Some null solutions of the Yang–Mills equations and Cauchy–Riemann structures

Pawel Nurowski
International School for Advanced Studies, S.I.S.S.A., Strada Costiera 11, 34014 Trieste, Italy

(Received 7 March 1991; accepted for publication 4 October 1991)

Some new null solutions of the Yang–Mills equations with congruences of twisting shear-free and null geodesics are obtained. These are test fields. Each of them is defined on a Lorentzian manifold with a metric tensor adapted to the Cauchy–Riemann geometry associated with the congruence. Some examples of solutions on Minkowski space-time are also given. Among them is a solution with a congruence of twisting shear-free and null geodesics different from the Robinson congruence.

I. INTRODUCTION. CAUCHY–RIEMANN STRUCTURES AND THE LORENTZ GEOMETRY

Cauchy–Riemann (CR) structures are related to null (algebraically special) solutions of the Einstein equations, Maxwell equations, and Yang–Mills equations.[1–5,10] The aim of this paper is to find some new null solutions of the Yang–Mills equations using the mathematical framework of the theory of CR structures. The work presented here is inspired by the paper by Tafel.[3] We develop some of the ideas mentioned there. Our considerations are purely local. It follows from the context which neighborhoods are taken into account.

A three-dimensional CR structure is defined on a real manifold N by a class of one-forms $[(\lambda,\mu)]$ (λ is real, μ is complex), given by the equivalence relation R, such that $(\lambda,\mu)R(\lambda',\mu')$ iff

$$\lambda'=f\lambda, \tag{1.1a}$$

$$\mu'=h\mu+e\lambda, \tag{1.1b}$$

where $f\neq0$ is any real function, and $h\neq0,e$ are arbitrary complex functions on N. Additionally, it is always assumed that

$$\lambda\wedge\mu\wedge\bar{\mu}\neq0. \tag{1.2}$$

We will consider Yang–Mills fields in space-time, which is (locally) the product

$$M=\mathbb{R}\times N, \tag{1.3}$$

equipped with the metric tensor

$$g=2p^2(\mu\bar{\mu}-\lambda\tau), \tag{1.4}$$

where p is any real function on M and τ is any real one-form on M, such that

$$\lambda\wedge\mu\wedge\bar{\mu}\wedge\tau\neq0. \tag{1.5}$$

It is known that any space-time admitting null Maxwell, Yang–Mills, and gravitational fields is of the form (1.3) and (1.4).[2–4] Hence, taking into consideration all CR structures $(N,[(\lambda,\mu)])$ and constructing Lorentzian space-times of the form (1.3) and (1.4) we can find all such fields.

This work is devoted to null Yang–Mills fields admitting some additional symmetries. (The analogous problem for null gravitational fields was investigated in Ref. 6).

II. NULL YANG–MILLS FIELDS AND SHEAR-FREE CONGRUENCES

From now on we consider (M,g) of the form (1.3) and (1.4). Let G be a gauge group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}_{\mathbb{C}}$ its complexification. The gauge potential A is a $\mathfrak{g}_{\mathbb{C}}$ valued one-form on M. The most general form of A is

$$A=b\mu+\overline{b\mu}+c\lambda+e\tau, \tag{2.1}$$

where b, c, and e are $\mathfrak{g}_{\mathbb{C}}$-valued functions on M and $c=\bar{c}$, $e=\bar{e}$.

The Yang–Mills field is said to be null iff

$$F=dA+A\wedge A=\lambda\wedge(\varphi\mu+\overline{\varphi\mu}), \tag{2.2}$$

where φ is a $\mathfrak{g}_{\mathbb{C}}$-valued function on M. Since all null Yang–Mills fields satisfying

$$\lambda\wedge d\lambda=0 \tag{2.3}$$

are known,[3] *we will consider only those for which*

$$\lambda\wedge d\lambda\neq0. \tag{2.4}$$

This means that CR structures associated with these fields are nondegenerate. A physical meaning of (2.3) and (2.4) is as follows. Let $k\neq0$ be a real vector field on M, such that

$$\lambda(k)=\mu(k)=0. \tag{2.5}$$

 0022-2488/92/020802-06$03.00

This vector field is transversal to N appearing in (1.3). Moreover, k is null in the metric (1.4) and it can be proved that it is geodesic and shear free.[2,4,7] It defines a congruence of shear-free and null geodesics in M. The condition (2.4) means that this congruence is *twisting*.

