\mathrm{T}^{*(0,1)}$ bundles. It is well known that we have the following decomposition of the bundle Λ^2 of all complex-valued 2-forms on $\mathcal{P}\mathcal{P}'$:

$$\Lambda^2 = \mathrm{T}^{*(2,0)} \oplus \mathrm{T}^{*(1,1)} \oplus \mathrm{T}^{*(0,2)}. \tag{80}$$

Consider now a natural almost complex structure $\mathcal{J}_{F'EE'}$ on $\mathcal{P}\mathcal{P}'$ and its corresponding decomposition of Λ^2. We analyse the differential $\mathrm{d}E$ of the spin connection 1-form from the point of view of this decomposition. To do this we consider a set $\mathcal{W}$ of 2-forms over $\mathcal{P}\mathcal{P}'$ defined by

$$\mathcal{W} = \{w \in \Gamma(\Lambda^2) \text{ s.t. } w = \Omega \wedge E + t\,\mathrm{d}E\},$$

where Ω and t are, respectively, any complex-valued 1-form and function on $\mathcal{P}\mathcal{P}'$. We decompose $\mathcal{W}$ according to (80).

Theorem 7.2. A real Euclidean- or a neutral-signature four-dimensional Weyl geometry $(\mathcal{M}, g, A)$ satisfies the Einstein–Weyl equations if and only if

$$\mathcal{W} \cap \Gamma(\mathrm{T}^{*(0,2)}) = \{0\}.$$

Proof. Observe that sections of $\mathrm{T}^{*(2,0)}$ and $\mathrm{T}^{*(1,1)}$ are in the ideal generated by the forms (Λ, F', E, E'). On the other hand the set of sections of $\mathrm{T}^{*(0,2)}$ has zero intersection with this ideal. Thus in the decomposition of the sections of $\mathrm{T}^{*(0,2)}$ onto a basis corresponding to $(F, F', \Lambda, T, E, \bar{E}, E', \bar{E}')$ there are no forms Λ, F', E, E'. Now the proof follows directly from the differential of E given by (C1) (see appendix C).

We close this section by considering a Weyl geometry which is not anti-self-dual. Fibres of its twistor bundle $\mathcal{P}$ have a discrete number of distinguished points. To see this consider such points in a fibre $\pi^{-1}(x)$, $x \in \mathcal{M}$, in which the expression $\mathrm{d}E \wedge E \wedge F \wedge B$ vanishes. Due to

$$\mathrm{d}E \wedge E \wedge F \wedge B = [-\Psi_0 + 4\Psi_1 z - 6\Psi_2 z^2 + 4\Psi_3 z^3 - \Psi_4 z^4] P \wedge K \wedge E \wedge F \wedge B \qquad (81)$$

we find that in the not anti-self-dual case there are at most four zs, corresponding to four points at the fibre $\pi^{-1}(x)$, in which the right-hand side of (81) is zero. These four points correspond to four maximal totally null self-dual planes at x. Thus, in a generic case, at every point of $\mathcal{M}$ we have four distinguished almost Hermitian structures. It further follows from reality conditions for Ψ_μ that in the not anti-self-dual case these four structures grouped in pairs of mutually conjugated structures. These two pairs may coincide in particular cases and, together with the additional two pairs associated with similar considerations on $\mathcal{P}'$, may be used to classify Weyl geometries. An interesting fact is that in the non-half-flat case these distinguished almost Hermitian structures are the only ones that may be integrable on $\mathcal{M}$ [2]. A less well known fact is that in a purely Euclidean case ($\varepsilon = -1$, $A = 0$), if the Einstein equations are satisfied, then any of the distinguished almost Hermitian structures is integrable [4, 7, 12].

7.3. Lorentzian signature

Due to the condition $\varepsilon = 0$ the fibres of $\mathcal{N}_{FEE'} \cap \bar{\mathcal{N}}_{FEE'}$, $\mathcal{N}_{FE\bar{E}'} \cap \bar{\mathcal{N}}_{FE\bar{E}'}$, $\mathcal{N}_{F\bar{E}E'} \cap \bar{\mathcal{N}}_{F\bar{E}E'}$ and $\mathcal{N}_{F\bar{E}\bar{E}'} \cap \bar{\mathcal{N}}_{F\bar{E}\bar{E}'}$ are all one-dimensional and, at every point $p \in \mathcal{P}\mathcal{P}'$, are spanned by a real vector

$$\kappa = \tilde{\tilde{k}} - z\tilde{\tilde{m}} - \bar{z}\tilde{\tilde{p}} + z\bar{z}\tilde{\tilde{n}}.$$

