MATHEMATICS

THEORY OF VECTOR-VALUED DIFFERENTIAL FORMS

PART I.

DERIVATIONS IN THE GRADED RING OF DIFFERENTIAL FORMS

BY

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§ 1. **Introduction.** The theory of vector-valued differential forms [1]) — briefly: *vector forms* — found its beginning in two papers published in 1951 [1], [2] [2]), each of which dealt with an application of a differential concomitant of a vector form of degree one. The differential concomitant itself was a vector form of degree two. In a paper of 1955 [3] the existence was proved of a differential concomitant $[L, M]$, where L and M are vector forms of degree l and m respectively. $[L, M]$ is then a vector form of degree $l+m$. Also, a Jacobi identity was proved for this bracket operation.

The present paper is the first of a series, and deals with general properties of vector forms. Contrary to the previous papers, the starting point is not the vector forms themselves. Instead, we consider the graded ring Φ of C^∞ differential forms — *scalar forms* — over a C^∞ manifold. It is commutative in the usual sense for graded rings: $\varphi_p \wedge \psi_q = (-1)^{pq}\, \psi_q \wedge \varphi_p$, where the subscripts indicate the degrees of the elements. The subring $\Phi_0 = F$ of elements of degree zero consists exactly of the C^∞ functions. The vector forms come up in connection with formal derivations in Φ. The mapping $D: \Phi \to \Phi$ is said to be a derivation of degree r if $D(\Phi_p) \subset \Phi_{p+r}$ and $D(\varphi_p \wedge \psi_q) = D\varphi_p \wedge \psi_q + (-1)^{rp} \varphi_p \wedge D\psi_q$ and $D(\varphi+\psi) = D\varphi + D\psi$. The analysis shows that there are two special types of derivations. One of them acts trivially on Φ_0, and its action on Φ_1 determines it completely. The other one is determined by its operation on Φ_0 and the fact that $Dd = (-1)^r dD$. Every other derivation D of degree r is the sum of two derivations of degree r, one of each kind. The derivations of the first kind ("type i_*") of degree r are in $1-1$ correspondence with the vector forms of degree $r+1$; in fact, they are given by a very simple algebraic operation (related to interior products) between the vector form and the scalar form. Among the derivations of the second kind ("type d_*") of degree 1 one finds the exterior differentiation d; those of type d_* and degree zero are Lie derivatives. More generally, the derivations of type d_* of degree r are in $1-1$ correspondence with the vector r-forms, and are differential concomitants of these with the scalar forms. They are shown to be generated by d and the derivations of type i_*.

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1) I.e. differential forms whose values are *tangent* vectors.

2) The bibliography is placed at the end of § 6.

Since derivations of type d_* commute with d, they can be carried over to the de Rham cohomology groups, but their operation there is trivial. Their action on the Dolbeault cohomology in complex manifolds is not trivial, however. This application will be discussed in a forthcoming paper.

The differential concomitant $[L, M]$ is determined as follows. If d_L and d_M are derivations of type d_* associated with vector l- and m-forms L and M, then $d_L d_M - (-1)^{lm} d_M d_L$ is shown to be again a derivation of type d_* of degree $l+m$, and therefore determines uniquely a vector $(l+m)$-form, which is what we have denoted by $[L, M]$. The Jacobi identity also finds its natural place in the whole structure. Numerous other identities are derived, all expressing relations between derivations of the types i_* and d_*.

The discussion of these derivations (§ 4) and the deduction of the relations between them (§ 5) is preceded first by a purely algebraic part (§ 2), where besides the exterior algebra of scalar forms on a module over a ring A with unit element we also introduce the module of vector forms. Then follows a resume on vector fields, scalar and vector forms over a manifold (§ 3), in which we deal mainly with two points of view of characterizing these objects; either defining them first in one point and then over a manifold as a differentiable cross-section in the corresponding fiber bundle; or alternatively, directly as global objects. At the end, in § 6, which is self-contained, we give a direct approach to vector one-forms, with a rather simple deduction of their basic properties. The corresponding formulas for general vector forms could have been derived in a somewhat similar fashion, but would require much more work. As an application of vector 1-forms we give a new proof of a theorem which, in its broadest form, was formulated by HAANTJES [4] and which was previously discussed by TONOLO [5], SCHOUTEN [6] and NIJENHUIS [1].

Further papers in the series will deal with applications of vector forms, including complex and almost-complex structures, and some cohomology aspects.

§ 2. **Scalar and vector forms on a module.** Let E be a module over a commutative ring A with unit element. If A is a field, E is a vector space. The case when A is not a field is essential in the following considerations.

Definition. An *A-scalar form* ω of degree q *on* E is an A-linear mapping

(2.1) $$\omega: \underbrace{E \wedge \ldots \wedge E}_{q \text{ times}} \to A$$

or, in other words, ω is defined on q-tuples of elements $u_1, \ldots, u_q$ of E, skew-symmetric in the arguments, A-linear in each of the arguments:

$$\omega(u_1, \ldots, \varphi u_i, \ldots, u_q) = \varphi\omega(u_1, \ldots, u_i, \ldots, u_q); \quad \varphi \in A, \tag{2.2}$$

with values in A. For $q=0$, ω is simply an element of A.

The *exterior product* $\omega \wedge \pi$ of two scalar forms, ω and π, of degrees q and p respectively, is a scalar form of degree $p+q$, defined by

$$\omega \wedge \pi(u_1, \ldots, u_{q+p}) = \frac{1}{p!\,q!} \sum_\alpha |\alpha|\, \omega(u_{\alpha_1}, \ldots, u_{\alpha_q})\, \pi(u_{\alpha_{q+1}}, \ldots, u_{\alpha_{q+p}}). \tag{2.3}$$

Here $\alpha = \binom{1, 2, \ldots, p+q}{\alpha_1, \alpha_2, \ldots, \alpha_{p+q}}$ runs over the symmetric group $\mathfrak{S}_{p+q}$. In general, in sums like (2.3), α will always be asumed to operate on all its subscripts by the appropriate full permutation group. We write $|\alpha|$ for sgn (α).

The notation $\frac{1}{p!\,q!} \sum_\alpha |\alpha| \ldots$ may seem ambiguous since no divisibility assumptions were made for A. It should be remarked that under the $\sum$ symbol in (2.3) there are $(p+q)!/p!q!$ groups of terms, each consisting of $p!q!$ terms, which are equal by virtue of the skew-symmetry properties of ω and π. The notation could therefore easily be modified to avoid the divisions. However, our main interest will be in rings which contain the field R of real numbers as a subring.

Let $S_{E,p}$ denote the A-module of all scalar forms of degree p on E. For $p<0$, $S_{E,p}$ reduces to $\{0\}$, and similarly when p exceeds the minimal number of generators of E. — If A is a field, and E a finite dimensional vector space, then the minimum number of generators of E is the dimension of E over A. Also, $S_{E,1}$ as the dual of E has dimension equal to that of E.