We say that this congruence is symmetric,[2-4] if there exists a real vector field X on M, called a symmetry of a congruence, such that

$$\underset{X}{\mathscr{L}}\lambda = t\lambda, \tag{2.6a}$$

$$\underset{X}{\mathscr{L}}\mu = w\mu + l\lambda, \tag{2.6b}$$

where t is a real function and w, l are complex functions on M. Taking the Lie derivative with respect to X of (2.5) we see that

$$\underset{X}{\mathscr{L}}k \sim k. \tag{2.7}$$

Since, in addition,

$$\underset{k}{\mathscr{L}}\lambda = \underset{k}{\mathscr{L}}\mu = 0, \tag{2.8}$$

then the Lie derivative with respect to k of (2.6) leads to

$$k(t) = k(w) = k(l) = 0. \tag{2.9}$$

This means that the functions t, w, and l in (2.6) are, in fact, defined on N.

It is worth noting that any vector field

$$k' = f \cdot k, \tag{2.10}$$

where f is a real nonvanishing function on M is a (trivial) symmetry of the congruence k.

III. SYMMETRIC CAUCHY–RIEMANN STRUCTURES

Let $(N,[(\lambda,\mu)])$ be a nondegenerate CR structure. We say that this structure is symmetric if there exists a real vector field on N such that

$$\underset{X}{\mathscr{L}}\lambda = t\lambda, \tag{3.1a}$$

$$\underset{X}{\mathscr{L}}\mu = w\mu + l\lambda, \tag{3.1b}$$

where t is a real function and w, l are complex functions on N.

Let us return to the manifold (M,g) and the congruence k considered in the previous section. Because of (2.7) any symmetry X of the congruence k is uniquely projected onto the CR manifold N included in M. This projection $\tilde{X}$ is a symmetry of N in the sense of (3.1) because of (2.9) [of course we exclude symmetries of the form (2.10), which reduce to points when projected onto N].

From now on we assume that the congruence k admits at least *three* (say $n \geqslant 3$) linearly independent symmetries $(X_i)_{i=1,2,3,\dots,n}$, such that

$$\forall i,j = 1,2,\dots,n, \quad [X_i,X_j] = \sum_k c^k{}_{ij} X_k, \quad c^k{}_{ij} = \text{const.} \tag{3.2}$$

In addition, we assume that none of these is a trivial symmetry in the sense of (2.10). Projections on N of these symmetries constitute n symmetries of the CR structure $(N,[(\lambda,\mu)])$. It is known[8] that we can always choose three of these symmetries, say, $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{X}_3$, which form a three-dimensional Lie algebra. It is also known[8] that a local three-dimensional Lie group so generated acts on N in a simply transitive fashion. This means that a three-dimensional group generated by the three symmetries X_1, X_2, X_3 on M (which are projected onto $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{X}_3$) has three-dimensional orbits in the space-time M. Choosing the r coordinate such that r is constant on these orbits, and forms (Ω,Ω_1) belonging to the class $[(\lambda,\mu)]$, and defining a CR structure on N, such that

$$\underset{X_i}{\mathscr{L}}\Omega = \underset{X_i}{\mathscr{L}}\Omega_1 = 0, \quad \forall i = 1,2,3, \tag{3.3}$$

$$d\Omega = i\Omega_1 \wedge \bar{\Omega}_1, \tag{3.4}$$

we can write the metric tensor (1.4) in the form

$$g = 2P^2[\Omega_1\bar{\Omega}_1 - \Omega(dr + W\Omega_1 + \overline{W\Omega_1} + H\Omega)], \tag{3.5}$$

where W is a complex function and $P \neq 0, H$ are real functions on M. Let us recall the following.[6,8]

Given a group G_3 associated with three symmetries $(X_i)_{i=1,2,3}$ of the CR structure $(N,[(\lambda,\mu)])$ there always exist forms Ω and Ω_1 belonging to the class $[(\lambda,\mu)]$ and satisfying (3.3) and (3.4). The explicit expressions for Ω and Ω_1 for any Bianchi type of G_3 are given in Ref. 6.