The fibres of $\mathcal{N}_{F'EE'} \cap \bar{\mathcal{N}}_{F'EE'}$, $\mathcal{N}_{F'E\bar{E}'} \cap \bar{\mathcal{N}}_{F'E\bar{E}'}$, $\mathcal{N}_{F'\bar{E}E'} \cap \bar{\mathcal{N}}_{F'\bar{E}E'}$, $\mathcal{N}_{F'\bar{E}\bar{E}'} \cap \bar{\mathcal{N}}_{F'\bar{E}\bar{E}'}$ are spanned by a real vector

$$\kappa' = (\tilde{\tilde{k}} - z'\tilde{\tilde{p}} - \bar{z}'\tilde{\tilde{m}} + z'\bar{z}'\tilde{\tilde{n}}).$$

These two real vectors are null and, together with their corresponding maximal totally null spaces, define eight distinguished optical geometries $\mathcal{O}_{FEE'}$, $\mathcal{O}_{F\bar{E}E'}$, $\mathcal{O}_{FE\bar{E}'}$, $\mathcal{O}_{F\bar{E}\bar{E}'}$, $\mathcal{O}_{F'EE'}$, $\mathcal{O}_{F'\bar{E}E'}$, $\mathcal{O}_{F'E\bar{E}'}$, $\mathcal{O}_{F'\bar{E}\bar{E}'}$ on $\mathcal{P}\mathcal{P}'$. According to section 2 any seven-dimensional submanifold of $\mathcal{P}\mathcal{P}'$ transversal to κ is equipped with four CR structures that correspond to $\mathcal{O}_{FEE'}$, $\mathcal{O}_{F\bar{E}E'}$, $\mathcal{O}_{FE\bar{E}'}$, $\mathcal{O}_{F\bar{E}\bar{E}'}$. Another four families of CR structures are associated with the seven-dimensional submanifolds of $\mathcal{P}\mathcal{P}'$ transversal to κ'. To get a Lorentzian version of theorem 7.1 we still need to note that if $\varepsilon = 0$ then $\Psi'_\mu = \bar{\Psi}_\mu$ and $\phi'_a = \bar{\phi}_a$ for all $\mu = 0, 1, 2, 3, 4$ and all $a = 0, 1, 2$. Thus in this case (anti)-self-duality of the Weyl tensor is equivalent to conformal flatness of the metric and (anti)-self-duality of $\mathcal{F}$ means that $\mathcal{F} \equiv 0$.

Theorem 7.3. Let $\mathcal{M}$ be a four-dimensional real manifold equipped with a Lorentzian Weyl geometry (g, A). Let $\mathcal{P}$ (respectively, $\mathcal{P}'$) be the corresponding twistor bundle of all self-dual (respectively, anti-self-dual) maximal totally null spaces over $\mathcal{M}$. Let $\mathcal{P}\mathcal{P}'$ be a fibre product of bundles $\mathcal{P}$ and $\mathcal{P}'$.

(i) There are eight natural almost optical geometries $\mathcal{O}_{FEE'}$, $\mathcal{O}_{FE\bar{E}'}$, $\mathcal{O}_{F\bar{E}E'}$, $\mathcal{O}_{F\bar{E}\bar{E}'}$, $\mathcal{O}_{F'EE'}$, $\mathcal{O}_{F'E\bar{E}'}$, $\mathcal{O}_{F'\bar{E}E'}$, $\mathcal{O}_{F'\bar{E}\bar{E}'}$ on $\mathcal{P}\mathcal{P}'$.

(ii) The following conditions are equivalent:

(a) $\mathcal{O}_{FEE'}$ (respectively, $\mathcal{O}_{F'EE'}$) is integrable;

(b) there exists a unique integrable seven-dimensional CR structure obtained from $\mathcal{P}\mathcal{P}'$ by identifying points lying on the same integral curve of κ (respectively, κ') and associated with $\mathcal{O}_{FEE'}$ (respectively, $\mathcal{O}_{F'EE'}$);

(c) the Weyl geometry is conformally flat and satisfies the Einstein–Weyl equations.

(iii) The following conditions are equivalent:

(a) $\mathcal{O}_{FE\bar{E}'}$ (respectively, $\mathcal{O}_{F'\bar{E}E'}$) is integrable;

(b) there exists a unique integrable seven-dimensional CR structure obtained from $\mathcal{P}\mathcal{P}'$ by identifying points lying on the same integral curve of κ (respectively, κ') and associated with $\mathcal{O}_{FE\bar{E}'}$ (respectively, $\mathcal{O}_{F'\bar{E}E'}$);

(c) the Weyl geometry is conformally flat, has zero scalar curvature R and a potential A such that $\mathrm{d}A \equiv 0$.