The *exterior algebra* S_E of E is the direct sum

$$S_E = \bigoplus_{p=-\infty}^{+\infty} S_{E,p} \tag{2.4}$$

with multiplication defined by the exterior product. — S_E is a commutative, associative graded ring: graded because $S_{E,p} \wedge S_{E,q} \subset S_{E,p+q}$ commutative in the usual sense for graded rings because $\omega \wedge \pi = (-1)^{pq}\pi \wedge \omega$; and associative because $(\omega \wedge \pi) \wedge \varrho = \omega \wedge (\pi \wedge \varrho)$.

Definition. An *A-vector form* L of degree l *on* E is an A-linear mapping

$$L\colon \underbrace{E \wedge \ldots \wedge E}_{l \text{ times}} \to E \tag{2.5}$$

or equivalently, L is defined on l-tuples of elements $u_1, \ldots, u_l \in E$, skew-symmetric in the arguments, A-linear in each of them, with values in E. For $l=0$, L is just an element of E itself.

For $l=1$ we have $L\colon E \to E$, so L is an endomorphism of E. In particular, I, the identity transformation on E, is a vector 1-form.

Let $V_{E,p}$ be the module of vector p-forms (hence $V_{E,0}=E$), and

$$V_E = \bigoplus_{p=-\infty}^{+\infty} V_{E,p}. \tag{2.6}$$

We define operation of S_E on V_E by an extension of the exterior multiplication: let $L \in V_{E,l}$, $\omega \in S_{E,q}$, then $\omega \wedge L \in V_{E,q+l}$ is defined by

$$\omega \wedge L(u_1, \ldots, u_{q+l}) = \frac{1}{q!\,l!} \sum_\alpha |\alpha|\, \omega(u_{\alpha_1}, \ldots, u_{\alpha_q})\, L(u_{\alpha_{q+1}}, \ldots, u_{\alpha_{q+l}}). \tag{2.7}$$

Similarly, $L \wedge \omega$ is defined as $(-1)^{lq}\omega \wedge L$. Obviously, we have $(\pi \wedge \omega) \wedge L = \pi \wedge (\omega \wedge L)$. — For $l=0$, $\omega \wedge L$ is the same as $\omega \otimes L$, where $\otimes$ is taken over the ring A.

There is also another operation, where V_E acts on S_E, denoted by $\omega \barwedge L$. Let again $\omega \in S_{E,q}$, $L \in V_{E,l}$, then $\omega \barwedge L \in S_{E,q+l-1}$:

$$\omega \barwedge L(u_1, \ldots, u_{q+l-1}) = \frac{1}{(q-1)!\,l!} \sum_\alpha |\alpha|\, \omega(L(u_{\alpha_1}, \ldots, u_{\alpha_l}), u_{\alpha_{l+1}}, \ldots, u_{\alpha_{l+q-1}}). \tag{2.8}$$

For $q=0$, $\omega \barwedge L$ is defined as zero. — Similarly, $\barwedge$ denotes also operation of V_E on V_E. In fact, let $L \in V_{E,l}$, $M \in V_{E,m}$, then $M \barwedge L \in V_{E,l+m-1}$:

$$\begin{cases} M \barwedge L(u_1, \ldots, u_{l+m-1}) = \\ = \dfrac{1}{(m-1)!\,l!} \sum_\alpha |\alpha|\, M(L(u_{\alpha_1}, \ldots, u_{\alpha_l}), u_{\alpha_{l+1}}, \ldots, u_{\alpha_{l+m-1}}). \end{cases} \tag{2.9}$$

Obviously, $\omega \barwedge L$ and $M \barwedge L$ have quite similar properties. When no confusion is possible, only those for $\omega \barwedge L$ will be stated.

For $v \in V_{E,0}=E$ we have $\omega \barwedge v(u_1, \ldots, u_{q-1}) = \omega(v, u_1, \ldots, u_{q-1})$; and for $h \in V_{E,1}$:

$$\omega \barwedge h(u_1, \ldots, u_q) = \sum_{i=1}^{q} \omega(u_1, \ldots, u_{i-1}, h \cdot u_i, u_{i+1}, \ldots, u_q). \tag{2.10}$$

In particular, $\omega \barwedge I = q\omega$; $M \barwedge I = mM$; $I \barwedge M = M$.

There are some relations between $\barwedge$ and $\wedge$. One easily shows:

$$\begin{cases} \text{a)} & (\omega \wedge \pi) \barwedge M = (\omega \barwedge M) \wedge \pi + (-1)^{q(m-1)}\omega \wedge (\pi \barwedge M); \\ \text{b)} & \omega \barwedge (\pi \wedge M) = \pi \wedge (\omega \barwedge M). \end{cases} \tag{2.11}$$

The operator $\barwedge$ is neither commutative nor associative, but the following commutative-associative rule holds, which states that the "non-associative part" of $\barwedge$ is commutative:

$$(\omega \barwedge M) \barwedge N - \omega \barwedge (M \barwedge N) = (-1)^{(m-1)(n-1)}\{(\omega \barwedge N) \barwedge M - \omega \barwedge (N \barwedge M)\}. \tag{2.12}$$

The proof goes as follows.

$$\begin{cases} (\omega \barwedge M) \barwedge N(u_1, \ldots, u_{q+m+n-2}) = \\ = \dfrac{1}{n!\,(m-1)!\,(q-1)!} \sum_\alpha |\alpha|\, \omega(M(N(u_{\alpha_1}, \ldots, u_{\alpha_n}), u_{\alpha_{n+1}}, \ldots, u_{\alpha_{n+m-1}}), u_{\alpha_{n+m}}, \ldots, u_{\alpha_{n+m+q-2}}) + \\ + \dfrac{(-1)^{n(m-1)}}{n!\,m!\,(q-2)!} \sum_\alpha |\alpha|\, \omega(M(u_{\alpha_1}, \ldots, u_{\alpha_m}), N(u_{\alpha_{m+1}}, \ldots, u_{\alpha_{m+n}}), u_{\alpha_{m+n+1}}, \ldots, u_{\alpha_{m+n+q-2}}). \end{cases} \tag{2.13}$$

The first term is $\omega \barwedge (M \barwedge N)$, and the last term equals $(-1)^{(m-1)(n-1)}$ times

$$(2.14)\qquad \frac{(-1)^{m(n-1)}}{m!\,n!\,(q-2)!}\sum_{\alpha}|\alpha|\,\omega\left(N(u_{\alpha_1},\ldots,u_{\alpha_n}),\,M(u_{\alpha_{n+1}},\ldots,u_{\alpha_{n+m}}),\,u_{\alpha_{n+m+1}},\ldots,u_{\alpha_{n+m+q-2}}\right)$$

which, by a similar computation, is found to equal $(\omega \barwedge N) \barwedge M - \omega \barwedge (N \barwedge M)$. This proves (2.12).