IV. GENERAL FORM OF THE GAUGE POTENTIAL AND THE METRIC

In this section G_3 denotes the local nontrivial [in the sense of (2.10)] symmetry group of congruence k included in the space-time M given by (1.3). In addition, we assume that G_3 is a local symmetry group of the null Yang–Mills field defined by (2.1) and (2.2). This means that transformations φ generated by the action of G_3 in M induce transformations φ^* of the gauge potential A, such that

$$\varphi^* A = \sigma A \sigma^{-1} + \sigma^{-1} d\sigma. \tag{4.1}$$

It follows from the preceding section that G_3 has three-dimensional orbits in M. This, together with the results of the paper by Harnad *et al.*,[9] means that there exists a gauge in which the gauge potential A is strictly invariant under G_3. Hence

$$\underset{X_i}{\mathscr{L}} A=0 \tag{4.2}$$

for any (X_i) $i=1,2,3$ satisfying (2.6) and generating G_3.

From the previous section we know that the Lorentzian structure on M can be given by (3.3)–(3.5). Hence the null Yang–Mills field (2.1) and (2.2) satisfying (4.2) can be written as

$$A=B\Omega_1+\overline{B\Omega_1}+C\Omega+E(dr+W\Omega_1+\overline{W\Omega_1}+H\Omega), \tag{4.3}$$

$$F=dA+A\wedge A=\Omega_1\wedge(\Phi\Omega_1+\overline{\Phi\Omega_1}), \tag{4.4}$$

where B, C, E, and Φ are $\mathfrak{g}_{\mathbb{C}}$-valued functions on M, such that

$$C=\overline{C},\quad E=\overline{E}.$$

The condition (4.2) is equivalent to

$$\forall i=1,2,3 \begin{cases} X_i(E)=0, & (4.5\text{a})\\ X_i(B)=-EX_i(W), & (4.5\text{b})\\ X_i(C)=-EX_i(H). & (4.5\text{c})\end{cases}$$

Since (X_i) $i=1,2,3$ generates N then (4.5a) means that

$$E=E(r), \tag{4.6}$$

where r is given by (3.5). Hence we can always find a gauge transformation $\sigma=\sigma(r)$ such that the transformed gauge potential A' has the form

$$A'=B'\Omega_1+\overline{B}'\overline{\Omega}_1+C'\Omega, \tag{4.7}$$

where B', C' are $\mathfrak{g}_{\mathbb{C}}$-valued functions on M and $C'=\overline{C}'$. After this transformation A' still satisfies the conditions

$$\forall i=1,2,3,\quad \underset{X_i}{\mathscr{L}} A'=0, \tag{4.8}$$

and (4.7) describes the null field (4.4).

Thus if G_3 (included in G_n, $n\geqslant 3$) is a local symmetry group of a null gauge field A (with G_n being a local symmetry group of null congruence k associated with A), then the most general form of A may be given as

$$A=B\Omega_1+\overline{B\Omega_1}+C\Omega, \tag{4.9}$$

where B, C are $\mathfrak{g}_{\mathbb{C}}$-valued functions on M, such that

$$B=B(r),\ C=C(r),\ C=\overline{C}, \tag{4.10}$$

and r is given by (3.5).