(iv) $\mathcal{O}_{F\bar{E}E'}$, $\mathcal{O}_{F\bar{E}\bar{E}'}$, $\mathcal{O}_{F'E\bar{E}'}$ and $\mathcal{O}_{F\bar{E}\bar{E}'}$ are never integrable.

Only point (iiic) of the theorem requires justification. It is, however, easy to see that this follows from the differentials of the basis 1-forms given in appendix C and from formula (C8) with $\varepsilon = 0$.

To find a passage from the above results to the description of the original Penrose bundle given in section 2 we proceed as follows.

First, we note that $\mathcal{P}$ defined as a bundle of all self-dual totally null 2-planes in $\mathcal{M}$ is naturally isomorphic to the Penrose bundle of all real null directions in $\mathcal{M}$. This is due to the fact that in the case of a Lorentzian metric any two-dimensional totally null plane in $\mathcal{M}$ is in one-to-one correspondence with a null direction.

Second, we note that the natural field of real null directions κ can be naturally pushed forward from $\mathcal{P}\mathcal{P}'$ to $\mathcal{P}$ by means of projection pr_*. This defines a field of real directions $X_L = \mathrm{pr}_*\kappa$ on $\mathcal{P}$ which is null in any metric $\tilde{g}$. Using X_L we define a field of directions of a real 1-form $\Lambda_L = \tilde{g}(X_L)$. Now, we can take $E_L = E$ and define 1-forms F_L (complex) and T (real) by $\tilde{g} = h\bar{h}E_L\bar{E}_L + \Lambda_L T + F_L\bar{F}_L$. It is easy to see that the forms (Λ_L, F_L, E_L) are precisely given up to transformations (3)–(5). The above information is sufficient to reconstruct on $\mathcal{P}$ all the structures of section 2. In particular, the annihilator $\mathcal{N}_{FE}$ of (Λ_L, F_L, E_L) defines an almost optical geometry $\mathcal{O}_{FE}$ on $\mathcal{P}$. Similarly the annihilators $\mathcal{N}_{\bar{F}E}$, $\mathcal{N}_{F\bar{E}}$ and $\mathcal{N}_{\bar{F}\bar{E}}$ of, respectively, $(\Lambda_L, \bar{F}_L, E_L)$, $(\Lambda_L, F_L, \bar{E}_L)$ and $(\Lambda_L, \bar{F}_L, \bar{E}_L)$ define optical geometries $\mathcal{O}_{\bar{F}E}$, $\mathcal{O}_{F\bar{E}}$ and $\mathcal{O}_{\bar{F}\bar{E}}$. Their integrability conditions are summarized in the following theorem.

Theorem 7.4. There are four natural optical geometries $\mathcal{O}_{FE}$, $\mathcal{O}_{\bar{F}E}$, $\mathcal{O}_{F\bar{E}}$ and $\mathcal{O}_{\bar{F}\bar{E}}$ on a six-dimensional Penrose twistor bundle of all null directions over a four-dimensional manifold $\mathcal{M}$ equipped with a Lorentzian Weyl geometry (g, A).

(i) The following conditions are equivalent:

(a) $\mathcal{O}_{FE}$ or $\mathcal{O}_{\bar{F}\bar{E}}$ is integrable;

(b) a five-dimensional manifold of lines of a congruence generated by X_L is naturally equipped with an integrable CR structure;
(c) the metric g is conformally flat.

(ii) $\mathcal{O}_{\bar{F}E}$ and $\mathcal{O}_{F\bar{E}}$ are never integrable.

We failed to find a geometrical interpretation of the following theorem.

Theorem 7.5. A Lorentzian four-dimensional Weyl geometry satisfies the Einstein–Weyl equations if the forms $(E_L, \bar{F}_L, \Lambda_L)$ satisfy an invariant equation

$$\mathrm{d}E \wedge \bar{F}_L \wedge \Lambda_L \wedge E_L = 0$$

everywhere on $\mathcal{P}$.

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Appendix A

In this appendix we present formulae which give decompositions of certain Weyl-geometric objects onto the Levi-Civita and A-dependent parts.