If $\omega \in S_{E,q}$ it follows from the vanishing of (2.14) that associativity holds. In fact, only in that case $\omega \barwedge M$ denotes a composite of two mappings:

$$(2.15)\qquad E \wedge \ldots \wedge E \xrightarrow{M} E \xrightarrow{\omega} A.$$

It is then sometimes useful to write $\omega \circ M$ for $\omega \barwedge M$, so the associativity

$$(2.16)\qquad \omega \circ M \barwedge N = (\omega \circ M) \barwedge N = \omega \circ (M \barwedge N)$$

will be obvious at one glance.

The formulas (2.11, 12, 16) also hold if $\omega \in S_{E,q}$ is replaced by $L \in V_{E,l}$, and q by l, with $l=1$ in (2.16).

§ 3. Manifolds; tangent vectors; scalar and vector forms.

The material in this section is of a nature which commonly carries the label "generally known". However, it is not generally available in the literature[3]) *). We have confined ourselves to the bare minimum of what is really needed in the subsequent sections.

Let X be a (HAUSDORFF) C^∞ manifold, and F the commutative ring of real-valued C^∞ functions on X, with addition and multiplication defined by

$$(3.1)\qquad (f+g)_x = f_x + g_x;\quad (fg)_x = f_x g_x,$$

where f_x is the value of f at $x \in X$. A subring is the field of constant functions, which we identify with the field R of real numbers itself.

Definition. A *tangent vector* u at $x_0 \in X$ is a functional whose domain is the set of real C^∞ functions, each defined in an open neighborhood of x_0, and whose range is the set R of real numbers. The sum and product of such functions are defined by (3.1) for all x belonging to the domains of both f and g. The conditions that u should satisfy are:

$$(3.2)\qquad \begin{cases} \text{a)}\ \ u(k)=0,\ k \in R, \\ \text{b)}\ \ u(f+g)=u(f)+u(g), \\ \text{c)}\ \ u(fg)=f_{x_0}u(g)+g_{x_0}u(f). \end{cases}$$

The following well-known lemma holds:

Lemma (3.1). Let $\mathscr{F}_{x_0}$ be a family of C^∞ functions defined on open neighborhoods of x_0, which includes the C^∞ functions on X, and which contains with any f, g also their sum and product. Let u be any mapping $u\colon \mathscr{F}_{x_0} \to R$ satisfying (3.2) for all $f, g \in \mathscr{F}_{x_0}$. Then for all $f, g \in \mathscr{F}_{x_0}$ the following properties hold:

a) If f vanishes in an open set $U \ni x_0$, then $u(f)=0$;
b) If f is constant in an open set $U \ni x_0$, then $u(f)=0$;
c) If f equals g in an open set $U \ni x_0$, then $u(f)=u(g)$.

Proof. Statements b) and c) follow easily from a). In order to prove a), take any function $\varphi \in F \subset \mathscr{F}_{x_0}$ with $\varphi_{x_0}=0$, and $\varphi_x=1$ outside U. Then $f=\varphi f$. Consequently, using (3.2c) we have

$$(3.3)\qquad u(f)=u(\varphi f)=\varphi_{x_0}u(f)+f_{x_0}u(\varphi)=0, \qquad \text{Q.E.D.}$$

Lemma (3.2). The set of tangent vectors at a point x_0 is a vector space of dimension equal to that of X.

Proof. Let U be an open star-shaped coordinate neighborhood of x_0 with C^∞ coordinates $(x^1, \ldots, x^n)$, and let x_0 have coordinates $(0, \ldots, 0)$. Let f be a C^∞ function defined in an open neighborhood of x_0; one may then assume that U is contained in this domain. Define f_i in U by

$$(3.4)\qquad \begin{cases} f_i(x^1,\ldots,x^n) = \dfrac{f(x^1,\ldots,x^i,0,\ldots,0)-f(x^1,\ldots,x^{i-1},0,\ldots,0)}{x^i}; \quad x^i \neq 0, \\[2ex] f_i(x^1,\ldots,x^{i-1},0,x^{i+1},\ldots,x^n) = \left(\dfrac{\partial f(x^1,\ldots,x^i,0,\ldots,0)}{\partial x^i}\right)_{x^i=0}; \end{cases}$$

then f_i, $i=1,\ldots,n$ are C^∞ functions in U, and one has $f_x = f_{x_0} + \sum_{i=1}^{n} f_i(x^1,\ldots,x^n)\,x^i$. Then $u(f_{x_0})=0$, because f_{x_0} is a constant; furthermore, because of the choice of coordinates, $x_0^i=0$. Hence, by (3.2b, c):

$$(3.5)\qquad u(f) = \sum_{i=1}^{n} f_i(x_0^1,\ldots,x_0^n)\,u(x^i) = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x^i}\right)_{x^1=\ldots=x^n=0} u(x^i).$$

Thus u is a linear combination of $\left(\frac{\partial}{\partial x^i}\right)_0$. The coefficients $u(x^i)$; $i=1,\ldots,n$, are called the *components* of u with respect to the coordinate system $(x^1, \ldots, x^n)$. Furthermore, the $\left(\frac{\partial}{\partial x^i}\right)_0$ themselves satisfy (3.2). The set $T_{x_0}(X)$ of tangent vectors at x_0 is a vector space over the field R of real numbers, with

$$(3.6)\qquad (au+bv)(f)=au(f)+bv(f);\quad a, b \in R.$$

The operators $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ form a basis for $T_{x_0}(X)$, and $T_{x_0}(X)$ is of the same dimension as X. This proves Lemma (3.2).

$T_x(X)$ is called the *tangent space* to X at x. The union $T(X) = \bigcup_{x \in X} T_x(X)$ with the natural differentiable structure is called the *tangent bundle* of X.

Definition. A (contravariant) *vector field* u over X is a mapping $u\colon X \to T(X)$ with $u_x \in T_x(X)$ such that, whenever $f \in F$, the function $[u, f]$ defined by $[u, f]_x = u_x(f)$, belongs to F.

If u is a vector field and $\varphi \in F$, then the vector field φu is defined by

[3]) Our set-up is related to e.g. that of CHEVALLEY [7], Ch. III, and of SPENCER [8], but ours is more general in some respects.

*) The reader is also referred to [14], § 1, which came out a few days after the presentation of this paper; especially for Proposition (3.4) and for the definition of d in § 4 (Added in proof).

$[\varphi u, f] = \varphi[u, f]$, or equivalently, by $(\varphi u)_x = \varphi_x u_x$. The sum $u+v$ of two vector fields is defined by $(u+v)_x = u_x + v_x$. Let Ψ_0 denote the set of vector fields over X, then Ψ_0 is a module over F, and simultaneously, an infinite dimensional vector space over R.