Taking into account the vector field k on M defined by (2.5) that is connected with the r coordinate by

$$k=f\partial_r, \tag{4.11}$$

where $f\neq 0$ is an arbitrary real function on M, we see that

$$k\,\lrcorner\, A=0. \tag{4.12}$$

If we want A to generate the Yang–Mills field F given by (4.4) we see that

$$\underset{k}{\mathscr{L}} A=k\,\lrcorner\, F=0, \tag{4.13}$$

which means that the functions B, C appearing in (4.3)–(4.10) are constants.

V. EQUATIONS AND SOLUTIONS

Let the space-time M be given by (1.3) with the metric tensor (3.3)–(3.5). Such a metric tensor is connected with the particular choice of $G_3\subset G_n$. G_n is a local symmetry group of congruence k included in M and G_3 is its subgroup, which simultaneously is a symmetry group of a null Yang–Mills field, which can be represented by (4.9) with B and C constants. It is known[6,8] that for any G_3 the forms Ω and Ω_1 satisfying (3.3) and (3.4) satisfy an equation

$$d\Omega_1=\bar{\alpha}\Omega_1\wedge\overline{\Omega}_1+i\beta\Omega\wedge\Omega_1-\vartheta\Omega\wedge\overline{\Omega}_1 \tag{5.1}$$

with *constant* quantities α, ϑ (complex), and β (real).

Choosing the volume form on M as

$$\eta=iP^4\Omega\wedge\Omega_1\wedge\overline{\Omega}_1\wedge dr, \tag{5.2}$$

where r is the same as in (3.5) we can see that the vacuum Yang–Mills equations for the gauge field (4.3) and (4.4) reduce to

$$[\overline{B},\Phi]-\bar{\alpha}\Phi=0, \tag{5.3a}$$

$$\Phi=i\beta B-\overline{\vartheta B}+[C,B], \tag{5.3b}$$

$$C=i(\bar{\alpha}B-\alpha\overline{B})+i[B,\overline{B}], \tag{5.3c}$$

$$F=\Omega_1\wedge(\Phi\Omega_1+\overline{\Phi\Omega_1}), \tag{5.3d}$$

where [,] denotes the commutator in $\mathfrak{g}_{\mathbb{C}}$.

We will solve these equations in the case of the gauge group $G=\mathrm{SU}(2)$. In this case the B appearing in (5.3) can be represented as the scalar product

$$B=\mathfrak{B}\mathrm{e}, \tag{5.4}$$

where $\mathfrak{B}\in\mathbb{C}^3$ and $\mathfrak{e}=(e_1,e_2,e_3)$ constitute a basis in the algebra $\mathfrak{g}=\mathrm{su}(2)$ with the following Lie brackets:

$$[e_i,e_j]=\epsilon^k{}_{ij}e_k.$$

The commutator $[K,L]$ of elements K, L of the form (5.4) is

$$[K,L]=(\mathfrak{K}\times\mathfrak{L})\mathfrak{e}, \tag{5.5}$$

where $\mathfrak{K}\times\mathfrak{L}$ is a standard vector product in $\mathbb{C}^3$.

In this formalism Eqs. (5.3) read as

$$i[\bar\alpha(\beta-\mathfrak{B}\bar{\mathfrak{B}})-\alpha\bar{\mathfrak{B}}^2]\mathfrak{B}+i[\bar\alpha(\mathfrak{B}^2+i\bar\vartheta)+\alpha\mathfrak{B}\bar{\mathfrak{B}}]\bar{\mathfrak{B}}$$
$$+i[-\mathfrak{B}\bar{\mathfrak{B}}+\beta+\alpha\bar\alpha]\mathfrak{B}\times\bar{\mathfrak{B}}=0, \tag{5.6a}$$

where for any $\mathfrak{K}$ and $\mathfrak{L}$ in $\mathbb{C}^3$, $\mathfrak{K}\mathfrak{L}$ denotes the standard scalar product of these vectors, and $\mathfrak{K}^2=\mathfrak{K}\mathfrak{K}$;