For the Riemann tensor 2-forms Ω_{ij} we have

$$\Omega_{ij} = \omega_{ij} + \tfrac{1}{2} g_{ij}\mathcal{F} + DA_{[j} \wedge \theta_{i]} + \tfrac{1}{2} A_{[j}\theta_{i]} \wedge A - \tfrac{1}{4} A^2 \theta_i \wedge \theta_j, \tag{A1}$$

where ω_{ij} denote curvature 2-forms of the Levi-Civita connection γ_{ij},

$$\mathrm{D}A_i = A_{i;j}\theta^j = \mathrm{d}A_i - A_j \gamma^j{}_i \qquad \theta_i = g_{ik}\theta^k \tag{A2}$$

and $A^2 = g^{ij} A_i A_j$. From these equations one easily gets curvature coefficients

$$R_{ijkl} = r_{ijkl} + \tfrac{1}{2} g_{ij}\mathcal{F}_{kl} + (A_{j;[k}g_{l]i} - A_{i;[k}g_{l]j}) + \tfrac{1}{2}(A_j g_{i[k}A_{l]} - A_i g_{j[k}A_{l]}) - \tfrac{1}{2} A^2 g_{i[k}g_{l]j}. \tag{A3}$$

Here r_{ijkl} are the usual components of the curvature tensor for the Levi-Civita connection $\gamma^i{}_j$.

The Ricci tensor and Ricci scalar decompositions read, respectively,

$$R_{jl} = r_{jl} + \tfrac{1}{2}\mathcal{F}_{jl} - A_{j;l} - \tfrac{1}{2} g^{ik} A_{i;k} g_{jl} + \tfrac{1}{2} A_j A_l - \tfrac{1}{2} A^2 g_{jl}, \tag{A4}$$

$$R = r - 3g^{jl} A_{j;l} - \tfrac{3}{2} A^2. \tag{A5}$$

Here quantities r_{ij} and r denote, respectively, the Ricci tensor and Ricci scalar of the Levi-Civita connection associated with g.

The symmetric part of the Ricci tensor decomposes according to

$$R_{(jl)} = r_{jl} - A_{(j;l)} - \tfrac{1}{2} g^{ik} A_{i;k} g_{jl} + \tfrac{1}{2} A_j A_l - \tfrac{1}{2} A^2 g_{jl}. \tag{A6}$$

S_{ij} has the following decomposition into the Levi-Civita (s_{ij}) and the A-dependent (σ_{ij}) part:

$$S_{ij} = s_{ij} + \sigma_{ij}, \tag{A7}$$

where

$$s_{ij} = r_{ij} - \tfrac{1}{4} r g_{ij} \tag{A8}$$

and

$$\sigma_{ij} = -A_{(i;j)} + \tfrac{1}{4} g^{kl} A_{k;l} g_{ij} + \tfrac{1}{2} A_i A_j - \tfrac{1}{8} A^2 g_{ij}. \tag{A9}$$

Decomposition of C_{ijkl} is given by

$$C_{ijkl} = w_{ijkl} + f_{ijkl}, \tag{A10}$$

where the Levi-Civita (w_{ijkl}) and A-dependent (f_{ijkl}) parts read, respectively,

$$w_{ijkl} = r_{ijkl} + \tfrac{1}{3} r g_{i[k} g_{l]j} + r_{j[k} g_{l]i} + r_{i[l} g_{k]j}, \tag{A11}$$

$$f_{ijkl} = \tfrac{1}{2}(\mathcal{F}_{j[k} g_{l]i} + \mathcal{F}_{i[l} g_{k]j}) + \tfrac{1}{2} g_{ij} \mathcal{F}_{kl}. \tag{A12}$$

Returning to the curvature forms Ω_{ij} we decompose it into antisymmetric and symmetric parts. We note that $\Omega_{[ij]}$ can be further decomposed into a part $\Omega^U_{[ij]}$ with coefficients having all the symmetries of the usual (i.e. Levi-Civita connection) Riemann tensor and the remaining part $\Omega^{NU}_{[ij]}$. Explicitly we have

$$\Omega_{ij} = \Omega^U_{[ij]} + \Omega^{NU}_{[ij]} + \Omega_{(ij)}, \tag{A13}$$

where

$$\Omega_{(ij)} = \tfrac{1}{2} g_{ij} \mathcal{F}, \tag{A14}$$

$$\Omega^{NU}_{[ij]} = -\tfrac{1}{4}(\theta_i \wedge \mathcal{F}_j - \theta_j \wedge \mathcal{F}_i), \tag{A15}$$

$$\Omega^U_{[ij]} = w_{ij} + \tfrac{1}{12} R \theta_i \wedge \theta_j + \tfrac{1}{2}(\theta_i \wedge S_j - \theta_j \wedge S_i), \tag{A16}$$

and $\mathcal{F}_i = \theta^k \mathcal{F}_{ki}$, $S_i = \theta^k S_{ki}$.