Definition. A *derivation* D on a commutative ring A relative to a subring K is a mapping $D\colon A \to A$, which satisfies

$$(3.7)\qquad \begin{cases} D(K) = \{0\}, \\ D(a_1 + a_2) = Da_1 + Da_2, \\ D(a_1 a_2) = a_1 Da_2 + a_2 Da_1. \end{cases}$$

Proposition (3.3). The derivations on the ring F of real valued C^∞ functions over X, relative to R, are the vector fields over X.

Proof. It is obvious from the definition of Ψ_0 that every vector field u is a derivation. Conversely, let u be a derivation, then denote by u_x the functional on F which assigns to $f \in F$ the real number $(uf)_x$. From the product definition in F and (3.7) it follows that the conditions (3.2) are satisfied for all $f \in F$. If g is a C^∞ function whose domain U contains x as an interior point, then there is a function $\bar{g} \in F$ which in a small neighborhood V, $\bar{V} \subset U$, of x agrees with g, for $\bar{g} = g\varphi$; with $\varphi \in F$, $\varphi|V = 1$, $\varphi|X - U = 0$ satisfies. If $\bar{g}$ and $\bar{\bar{g}}$ are two extensions of g in a neighborhood of x, then by Lemma (3.1) $u_x(\bar{g}) = u_x(\bar{\bar{g}})$. In defining $u_x(g) = u_x(\bar{g})$ we then have obtained an operator u_x which satisfies all conditions for a tangent vector. Thus u_x represents a unique tangent vector at x. The mapping $x \to u_x$ is a vector field over X because for every $f \in F$ we have $(uf)_x^{\#} = u_x(f)$, and uf was given to be a C^∞ function. This completes the proof.

The remarks of § 2 on scalar and vector forms on a module E over a ring A can be applied either to the tangent space $T_x(X)$, which is a vector space over the field R of real numbers, or to the set Ψ_0 of all vector fields over X, which is a module over the ring F of C^∞ functions. The vector space of R-scalar q-forms on $T_x(X)$ is denoted by $T_x^{*q}(X)$, and the vector space of R-vector l-forms on $T_x(X)$ by $V_x^l(X)$. The unions $T^{*q}(X) = \bigcup_{x \in X} T_x^{*q}(X)$ and $V^l(X) = \bigcup_{x \in X} V_x^l(X)$ are the *bundles* of *scalar* and *vector forms* on $T(X)$. The module of F-scalar q-forms on Ψ_0 is denoted by Φ_q; and the module of F-vector l-forms on Ψ_0 by Ψ_l. — The elements of $T_x^*(X) = T_x^{*1}(X)$ are called the *covectors* at x.

Let u be a tangent vector at x; f a C^∞ function defined in an open neighborhood of x. Then u acts on f as $u(f)$, and this operation is R-linear in u and in f. Therefore f also acts on $T_x(X)$, and this linear functional is denoted by df_x, as follows:

$$(3.8)\qquad df_x(u) = u(f).$$

df_x is a covector at x. If, in particular, $x^1, \ldots, x^n$ are coordinates in a neighborhood of x, then $dx^1, \ldots, dx^n$ are covectors at x; and because $dx^i\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial x^k}(x^i) = \delta^i_k$, they are linearly independent. Since $\dim T_x^*(X) = \dim T_x(X)$ it follows that $dx^1, \ldots, dx^n$ form a basis of $T_x^*(X)$. Using the definition (3.8) in (3.2, 5) one finds

$$(3.9)\qquad \begin{cases} \text{a)}\quad dk = 0, \quad k \in R, \\ \text{b)}\quad d(f+g) = df + dg, \\ \text{c)}\quad d(fg) = f\,dg + g\,df, \\ \text{d)}\quad df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i. \end{cases}$$

Definition. A *scalar q-form ω over X* is a mapping $\omega\colon X \to T^{*q}(X)$ with $\omega_x \in T_x^{*q}(X)$, such that, whenever $u_1, \ldots, u_q \in \Psi_0$, the function $\omega(u_1, \ldots, u_q)$ defined by $\omega(u_1, \ldots, u_q)_x = \omega_x(u_{1x}, \ldots, u_{qx})$, belongs to F. From the definition of $\omega(u_1, \ldots, u_q)$ follows that

$$(3.10)\qquad \omega(u_1, \ldots, \varphi u_i, \ldots, u_q) = \varphi\omega(u_1, \ldots, u_i, \ldots, u_q), \quad \varphi \in F.$$

Consequently, ω is an F-scalar q-form on the module Ψ_0, hence $\omega \in \Phi_q$.

Proposition (3.4). The module Φ_q of F-scalar q-forms on Ψ_0 consists of all scalar q-forms over X.

Proof. If $\omega \in \Phi_q$ we have to show that ω is a scalar q-form over X. Let $u_1, \ldots, u_q \in \Psi_0$, then $\omega(u_1, \ldots, u_q) \in F$. Define ω_x by $\omega_x(u_1, \ldots, u_q) = \omega(u_1, \ldots, u_q)_x$. We have to show that if $u_i' \in \Psi_0$ is such that $u_{ix}' = u_{ix}$, then $\omega_x(u_1, \ldots, u_i, \ldots, u_q) = \omega_x(u_1, \ldots, u_i', \ldots, u_q)$, or equivalently, that if $u_{ix} = 0$, then $\omega_x(u_1, \ldots, u_i, \ldots, u_q) = 0$. Let $v_1', \ldots, v_n'$ be C^∞ vector fields over $\bar{V}$, V open, $V \ni x$, and let $v_{1y}', \ldots, v_{ny}'$ be a basis for $T_y(X)$, $y \in \bar{V}$. Then $u_i = \psi_1' v_1' + \ldots + \psi_n' v_n'$ over $\bar{V}$. Take $\varphi \in F$, with $\varphi_x = 1$, and $\varphi = 0$ outside V. Then $u_i = (1 - \varphi^2) u_i + \sum_{j=1}^{n} \psi_j v_j$, over all of X, where in V $\psi_j = \varphi\psi_j'$, $v_j = \varphi v_j'$, and outside V $\psi_i = 0$, $v_j = 0$; hence $\psi_j \in F$; $v_j \in \Psi_0$. The fact that $u_{ix} = 0$ implies $\psi_{jx} = 0$, $j = 1, \ldots, n$. Since $\omega_x(u_1, \ldots, \varphi u_i, \ldots, u_q) = \varphi_x \omega_x(u_1, \ldots, u_i, \ldots, u_q)$ for all $\varphi \in F$ it follows

$$(3.12)\qquad \begin{cases} \omega_x(u_1, \ldots, u_i, \ldots, u_q) = (1 - \varphi_x^2)\, \omega_x(u_1, \ldots, u_i, \ldots, u_q) + \\ + \sum_{j=1}^{n} \psi_{jx}\, \omega_x(u_1, \ldots, u_{i-1}\, v_j, u_{i+1}, \ldots, u_q) = 0, \quad \text{Q.E.D.} \end{cases}$$

Definition. A *vector l-form L over X* is a mapping $L\colon X \to V^l(X)$ with $L_x \in V_x^l(X)$, and $L(u_1, \ldots, u_l) \in \Psi_0$, for $u_1, \ldots, u_l^1 \in \Psi_0$, where $L(u_1, \ldots, u_l)$ is defined by $L(u_1, \ldots, u_l)_x = L_x(u_{1x}, \ldots, u_{lx})$.