$$\Phi=[i(\beta-\mathfrak{B}\bar{\mathfrak{B}})\mathfrak{B}+(i\mathfrak{B}^2-\bar\vartheta)\bar{\mathfrak{B}}+i\alpha\mathfrak{B}\times\bar{\mathfrak{B}}]\mathfrak{e} \tag{5.6b}$$

and

$$F=\Omega\wedge(\Phi\Omega_1+\overline{\Phi\Omega_1}). \tag{5.6c}$$

We solve Eqs. (5.6) by considering two different cases characterized by

$$\mathfrak{B}\times\bar{\mathfrak{B}}=0 \tag{5.7a}$$

or

$$\mathfrak{B}\times\bar{\mathfrak{B}}\neq0. \tag{5.7b}$$

First, we give solutions to (5.6) for which (5.7a) is satisfied. These solutions correspond to Abelian gauge fields. The condition (5.7a) shows that

$$\mathfrak{B}=\rho e^{i\phi}\mathfrak{n}, \tag{5.8}$$

where ρ, ϕ are constants such that $\rho\in\mathbb{R}$, $\phi\in[0,2\pi]$ and $\mathfrak{n}$ is a constant unit vector in $\mathbb{R}^3$. Using such a $\mathfrak{B}$ we show that Eqs. (5.6) are equivalent to

$$\alpha=0 \tag{5.9}$$

and that the gauge fields have the form

$$A=\rho\mathfrak{n}\mathfrak{e}(e^{i\phi}\Omega_1+e^{-i\phi}\bar\Omega_1). \tag{5.10}$$

These gauge fields live in the space-time

$$M=\mathbb{R}\times N, \tag{5.11a}$$

with the metric tensor

$$g=2P^2[\Omega_1\bar\Omega_1-\Omega(dr+W\Omega_1+\overline{W\Omega_1}+H\Omega)]. \tag{5.11b}$$

Looking at the list of all CR structures admitting the three-dimensional symmetry group given in Ref. 6 we see that the solutions (5.9)–(5.11) exist only for the CR structures listed below. They are ordered according to the Bianchi type of symmetry group G_3. Here, $(u,z=x+iy,\bar z)$ is a chart on N. Any Abelian solution of (5.6) is given by (5.10) and (5.11) with Ω and Ω_1 given below.

Bianchi type VI_0:

$$\Omega_1=\tfrac12 y^{-1}\,dx+\tfrac12 y\,du+iy^{-1}\,dy,$$
$$\Omega=y^{-1}\,dx-y\,du. \tag{5.12}$$

Bianchi type VII_0:

$$\Omega_1=\tfrac12(e^{iu}\,dz-e^{-iu}\,d\bar z-du),$$
$$\Omega=\tfrac12(du+e^{iu}\,dz+e^{-iu}\,d\bar z). \tag{5.13}$$

Bianchi type VIII (lower sign) and IX (upper sign):

$$\Omega_1=\frac{e^{iu}}{z\bar z\pm1}dz-\frac{k}{k^2\pm1}\left(du+\frac{ke^{iu}-i\bar z}{z\bar z\pm1}dz+\frac{ke^{-iu}+iz}{z\bar z\pm1}d\bar z\right),$$
$$\Omega=\frac{2}{k^2\pm1}\left(du+\frac{ke^{iu}-i\bar z}{z\bar z\pm1}dz+\frac{ke^{-iu}+iz}{z\bar z\pm1}d\bar z\right), \tag{5.14}$$
$$k^2\pm1\neq0,\quad k\geqslant0.$$

Solutions of (5.6) satisfying (5.7b) generate non-Abelian gauge fields. It is easy to see that they are not gauge equivalent to any Abelian gauge field.

Since (5.7b) is equivalent to

$$\mathfrak{B}=a\mathfrak{n}+ib\mathfrak{m},\quad ab\mathfrak{n}\times\mathfrak{m}\neq0, \tag{5.15}$$

where a, b are real constants and $\mathfrak{n}$, $\mathfrak{m}$ are constant unit vectors in $\mathbb{R}^3$, we give solutions of (5.6) and (5.7b) in terms of a, b, $\mathfrak{n}$, and $\mathfrak{m}$.