Appendix B

Given a Weyl geometry $(\mathcal{M}, g, A)$ consider null cotetrad (26). Then the decomposition of the antisymmetric part of the curvature 2-forms $\Omega_{[ij]}$ onto basis of self-dual and anti-self-dual 2-forms read as follows:

$$\begin{aligned}
\Omega_{[14]} &= \Psi'_0 M \wedge K + \left(\Psi'_1 + \tfrac{1}{4}\phi'_2\right)(N \wedge K + M \wedge P) + \left(\Psi'_2 + \tfrac{1}{12} R + \tfrac{1}{2}\phi'_1\right) N \wedge P \\
&\quad + \tfrac{1}{2} S_{44} P \wedge K + \tfrac{1}{2} S_{41}(N \wedge K - M \wedge P) + \tfrac{1}{2} S_{11} N \wedge M \\
\Omega_{[23]} &= \left(-\Psi'_2 + \tfrac{1}{12} R + \tfrac{1}{2}\phi'_1\right) M \wedge K + \left(-\Psi'_3 + \tfrac{1}{4}\phi'_0\right)(N \wedge K + M \wedge P) - \Psi'_4 N \wedge P \\
&\quad - \tfrac{1}{2} S_{22} P \wedge K - \tfrac{1}{2} S_{32}(N \wedge K - M \wedge P) - \tfrac{1}{2} S_{33} N \wedge M \\
\tfrac{1}{2}(\Omega_{[34]} + \Omega_{[12]}) &= \left(\Psi'_1 - \tfrac{1}{4}\phi'_2\right) M \wedge K + \left(\Psi'_2 - \tfrac{1}{24} R\right)(N \wedge K + M \wedge P) \\
&\quad + \left(\Psi'_3 + \tfrac{1}{4}\phi'_0\right) N \wedge P + \tfrac{1}{2} S_{42} P \wedge K \\
&\quad + \tfrac{1}{4}(S_{12} + S_{34})(N \wedge K - M \wedge P) + \tfrac{1}{2} S_{31} N \wedge M
\end{aligned} \tag{B1}$$

$$\Omega_{[24]} = \Psi_0 P\wedge K + \left(\Psi_1 + \tfrac{1}{4}\phi_2\right)(N\wedge K - M\wedge P) + \left(\Psi_2 + \tfrac{1}{12}R + \tfrac{1}{2}\phi_1\right)N\wedge M$$
$$+\tfrac{1}{2}S_{44}M\wedge K + \tfrac{1}{2}S_{42}(N\wedge K + M\wedge P) + \tfrac{1}{2}S_{22}N\wedge P$$
$$\Omega_{[13]} = \left(-\Psi_2 + \tfrac{1}{12}R + \tfrac{1}{2}\phi_1\right)P\wedge K + \left(-\Psi_3 + \tfrac{1}{4}\phi_0\right)(N\wedge K - M\wedge P) - \Psi_4 N\wedge M$$
$$-\tfrac{1}{2}S_{11}M\wedge K - \tfrac{1}{2}S_{31}(N\wedge K + M\wedge P) - \tfrac{1}{2}S_{33}N\wedge P \tag{B2}$$
$$\tfrac{1}{2}(\Omega_{[34]} - \Omega_{[12]}) = \left(\Psi_1 - \tfrac{1}{4}\phi_2\right)P\wedge K + \left(\Psi_2 - \tfrac{1}{24}R\right)(N\wedge K - M\wedge P)$$
$$+\left(\Psi_3 + \tfrac{1}{4}\phi_0\right)N\wedge M + \tfrac{1}{2}S_{41}M\wedge K$$
$$+\tfrac{1}{4}(S_{12} + S_{34})(N\wedge K + M\wedge P) + \tfrac{1}{2}S_{32}N\wedge P.$$

Appendix C

In this appendix we give differentials of basis 1-forms $(E, \bar{E}, E', \bar{E}', F, F', \Lambda, T)$ on $\mathcal{P}\mathcal{P}'$. For a given Weyl geometry $(\mathcal{M}, g, A)$ we use a null cotetrad (26) and represent the basis 1-forms on $\mathcal{P}\mathcal{P}'$ according to (55).