The proof of the following proposition is a copy of the one above.

Proposition (3.5). The module Ψ_l of all F-vector l-forms on Ψ_0 consists of all vector l-forms over X.

§ 4. **Derivations on the ring of scalar forms.** The graded ring Φ of scalar forms on X is commutative; i.e.

$$(4.1)\qquad \pi \wedge \omega = (-1)^{pq}\, \omega \wedge \pi, \quad \omega \in \Phi_q,\ \pi \in \Phi_p.$$

The subring $\Phi_0 = F$ consists of the C^∞ scalar functions over X. Each Φ_p is a module over F.

Definition. The mapping $D\colon \Phi \to \Phi$ is called a *derivation* of degree r on Φ, (relative to R) if

$$(4.2)\quad \begin{cases} \text{a)} & Dk = 0, \quad k \in R, \\ \text{b)} & D\Phi_p \subset \Phi_{p+r}, \\ \text{c)} & D(\varphi+\psi) = D\varphi + D\psi, \\ \text{d)} & D(\pi \wedge \omega) = D\pi \wedge \omega + (-1)^{pr}\pi \wedge D\omega; \quad \pi \in \Phi_p, \omega \in \Phi_q. \end{cases}$$

The following lemma states that all derivations are 'local' operators.

Lemma (4.1). Let D be a derivation on Φ, and let $\varphi, \psi \in \Phi_p$ have the property that $\varphi|U = \psi|U$, where U is an open set of X. Then for $x \in U$ one has $(D\varphi)_x = (D\psi)_x$.

Proof. Let $\chi = \varphi - \psi$; then $\chi|U = 0$. Choose $x \in U$, then x has an open neighborhood V with $\bar{V} \subset U$, and there is a $\varrho \in F$ with $\varrho|V = 0$, $\varrho|X-U = 1$. Then $\chi = \varrho\chi$, and therefore

$$(4.3)\quad (D\chi)_x = (D\varrho)_x \wedge \chi_x + \varrho_x (D\chi)_x = (D\varrho)_x \wedge \chi_x.$$

$(D\chi)_x$ vanishes if we can prove that $(D\varrho)_x$ vanishes. There is a $\sigma \in F$ with $\sigma|X-V = 1$ and $\sigma_x = 0$. Then $\varrho = \sigma\varrho$, and

$$(4.4)\quad (D\varrho)_x = \varrho_x(D\sigma)_x + (D\varrho)_x\sigma_x = 0, \qquad \text{Q.E.D.}$$

Lemma (4.2). Every derivation D on Φ is determined by its action on $\Phi_0 = F$ and Φ_1 alone.

Proof. Since scalar forms that are equal on an open set have the same derivative on that set, one can confine oneself to an arbitrarily small open neighborhood of the point x at which one wants to prove that D, on forms restricted to U, evaluated at x, is determined by D on the 0-forms and 1-forms restricted to U. Let U be so small that there are n 1-forms $\omega^1, \ldots, \omega^n$ over U, such that $\omega^1_y, \ldots, \omega^n_y$ form a basis in each $T^*_y(X)$, $y \in U$. Every C^∞ form φ of degree p over U is then uniquely representable as

$$(4.5)\quad \varphi = \sum_{i_1 < \ldots < i_p} a_{i_1 \ldots i_p}\, \omega^{i_1} \wedge \ldots \wedge \omega^{i_p}$$

where the $a_{i_1 \ldots i_p}$ are C^∞ functions on U. — Applying D we find

$$(4.6)\quad \begin{cases} D\varphi = \sum_{i_1 < \ldots < i_p} (Da_{i_1 \ldots i_p}) \wedge \omega^{i_1} \wedge \ldots \wedge \omega^{i_p} + \\ + \sum_{i_1 < \ldots < i_p} \sum_{j=1}^{p} (-1)^{r(j-1)} a_{i_1 \ldots i_p}\, \omega^{i_1} \wedge \ldots \wedge D\omega^{i_j} \wedge \ldots \wedge \omega^{i_p}. \end{cases}$$

The left-hand side indicates the operation of D on an arbitrary element of Φ_p; the right-side is determined by the operation of D on Φ_0 and Φ_1. Since the open set U was a neighborhood of x, and x was an arbitrary point of X, the lemma is proved for all $x \in X$.

Lemma (4.3). Every mapping $D\colon \Phi_0 \oplus \Phi_1 \to \Phi$ satisfying (4.2*a–c*) and (4.2d) for $p+q \leqslant 1$ can be uniquely extended to Φ so that D is a derivation of degree r on Φ.

Proof. The argument in the proof of Lemma (4.1) shows that D is a 'local' operator on $\Phi_0 \oplus \Phi_1$. Let x be any point $\in X$, and U an open neighborhood of x such that there are forms $\omega^1, \ldots, \omega^n$ over U, as in the proof of the previous lemma. Every φ over U then has a local decomposition (4.5); and (4.6) defines a derivative $D\varphi$, which is a C^∞ form over U. $D\varphi$ is defined with respect to the forms $\omega^1, \ldots, \omega^n$; and it is not yet clear that other forms $\omega'^1, \ldots, \omega'^n$ defined over an open set V would have given the same $D\varphi$ over $U \cap V$. To show this, we remark that on $U \cap V$ there are C^∞ functions α^j_i with $\omega^j = \sum_{i=1}^{n} \alpha^j_i \omega'^i$. Then we have

$$(4.7)\quad \varphi = \sum_{i_1, \ldots, i_p = 1}^{n} \sum_{j_1 < \ldots < j_p} a_{j_1 \ldots j_p}\, \alpha^{j_1}_{i_1} \ldots \alpha^{j_p}_{i_p}\, \omega'^{i_1} \wedge \ldots \wedge \omega'^{i_p}.$$

The actual computation to show that $D\varphi$ computed from (4.7) is the same as that of (4.5) is a simple and straightforward application of (4.2) for $p+q \leqslant 1$. Thus $D\varphi$ is globally defined, and $D\varphi \in \Phi_{p+r}$; Q.E.D.

Corollary of Lemma (4.3). There are no non-trivial derivations on Φ of degree $r \leqslant -2$. — All derivations of degree -1 vanish on F.

For, if $r \leqslant -2$, then $D\Phi_0 \subset \Phi_r = \{0\}$, and $D\Phi_1 \subset \Phi_{1+r} = \{0\}$; hence $D\Phi = \{0\}$. And if $r = -1$, then $D\Phi_0 \subset \Phi_{-1} = \{0\}$.