Using a, b, $\mathfrak{n}$, and $\mathfrak{m}$ we see that (5.6) and (5.7b) are equivalent to

$$\bar\alpha(a^2-b^2+2iab\mathfrak{m}\mathfrak{n}+\alpha^2)=0, \tag{5.16a}$$

$$a^2+b^2=\beta+\alpha\bar\alpha. \tag{5.16b}$$

We find all the solutions of these equations using the list of all CR structures with three-dimensional symmetry group given in Ref. 6. They are ordered according to the Bianchi type of the three-dimensional symmetry group G_3 of the CR structure (it is also a symmetry group of the gauge field). We use $(u, z=x+iy, \bar{z})$ as a chart on N. Any solution lives in the space-time

$$M=\mathbb{R}\times N, \tag{5.17a}$$

with the metric

$$g=2P^2[\Omega_1\bar{\Omega}_1-\Omega(dr+W\Omega_1+\overline{W\Omega}_1+H\Omega], \tag{5.17b}$$

where Ω_1 and Ω are given below.

Bianchi type IV:

$$A=a\mathrm{n}e(\Omega_1+\bar{\Omega}_1-2\Omega)+ib\mathrm{m}e(\Omega_1-\bar{\Omega}_1)$$
$$+2ab(\mathrm{n}\times\mathrm{m})e\Omega,$$

where

$$\mathrm{mn}=0,\quad a^2=\tfrac{9}{8},\quad b^2=\tfrac{1}{8}, \tag{5.18}$$

$$\Omega_1=y^{-1}(du+(1-\ln y)dx+i\,dy),$$

$$\Omega=2y^{-1}(du-\ln y\,dx).$$

Bianchi types VI_h:
Solutions exist only for $h<-1$.

$$A=a\mathrm{n}e(\Omega_1+\bar{\Omega}_1+(d-1)\Omega)+ib\mathrm{m}e(\Omega_1-\bar{\Omega}_1)$$
$$+2ab(\mathrm{n}\times\mathrm{m})e\Omega,$$

where

$$d:=\frac{1-\sqrt{-h}}{1+\sqrt{-h}},\quad a^2=\frac{1}{4}(1-d)^2-\frac{d}{8},$$
$$b^2=-\frac{d}{8},\quad \mathrm{nm}=0, \tag{5.19}$$

$$\Omega_1=y^{-1}\,dz+[d/(d+1)](y^d\,du-y^{-1}\,dx),$$

$$\Omega=-[2/(d+1)](y^d\,du-y^{-1}\,dx).$$

Bianchi type VII_0:

$$A=\cos\phi\,\mathrm{n}e(\Omega_1+\bar{\Omega}_1)+i\sin\phi\,\mathrm{m}e(\Omega_1-\bar{\Omega}_1)$$
$$+\sin 2\phi(\mathrm{n}\times\mathrm{m})e\Omega,$$

where ϕ is a constant such that $\phi\in[0,2\pi[]]$, $\sin 2\phi\neq 0$, n and m are not parallel,

$$\Omega_1=\tfrac{1}{2}(e^{iu}\,dz-e^{-iu}\,d\bar{z}-du),$$
$$\Omega=\tfrac{1}{2}(du+e^{iu}\,dz+e^{-iu}\,d\bar{z}). \tag{5.20}$$