Let

$$2\gamma = \Gamma_{211} + \Gamma_{341} + 2z\Gamma_{131} - z'(\Gamma_{213} + \Gamma_{343}) - 2zz'\Gamma_{133},$$

$$2\omega = \Gamma_{214} + \Gamma_{344} + z(2\Gamma_{134} - \Gamma_{211} - \Gamma_{341}) - z'(\Gamma_{212} + \Gamma_{342})$$
$$+zz'(\Gamma_{213} + \Gamma_{343} - 2\Gamma_{132}) - 2z^2\Gamma_{131} + 2z^2z'\Gamma_{133},$$

$$2\Phi = \tfrac{1}{2}S_{44} - z'S_{24} - zS_{14} + zz'(S_{12} + S_{34}) + \tfrac{1}{2}z'^2S_{22} + \tfrac{1}{2}z^2S_{11}$$
$$-z'^2zS_{23} - z^2z'S_{13} + \tfrac{1}{2}z^2z'^2S_{33},$$

$$4\Psi = \Psi_0 - 4\Psi_1 z + 6\Psi_2 z^2 - 4\Psi_3 z^3 + \Psi_4 z^4,$$

$$4\phi = -\phi_2 + 2z\phi_1 - z^2\phi_0,$$

$$2a = A_1 z + A_2 z' - A_3 zz' - A_4.$$

Applying transformations (56) we also get the quantities γ', ω', Ψ' and ϕ'†.

Vanishing of some of the above coefficients has a well defined meaning in terms of the Weyl geometry on $\mathcal{M}$. Note, for example, that $\Phi \equiv 0$ means that the Weyl geometry satisfies the Einstein–Weyl equations, $\Psi \equiv 0$ means that the Weyl geometry is anti-self-dual, $\phi' \equiv 0$ means that curvature $\mathcal{F}$ of A is self-dual.

Using the above quantities and denoting their derivatives along z or z' by a subscript $_z$ or $_{z'}$ respectively, we find that the differentials of the basis 1-forms read as follows:

$$\mathrm{d}E = 2\gamma E\wedge F - 2\gamma_{z'}\Lambda\wedge E - 2\omega_{z'}E\wedge F' + 2\omega E\wedge K + 2\Phi K\wedge F + 4\Psi K\wedge F'$$
$$+\Phi_{z'z'}\Lambda\wedge F' + \left[\tfrac{1}{3}\Psi_{zz} + \tfrac{1}{12}R + \phi_z\right]\Lambda\wedge F$$
$$+[\Psi_z + \phi][K\wedge\Lambda - F\wedge F'] + \Phi_{z'}[K\wedge\Lambda + F\wedge F'], \tag{C1}$$

$$\mathrm{d}F = E\wedge K + [\gamma_{z'} - 2\gamma'_z - \omega'_{zz'} - a_{zz'}]F\ \mathrm{d}z\Lambda + \gamma'_{z'}F'\wedge\Lambda$$
$$+[\omega'_{z'} + \gamma']K\wedge\Lambda + [\gamma' + \omega_{z'} + a_{z'}]F'\wedge F + [\omega - \omega' + a]F\wedge K, \tag{C2}$$

$$\mathrm{d}\Lambda = F\wedge E' + F'\wedge E + [\gamma - \omega'_z - a_z]\Lambda\wedge F$$
$$+[\gamma' - \omega_{z'} - a_{z'}]\Lambda\wedge F' + [\omega + \omega' + a]\Lambda\wedge K, \tag{C3}$$

† Note that these transformations do not affect Φ and a.

$$\mathrm{d}K = \gamma_{zz'}F\wedge\Lambda + \gamma'_{zz'}F'\wedge\Lambda + [\gamma_{z'}+\gamma'_z + a_{zz'}]\Lambda\wedge K + [\omega'_z - \omega_z - 2\gamma - a_z]K\wedge F$$
$$+[\gamma'_z + a_{zz'} - \gamma_{z'} + \omega'_{zz'} - \text{ø}m_{zz'}]F\wedge F' + [\omega_{z'} - \omega'_{z'} - 2\gamma' - a_{z'}]K\wedge F'. \tag{C4}$$

Differentials of E' and F' are obtained from the above equations by means of transformations (56).

To calculate differentials of $\bar{E}$ and $\bar{E}'$ we need to know what the reality conditions are for the null cotetrad (26) on $\mathcal{M}$. If we assume the reality conditions (75) then we find that

$$\bar{F} = (1-\varepsilon z\bar{z})F' - \varepsilon\bar{z}z'F - \varepsilon\bar{z}\Lambda + (\bar{z} - z' + \varepsilon\bar{z}(zz'-\varepsilon))K, \tag{C5}$$

$$\bar{\Lambda} = (1-|\varepsilon| + \varepsilon\bar{z}\bar{z}')\Lambda + ((1-|\varepsilon| + \varepsilon\bar{z}\bar{z}')z - \bar{z}')F' + ((1-|\varepsilon|+\varepsilon\bar{z}\bar{z}')z' - \bar{z})F$$
$$+[(z'-\bar{z})(\bar{z}'-z) - \varepsilon(\varepsilon - \bar{z}\bar{z})(\varepsilon - zz')]K, \tag{C6}$$

$$\bar{K} = (1-|\varepsilon| + \varepsilon zz')K - \varepsilon z'F - \varepsilon zF' - \varepsilon\Lambda. \tag{C7}$$

The corresponding formula for $\bar{F}'$ may be obtained from $\bar{F}$ by applying (56).