Proposition (4.4). Every derivation D of degree r, $r \geqslant -1$, which acts trivially on F is of the form $D\colon \varphi \to \varphi \barwedge L$, where L is a vector form over X of degree $r+1$, uniquely determined by D. Conversely, every mapping $D\colon \varphi \to \varphi \barwedge L$, where L is a vector $(r+1)$-form over X, is a derivation of degree r acting trivially on F.

Definition. Under the conditions of Proposition (4.4), D is called a derivation of *type* i_*, and is denoted by i_L.

Proof of Proposition (4.4). If D acts trivially on F, one has

$$(4.8)\quad (D(\varphi\psi))_x = \varphi_x (D\psi)_x, \quad \varphi \in F,\ \psi \in \Phi_p.$$

An argument similar to that in the proof of Proposition (3.4) then shows that $(D\psi)_x$ depends on ψ_x, but not on the continuation of ψ over X. Let U be an open set, and let $\omega^1_x, \ldots, \omega^n_x$ be a basis of $T^*_x(X)$, $x \in U$. Let $u_{1x}, \ldots, u_{nx}$ be a dual basis of $T_x(X)$; i.e. $\omega^i(u_j) = \delta^i_j$; then $(D\omega^i)_x$, for each $i = 1, \ldots, n$, is an $(r+1)$-form on $T_x(X)$; and $L_x = \sum_{i=1}^{n} u_{ix} \otimes (D\omega^i)_x$ gives a vector $(r+1)$-form L_x on $T_x(X)$ which by (4.8) is independent of the bases ω^i and u_i. We have $(D\omega^i)_x = \omega^i_x \barwedge L_x$. — Let $\varphi \in \Phi_p$; then φ has a local decomposition (4.5), and $(D\varphi)_x$ is given according to (4.6):

$$(4.9)\quad (D\varphi)_x = \sum_{i_1 < \ldots < i_p} \sum_{j=1}^{p} (-1)^{r(j-1)} a_{i_1 \ldots i_p x}\, \omega^{i_1}_x \wedge \ldots \wedge (\omega^{i_j}_x \barwedge L_x) \wedge \ldots \wedge \omega^{i_p}_x = \varphi_x \barwedge L_x.$$

Conversely, (2.11a) shows that the mapping $\varphi \to \varphi \barwedge L$ satisfies the conditions (4.2), and by definition acts trivially on F.

The set of derivations that are not of type i_*, is not empty, because the exterior derivative d is a derivation, but does not vanish on F. We defined d on F in § 3, and some of its properties are listed in (3.9). The operation of d on Φ_1 then is determined by the condition $ddf=0$, because the operation of d on any 1-form $\varphi=\sum_i a_i dx^i$ is by (4.6) $d\varphi=\sum_i da_i \wedge dx^i$.

The verification that $d\varphi$ does not depend on the coordinates is obvious: if $\varphi=\sum a_i \frac{\partial x^i}{\partial y^j} dy^j$, then

$$(4.10)\qquad d\varphi = \sum_{i,j} d\left(a_i \frac{\partial x^i}{\partial y^j}\right) \wedge dy^j = \sum_{i,j} da_i \wedge \frac{\partial x^i}{\partial y^j} dy^j + \sum_{i,j} a_i\, d\frac{\partial x^i}{\partial y^j} \wedge dy^j .$$

The first term equals $\sum da_i \wedge dx^i$ (by (3.9d)), and the second term vanishes because

$$\sum d\frac{\partial x^i}{\partial y^j} \wedge dy^j = \sum d\left(\frac{\partial x^i}{\partial y^j} dy^j\right) - \sum \frac{\partial x^i}{\partial y^j} ddy^j = d(dx^i) = 0.$$

Thus d is well-defined on $\Phi_0 \oplus \Phi_1$. A computation similar to (4.13) below, with $D=dd$, $i_L=d$, $r=1$ shows that dd is a derivation. Since $dd\varphi=0$, dd vanishes on all of Φ. – The derivation d is of type d_*, in the following sense:

Definition. A derivation D of degree r on Φ is of *type* d_* if $Dd=(-1)^r dD$.

Proposition (4.5). Every derivation D on Φ of type d_* is determined uniquely by its action on F. With every derivation D of degree r of type d_* there is associated a vector form L of degree r, and

$$(4.11)\qquad D\omega = [L, \omega] = d\omega \barwedge L + (-1)^r d(\omega \barwedge L) = (i_L d - (-1)^{r-1} d i_L)\cdot \omega .$$

Conversely, every mapping $D\colon \Phi \to \Phi$ of the form (4.11) is a derivation of type d_*.

Remark. The derivation of type d_* associated with $L \in \Psi_r$ is denoted by d_L.

Proof of the proposition. Let $\omega \in \Phi_1$, and let $x^1, \ldots, x^n$ be local coordinates. Then, by the assumption that D, of degree r, be of type d_*, we find

$$(4.12)\qquad D\omega = \sum_i Da_i \wedge dx^i + \sum_i a_i\, Ddx^i = \sum_i Da_i \wedge dx^i + (-1)^r \sum_i a_i\, dDx^i .$$

The right-hand side is determined by the action of D on a_i, x^i, which are real-valued C^∞ functions. This proves the first statement. The second statement (not including (4.11)) will be proved without using that D is of type d_*. Let $u_1, \ldots, u_r \in \Psi_0$ and $\varphi \in F$, then the mapping on F of $\varphi \to D\varphi(u_1, \ldots, u_r)$, in virtue of the definition of a derivation of degree r, (cf. (4.2)) satisfies the conditions for a derivation $v\colon F \to F$; cf. Proposition (3.3). Hence $D\varphi(u_1, \ldots, u_r)=v(\varphi)=d\varphi(v)$, where $v \in \Psi_0$ depends on the choice of $u_1, \ldots, u_r \in \Psi_0$. Since $D\varphi$ is an F-scalar r-form, it follows that $v=L(u_1, \ldots, u_r)$, where L is an F-vector r-form on Ψ_0; i.e. by Proposition (3.5) L is a vector r-form over X; and we have $D\varphi=d\varphi \barwedge L$. – In order to verify (4.11) we remark that, because of the statements just proved

we only have to show: 1) The formula (4.11) for $\varphi \in F$ agrees with $D\varphi=d\varphi \barwedge L$; which is obvious; 2) The formula (4.11) really defines a derivation; 3) $Dd=(-1)^r dD$, which follows because by (4.11) both sides equal $(-1)^r d(d\omega \barwedge L)$. Remains the proof that (4.11) defines a derivation. Conditions (4.2a–c) are obviously satisfied; (4.2d) follows thus:

$$(4.13)\qquad \begin{cases} D(\omega \wedge \pi) = d(i_L\omega \wedge \pi + (-1)^{(r-1)q} \omega \wedge i_L \pi) + (-1)^r i_L (d\omega \wedge \pi + (-1)^q \omega \wedge d\pi) = \\ \qquad = d i_L \omega \wedge \pi + (-1)^r i_L\, d\omega \wedge \pi + (-1)^{rq} \omega \wedge d i_L \pi + (-1)^{qr+r} \omega \wedge i_L\, d\pi = \\ \qquad = D\omega \wedge \pi + (-1)^{rq} \omega \wedge D\pi, \quad \text{Q. E. D.} \end{cases}$$

Proposition (4.6). Every mapping $D\colon F \to \Phi_r$ which 1) is trivial on R, 2) is linear, and 3) satisfies $D(\varphi\psi)=\varphi D\psi + \psi D\varphi$, is uniquely extendable to a derivation of type d_* on the whole of Φ.