Bianchi type VII_h:

$$A=a\mathrm{n}e(\Omega_1+\bar{\Omega}_1)+ib\mathrm{m}e(\Omega_1-\bar{\Omega}_1+4f\Omega)$$
$$+2ab(\mathrm{n}\times\mathrm{m})e\Omega,$$

$$f=\sqrt{h},\quad a^2=(1+h)/2,$$

$$b^2=(1+9h)/2,\quad \mathrm{nm}=0, \tag{5.21}$$

$$\Omega_1=e^{(f+i)u}\,dz+[(f-i)/2i](du+e^{(i+f)u}\,dz$$
$$+e^{(f-i)u}\,d\bar{z}),$$

$$\Omega=\tfrac{1}{2}(du+e^{(f+i)u}\,dz+e^{(f-i)u}\,d\bar{z}).$$

Bianchi type VIII (lower sign) and IX (upper sign):

$$A=(\sqrt{k^2\pm 2}/2)\cos\phi\,\mathrm{n}e(\Omega_1+\bar{\Omega}_1)$$
$$+i(\sqrt{k^2\pm 2}/2)\sin\phi\,\mathrm{m}e(\Omega_1-\bar{\Omega}_1)$$
$$+[(k^2\pm 2)/4]\sin 2\phi(\mathrm{n}\times\mathrm{m})e\Omega,$$

where ϕ is a constant, such that

$$\phi\in[0,2\pi[,\quad \sin 2\phi\neq 0,\quad \mathrm{n}\times\mathrm{m}\neq 0,$$

$$\Omega_1=\frac{2e^{iu}}{z\bar{z}\pm 1}\,dz-\frac{k}{k^2\pm 1}\left(du+\frac{ke^{iu}-i\bar{z}}{z\bar{z}\pm 1}\,dz\right.$$
$$\left.+\frac{ke^{-iu}+iz}{z\bar{z}\pm 1}\right), \tag{5.22}$$

$$\Omega=\frac{2}{k^2\pm 1}\left(du+\frac{ke^{iu}-i\bar{z}}{z\bar{z}\pm 1}\,dz+\frac{ke^{-iu}+iz}{z\bar{z}\pm 1}\,d\bar{z}\right),$$

and $k>\sqrt{2}$ for Bianchi type VIII, $k\geqslant 0$ for Bianchi type IX.

VI. CONCLUSIONS AND OUTLOOK

We have considered twisting null solutions to the Yang–Mills equations with gauge group $G=\mathrm{SU}(2)$. We assumed that congruences of shear-free and null geodesics associated with these solutions admitted n symmetries ($n\geqslant 3$). We applied the fact that among those n symmetries there always exist three symmetries that form three-dimensional Lie algebra. Assuming that none of those three symmetries is trivial [in the sense of (2.10)] we have shown that there were always symmetries of the CR structure N defined on M by the shear-free geodetic and null congruence. Finally, we found all twisting null Yang–Mills fields with gauge group SU(2) for which G_3 generated by above-mentioned three symmetries was a symmetry group. We obtained Abelian solutions with G_3

of Bianchi types VI_0, VII_0, VIII, and IX. More interesting, non-Abelian solutions were obtained for G_3 of Bianchi types IV, VI_h $(h<-1)$, VII_h, VIII, and IX. All these solutions can exist in any Lorentzian manifold of the form

$$M=\mathbf{R}\times N, \tag{6.1}$$

equipped with the metric tensor

$$g=2P^2[\Omega_1\bar{\Omega}_1-\Omega(dr+W\Omega_1+\overline{W\Omega}_1+H\Omega)], \tag{6.2}$$

with Ω and Ω_1 related to a particular Bianchi type of G_3 and $0\neq P$, W, H being absolutely arbitrary.

The question arises as to which of the obtained solutions can live in physically interesting space-times. Even if we restrict this question to the Minkowski space-time very little is known about the answer. (This particular case of the question is closely related to the problem of the formulation of the Kerr theorem[1,10] in terms of forms Ω and Ω_1. It seems to be unsolved so far). However, it is known that, for example, if Ω and Ω_1 correspond to the Robinson congruence[11] then there exist functions P, W, and H in (6.2) such that (6.2) is the Minkowski metric. In our list of solutions Ω and Ω_1 related to the Robinson congruence are given by (i) (5.14) for $k=0$ and $k=\sqrt{2}$ (Bianchi type VIII) and $k=0$ (Bianchi type IX), (ii) (5.19) for $h=-9$, and (iii) (5.22) for $k=0$ (Bianchi type IX). This means that all these solutions can live in Minkowski space-time.