Now, using the above expressions and (C1), (76) we easily get formulae for $\mathrm{d}\bar{E}$ and $\mathrm{d}\bar{E}'$. These, in particular, imply that

$$\mathrm{d}\bar{E}\wedge\bar{E}\wedge\Lambda\wedge F'\wedge E'$$
$$= \Big\{(1-|\varepsilon|)\Big[4\bar{\Psi} + 2(\bar{\Psi}_{\bar{z}} + \bar{\phi})(\bar{z}-z') + \big(\tfrac{1}{3}\bar{\Psi}_{\bar{z}\bar{z}} + \tfrac{1}{12}R + \bar{\phi}_{\bar{z}}\big)(\bar{z}-z')^2\Big]$$
$$+\varepsilon\Big[-2\bar{\Phi} + 2\bar{\Phi}_{\bar{z}'}z'(\varepsilon - z'\bar{z}') - \bar{\Phi}_{\bar{z}'\bar{z}'}(\varepsilon - z'\bar{z}')^2\Big]\Big\}K\wedge F\wedge\bar{E}\wedge\Lambda\wedge F'\wedge\bar{E}'. \tag{C8}$$

As usual, the primed counterpart of (C8) is obtained by applying (56).

Appendix D

Differentials of the basis 1-forms on $\mathcal{P}$ are obtained by using their representation (26), (43).

$$\mathrm{d}F = E\wedge K + [\Gamma_{212} + z(\Gamma_{132} - \Gamma_{213}) - z^2\Gamma_{133}]F\wedge P$$
$$+[\Gamma_{213} - \Gamma_{231} + z\Gamma_{133}]F\wedge B + [\Gamma_{214} + z(\Gamma_{134} - \Gamma_{211}) - z^2\Gamma_{131}]F\wedge K$$
$$+[\Gamma_{234} - z\Gamma_{231}]B\wedge K + [\Gamma_{232} - z\Gamma_{233}]B\wedge P$$

$$\mathrm{d}B = E\wedge P + [-\Gamma_{434} + z(\Gamma_{431} - \Gamma_{314}) + z^2\Gamma_{311}]B\wedge K$$
$$+[-\Gamma_{431} + \Gamma_{413} - z\Gamma_{311}]B\wedge F + [-\Gamma_{432} + z(\Gamma_{433} - \Gamma_{312}) + z^2\Gamma_{313}]B\wedge P$$
$$+[-\Gamma_{412} + z\Gamma_{413}]F\wedge P + [-\Gamma_{414} + z\Gamma_{411}]F\wedge K$$

$$\mathrm{d}E = [\Gamma_{213} + \Gamma_{343} + 2z\Gamma_{133}]E\wedge B$$
$$+[\Gamma_{341} + \Gamma_{211} + 2z\Gamma_{131}]E\wedge F$$
$$+[\Gamma_{212} + \Gamma_{342} + z(2\Gamma_{132} - \Gamma_{213} - \Gamma_{343}) - 2z^2\Gamma_{133}]E\wedge P$$
$$+[\Gamma_{214} + \Gamma_{344} + z(2\Gamma_{134} - \Gamma_{211} - \Gamma_{341}) - 2z^2\Gamma_{131}]E\wedge K$$
$$+\big[\tfrac{1}{2}S_{44} - zS_{41} + \tfrac{1}{2}z^2S_{11}\big]K\wedge F$$
$$+[\Psi_0 - 4\Psi_1 z + 6\Psi_2 z^2 - 4\Psi_3 z^3 + \Psi_4 z^4]K\wedge P$$
$$+\big[\Psi_2 + \tfrac{1}{12}R + \tfrac{1}{2}\phi_1 - z(2\Psi_3 + \tfrac{1}{2}\phi_0) + z^2\Psi_4\big]K\wedge P$$
$$+\big[\tfrac{1}{2}S_{22} - zS_{32} + \tfrac{1}{2}z^2S_{33}\big]P\wedge B$$
$$+\big[-\Psi_1 - \tfrac{1}{4}\phi_2 + z\big(3\Psi_2 + \tfrac{1}{2}\phi_1\big)$$
$$-z^2\big(3\Psi_3 + \tfrac{1}{4}\phi_0\big) - z^3\Psi_4\big](P\wedge F - K\wedge B)$$