Proof. The proof of the second statement of Proposition (4.5) shows precisely that $D\varphi=d\varphi \barwedge L$. The extension to the whole of Φ is defined by (4.11). According to Proposition (4.5) the extension is unique.

Remark. The derivations of type d_* include the following operations.

a) $r=0$; $L=v$, a vector field. Then $d_v\omega=[v, \omega]$ is the Lie derivative of ω with respect to v;

b) $r=1$; $L=I$, the identity transformation on Ψ_0. Then

$$(4.14)\qquad d_I\omega = d\omega \barwedge I - d(\omega \barwedge I) = (q+1)\, d\omega - q d\omega = d\omega .$$

c) $r=1$; $L=p$, a projection operator: $p \circ p=p$; then $d_p\omega=[p, \omega]$ denotes a "partial" exterior derivative of ω with respect to the decomposition of $T(X)$ defined by p (cf. Spencer [9]) *).

Remark. The expression $[L, \omega]$ [4]) is a scalar form, hence a tensor field, which is determined by the tensor fields L and ω. $[L, \omega]_x$ depends only on the "germs" of L and ω at x; i.e. it depends on the behavior of L and ω in an arbitrarily small neighborhood of x. Therefore $[L, \omega]$ is called a *differential concomitant* [5]) of L and ω.

Proposition (4.7). Every derivation D on Φ has a unique decomposition in a sum of two derivations; one of type i_* and one of type d_*.

Proof. According to Proposition (4.6) the action of D on F determines a derivation d_M. The action of $D-d_M$ on F (which is trivial) and on Φ_1 determines a derivation i_L (Proposition (4.4)). Then D and i_L+d_M act in the same way on Φ_0 and Φ_1, and by Lemma (4.3) this means $D=i_L+d_M$; Q.E.D.

The main results of this section are summarized in the following theorem.

Theorem I. The derivations on the graded ring Φ of C^∞ scalar forms over a C^∞ manifold X are all locally defined; i.e. the derivative of

[4]) The double notation $[L, \omega]$, $d_L\omega$ was adopted because the latter stresses the operator character of L on ω, while the former indicates that $[L, \omega]$ is an object determined by L and ω alone. It may seem surprising that square brackets are also used for $[L, M]$ in § 5, but, in fact, it is convenient in view of the formal analogies of identities in § 5, cf. Theorem II.

[5]) An exposition on differential concomitants was given by Schouten [10].

*) See also [14], § 2 (Added in proof).

ω at any point x is determined by the behavior of ω in the neighborhood of x. Each derivation D of degree r that acts trivially on $\Phi_0 = F$ is uniquely determined by a vector form L of degree $r+1$ over X, and $D\omega = \omega \barwedge L$. Each derivation D of degree r which commutes with d: $Dd = (-1)^r dD$, is uniquely determined by a vector form L of degree r, and (4.11) holds. Every derivation of degree r is the sum of two derivations of degree r, one of each of the kinds mentioned.

(To be continued)

MATHEMATICS

THEORY OF VECTOR-VALUED DIFFERENTIAL FORMS
PART I (CONTINUED).
DERIVATIONS IN THE GRADED RING OF DIFFERENTIAL FORMS

BY

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§ 5. **The identities for scalar and vector forms.** In § 4 the vector forms have been identified as the objects determining derivations on the graded ring of scalar forms. The further development of the general theory follows a rather standard pattern, and might therefore be of little value by itself. It appears, however, that from a different point of view the identities thus arising are of value. The expression $[L, \omega]$ was found to be a differential concomitant of L and ω. Similarly, the standard development of the derivations leads to another differential concomitant, $[L, M]$, of two vector forms. By writing in terms of $[L, \omega]$ and $[L, M]$ all identities for derivations one obtains identities for those differential concomitants. A direct, computational deduction of those identities would have been a more laborious and less enlightening exercise than the study of derivations. – The following properties of $[L, \omega]$ are already known from the previous discussion:

$$(5.1)\quad \begin{cases} \text{a)} & [L, \omega \wedge \pi] = [L, \omega] \wedge \pi + (-1)^{lq} \omega \wedge [L, \pi], \\ \text{b)} & d\,[L, \omega] = (-1)^l\,[L, d\omega] \end{cases} \qquad \begin{cases} L \in \Psi_l \\ \omega \in \Phi_q \end{cases}$$

Let $\mathscr{D}_r^i$ ($\mathscr{D}_r^d$) denote the set of derivations of type i_* (type d_*) of degree r, and let

$$(5.2)\quad \begin{cases} \mathscr{D}_r = \mathscr{D}_r^i \oplus \mathscr{D}_r^d, & \mathscr{D}^i = \bigoplus_{r=-1}^{n} \mathscr{D}_r^i, \\ \mathscr{D}^d = \bigoplus_{r=0}^{n} \mathscr{D}_r^d & \mathscr{D} = \mathscr{D}^i \oplus \mathscr{D}^d. \end{cases}$$

Proposition (5.1). Under the multiplication ωD defined by $(\omega D)\varphi = \omega \wedge D\varphi$, $\mathscr{D}$ is a graded module over the graded ring Φ: $\Phi_q \mathscr{D}_r \subset \mathscr{D}_{r+q}$. Restricted to $\mathscr{D}^i$ one has $\Phi \mathscr{D}^i \subset \mathscr{D}^i$, but on $\mathscr{D}^d$, $\omega \mathscr{D}^d \subset \mathscr{D}^d$ only if ω is *closed*, i.e. if $d\omega = 0$. In fact:

$$(5.3)\quad \begin{cases} \text{a)} & \omega\, i_L = i_{\omega \wedge L} \\ \text{b)} & \omega\, d_L = d_{\omega \wedge L} + (-1)^{q+l-1}\, i_{d\omega \wedge L}. \end{cases}$$

*) Supported in part by a National Science Foundation Grant at the University of Chicago.

Proof. The full statement is contained in (5.3). (5.3a) is equivalent to (2.11b), and (5.3b) follows from (4.11) by a most elementary computation.