Another nontrivial example of a null solution of the Yang–Mills equations with twisting rays that lives in Minkowski space-time can be obtained by using the results of Ref. 6. It follows from that paper that solutions (5.12) as well as (5.19) for $h=-4$ can live in Minkowski space-time. For the non-Abelian solution (5.19) expressions for functions P, W, H appearing in (6.2) that correspond to the Minkowski metric are[6]

$$P=\frac{wy^{2/3}}{\cos(r/2)}, \quad W=-\frac{1}{6}e^{ir}+\frac{1}{3},$$
$$H=\frac{1}{12}(e^{ir}+e^{-ir}-1). \tag{6.3}$$

Here, $w\neq 0$ is an arbitrary constant and Ω, Ω_1 are given by (5.19) for $h=-4$.

Expressions (5.19) for $h=-4$ and (6.2) and (6.3) give an explicit example of a null Yang–Mills field with twisting rays living in Minkowski space-time. The congruence of shear-free and null geodesics appearing in this solution is twisting and not equivalent to the Robinson congruence.[8] According to the results of Tafel's paper,[3] it seems to be the only known solution of this type.

Besides the solutions discussed in this section we do not know whether other solutions from our list (5.18)–(5.22) can also be imbedded in Minkowski space-time. It is also interesting to ask whether our solutions (5.18)–(5.22) generate any solution to the coupled Einstein–Yang–Mills equations with twisting rays.

ACKNOWLEDGMENTS

I am grateful to Professor Jacek Tafel for remarks concerning relevance of the paper[9] for this work. I also wish to thank Professor Andrzej Trautman and Dr. Jerzy Lewandowski for helpful discussions. Special thanks are due to Phil Cuddeford for help with the English.

[1] J. Tafel, "On the Robinson theorem and shear-free geodesic null congruences," Lett. Math. Phys. **10**, 33 (1985).
[2] I. Robinson and A. Trautman, *Cauchy–Riemann Structures in Optical Geometry*, Proceedings of the 4th Marcel Grossman Meeting on General Relativity, edited by R. Ruffini (Elsevier, New York, 1985).
[3] J. Tafel, "Null solutions of the Yang–Mills equations," Lett. Math. Phys. **12**, 163 (1986).
[4] I. Robinson and A. Trautman, "Optical Geometry," in *New Theories in Physics*, edited by Z. Ajduk, S. Pokorski, and A. Trautman (Scientific, Singapore, 1989), p. 454.
[5] P. Sommers, "Properties of shear-free congruences of null geodesics," Proc. R. Soc. London Ser. A **349**, 309 (1976); Type *N* Spacetimes as Special Functions on $\mathbb{C}^2$, GRG 8, 1977, p. 855; R. O. Wells, Jr., "Complex manifolds and mathematical physics," Bull. Am. Math. Soc. **1**, 296 (1979).
[6] J. Lewandowski and P. Nurowski, "Algebraically Special Twisting Gravitational Fields and Three-dimensional Cauchy–Riemann Structures," Classical Quantum Grav. **7**, 309 (1990).
[7] I. Robinson and A. Trautman, "Conformal geometry of flows in n dimensions," J. Math. Phys. **24**, 1425 (1983).
[8] P. Nurowski and J. Tafel, "Symmetries of Cauchy–Riemann spaces," Lett. Math. Phys. **15**, 31 (1988).
[9] J. Harnad, S. Shnider, and L. Vinet, "Group actions on principal bundles and invariance conditions for gauge fields," J. Math. Phys. **21**, 2719 (1980).
[10] R. Penrose and W. Rindler, *Spinors and Spacetime* (Cambridge U.P., Cambridge, 1986), Vol. II.
[11] R. Penrose, "On the Origin of Twistor Theory," in *Gravitation and Geometry*, edited by W. Rindler and A. Trautman (Bibliopolis, Napoli, 1987), p. 341.