$$
\begin{aligned}
&+\left[\tfrac{1}{2}S_{42} - \tfrac{1}{2}z(S_{12}+S_{34}) + \tfrac{1}{2}z^2 S_{31}\right](P\wedge F + K\wedge B)\\
\mathrm{d}K &= \Gamma_{313}B\wedge F + \Gamma_{323}B\wedge P + [\Gamma_{321}-\Gamma_{312}+z\Gamma_{313}]F\wedge P\\
&\quad +[\Gamma_{341}-\Gamma_{314}]F\wedge K + [\Gamma_{343}-z\Gamma_{313}]B\wedge K\\
&\quad +[-\Gamma_{324}+\Gamma_{342}+z(-\Gamma_{312}+\Gamma_{321}-\Gamma_{343})+z^2\Gamma_{313})]P\wedge K\\
\mathrm{d}P &= [-\Gamma_{132}+\Gamma_{123}]P\wedge B + [\Gamma_{121}-z\Gamma_{131}]P\wedge F\\
&\quad +\Gamma_{131}B\wedge F + \Gamma_{141}K\wedge F + [\Gamma_{134}-\Gamma_{143}-z\Gamma_{131}]B\wedge K\\
&\quad +[\Gamma_{124}-\Gamma_{142}+z(\Gamma_{143}-\Gamma_{134}-\Gamma_{121})+z^2\Gamma_{131}]P\wedge K.
\end{aligned}
$$

The complex conjugate of $\mathrm{d}E$ is easily calculable using (75) and (76).

Finally, differentials of the basis 1-forms on $\mathcal{P}'$ follow from the above by applying (56).

Appendix E

In this appendix we study the properties of the basis 1-forms on $\mathcal{PP}'$ from the point of view of the Weyl transformations

$$(g, A) \to (\hat{g}, \hat{A}) = (\mathrm{e}^{2\varphi}g, A - 2\mathrm{d}\varphi). \qquad (*)$$

We start with the local representation (55) of the forms. Then we note that E has the following decomposition onto the Levi-Civita and A-dependent part†.

$$E = E^{LC} + E^{A},$$

where

$$E^{LC} = \mathrm{d}z - \gamma_2^3 + z(\gamma_1^1 - \gamma_4^4) + z^2\gamma_3^2,$$

and

$$E^{A} = \tfrac{1}{2}(A_2 - zA_3)\Lambda + \tfrac{1}{2}(zA_1 + z'A_2 - zz'A_3 - A_4)F.$$

We now represent the Weyl transformations as transformations that change coefficients of the metric and do not affect the null cotetrad. Thus we have

$$(g_{ij}, A_i) \to (\hat{g}_{ij}, \hat{A}_i) = (\mathrm{e}^{2\varphi}g_{ij}, A_i - 2\varphi_{|i}),$$

where the subscript $_{|i}$ means a derivative along the null tetrad vector e_i. Now, it is easy to see that under the above transformations the forms E^{LC} and E^{A} transform as follows.

$$
\begin{aligned}
\hat{E}^{LC} &= E^{LC} + (\varphi_{|2} - z\varphi_{|3})\Lambda + (z\varphi_{|1} + z'\varphi_{|2} - zz'\varphi_{|3} - \varphi_{|4})F,\\
\hat{E}^{A} &= E^{A} - (\varphi_{|2} - z\varphi_{|3})\Lambda - (z\varphi_{|1} + z'\varphi_{|2} - zz'\varphi_{|3} - \varphi_{|4})F.
\end{aligned}
$$

This shows that E is invariant under the Weyl transformations. The same is also true for F, F' and Λ. This, in particular, implies the invariance of the (anti)-self-duality equations (67) under the Weyl transformations. The invariance of the Einstein–Weyl equations (68) under these transformations also follows.

Finally, we comment on the purely Riemannian case. In this case we have a given metric g and $A = 0$. Since our twistor bundles are constructed out of null objects then it is reasonable to ask how the constructions change under the conformal transformations $g \to \hat{g} = \mathrm{e}^{2\varphi}g$ of the metric. It follows from the above transformations of E^{LC} and from the conformal invariance of Λ, F and F' that although the twistor bundles for any metric from a given conformal class are the same, the horizontal spaces for different base metrics are different. This difference is not essential for the system of (anti)-self-duality equations (61),

† Analogous formulae may also be obtained for E'.

(62), (67) which is invariant under the conformal transformations, but it is essential for the Einstein equations (68). In this latter case, we need to pick up a particular metric on $\mathcal{M}$ and then use it to define the forms E, Λ, F'. Using them we can encode the Einstein equations for g in $\mathcal{PP}'$.

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