In bracket notation, (5.3b) signifies

$$[\omega \wedge L, \pi] = \omega \wedge [L, \pi] + (-1)^{q+l} d\omega \wedge (\pi \barwedge L). \tag{5.4}$$

Proposition (5.2). The commutator of two derivations:

$$[D_1, D_2] = D_1 D_2 - (-1)^{r_1 r_2} D_2 D_1; \qquad D_1 \in \mathcal{D}_{r_1}, D_2 \in \mathcal{D}_{r_2}. \tag{5.5}$$

is again a derivation, and $[D_1, D_2] \in \mathcal{D}_{r_1+r_2}$. The subsets $\mathcal{D}^i$ and $\mathcal{D}^d$ are closed under the formation of commutators. The commutators of $\mathcal{D}^i$ and $\mathcal{D}^d$ are linear under multiplication by any elements of Φ; in fact:

$$\begin{cases} \text{a)} & [i_L, i_M] = i_{M \barwedge L} - (-1)^{(l-1)(m-1)} i_{L \barwedge M}, \\ \text{b)} & [\omega i_L, i_M] = \omega [i_L, i_M] - (-1)^{(l+q-1)(m-1)} (i_M \omega) i_L \\ \text{c)} & [\omega d_L, d_M] = \omega [d_L, d_M] - (-1)^{(l+q)m} (d_M \omega) d_L. \end{cases} \tag{5.6}$$

Proof. The fact that $[D_1, D_2] \in \mathcal{D}$ follows by an elementary computation almost identical with (4.13), replacing i_L and d by D_1 and D_2, and changing signs where necessary. The fact that $\mathcal{D}^i$ and $\mathcal{D}^d$ are closed under commutator operations is equivalent with the fact that $D_1 F = \{0\}$, $D_2 F = \{0\}$ implies the same for $[D_1, D_2]$; and that $[d, D_1] = 0$, $[d, D_2] = 0$ implies $[d, [D_1, D_2]] = 0$. Relation (a) follows from (2.12), and (5.6b, c) are direct applications of the product rule (4.2d).

The following proposition is a corollary of the Propositions (5.2) and (4.5).

Proposition (5.3). Given any two vector forms $L \in \Psi_l$, $M \in \Psi_m$, there exists a vector form $[L, M] \in \Psi_{l+m}$ which is uniquely determined by the condition

$$[d_L, d_M] = d_{[L, M]}. \tag{5.7}$$

The vector form $[L, M]$ is a differential concomitant of L and M, and satisfies the following rules:

$$\begin{cases} \text{a)} & [M, L] = (-1)^{lm+1} [L, M], \\ \text{b)} & [L, [M, \omega]] - (-1)^{lm} [M, [L, \omega]] = [[L, M], \omega], \\ \text{c)} & [I, M] = 0. \end{cases} \tag{5.8}$$

The last formula follows from $d_{[I, M]} = [d, d_M] = 0$.

Proposition (5.4). The commutator $[i_M, d_L]$ has the following decomposition into parts in $\mathcal{D}^i$ and $\mathcal{D}^d$:

$$[i_M, d_L] = d_{L \barwedge M} + (-1)^l i_{[M, L]}. \tag{5.9}$$

Its equivalent for the differential concomitants $[L, \omega]$ and $[L, M]$ is

$$[L, \omega] \barwedge M - (-1)^{l(m-1)} [L, \omega \barwedge M] = [L \barwedge M, \omega] + (-1)^l \omega \barwedge [M, L]. \tag{5.10}$$

Proof. Let $[i_M, d_L] = d_X + i_Y$. Since i_Y acts trivially on F, X can be determined by having $[i_M, d_L]$ act on $\varphi \in F$. We find:

$$\begin{cases} [i_M, d_L] \varphi = i_M d_L \varphi = i_M (d\varphi \barwedge L) = (d\varphi \barwedge L) \barwedge M = \\ \qquad = d\varphi \barwedge (L \barwedge M) = d_{L \barwedge M} \varphi. \end{cases} \tag{5.11}$$

Hence $X = L \barwedge M$. The operator $i_Y = [i_M, d_L] - d_{L \barwedge M}$ is of type i_*, and it is determined by its action on Φ_1. Since $(i_Y \omega)_x^q$ depends on ω_x, but not on the continuation of the field ω we may assume that (locally) $\omega = d\varphi$, $\varphi \in F$. Then we have

$$\begin{cases} i_Y \omega = i_M d_L d\varphi - (-1)^{(m-1)l} d_L i_M d\varphi - d_{L \barwedge M} d\varphi = \\ \quad = i_M (-1)^l d (d\varphi \barwedge L)) - (-1)^{(m-1)l} d_L d_M \varphi - (-1)^{l+m-1} d (d\varphi \barwedge (L \barwedge M)) = \\ \quad = (-1)^l i_M d\, d_L \varphi - (-1)^{l+m-1} d i_M d_L \varphi - (-1)^{(m-1)l} d_L d_M \varphi = \\ \quad = (-1)^l (d_M d_L \varphi - (-1)^{ml} d_L d_M \varphi) = \\ \quad = (-1)^l d_{[M, L]} \varphi = (-1)^l i_{[M, L]} d\varphi = (-1)^l i_{[M, L]} \omega. \end{cases} \tag{5.12}$$

This completes the proof of (5.9). Formula (5.10) follows trivially.

Proposition (5.5). The differential concomitant $[L, M]$ comprises the following special cases:

1) If $L, M \in \Psi_0$, $[L, M]$ is the usual bracket of vector fields;
2) If $L \in \Psi_0$, $[L, M]$ is the Lie derivative of M with respect to L.

Proof. (1) Let $\varphi \in F$; then $[L, \varphi] = L(\varphi)$, and by (5.8*b*) we have

$$L(M(\varphi)) - M(L(\varphi)) = [L, M](\varphi), \tag{5.13}$$

but this is exactly the usual defining equation for the bracket of two vector fields. — (2) Let $\omega \in \Phi_1$, then by (5.10*b*) we have

$$[L, \omega \barwedge M] = [L, \omega] \barwedge M + \omega \barwedge [L, M], \tag{5.14}$$

where in the first two terms both brackets are Lie derivatives with respect to L. Then, by virtue of the product rule for Lie derivatives the last term must be the third term in the product rule.

Proposition (5.6). The bracket $[L, M]$ satisfies the identity

$$[\omega \wedge L, M] = \omega \wedge [L, M] + (-1)^{q+l} d\omega \wedge (M \barwedge L) - (-1)^{(l+q)m} [M, \omega] \wedge L. \tag{5.15}$$

Proof. (5.15) follows from (5.6c) by applying (5.3b, 7, 9). The left-hand side is:

$$\begin{cases} [\omega d_L, d_M] = [d_{\omega \wedge L} + (-1)^{q+l-1} i_{d\omega \wedge L}, d_M] = \\ \qquad = d_{[\omega \wedge L, M]} + (-1)^{q+l-1} d_{M \barwedge (d\omega \wedge L)} + (-1)^{q+l+m-1} i_{[d\omega \wedge L, M]}; \end{cases} \tag{5.16}$$

and the right-hand side of (5.6c) equals

$$\begin{cases} \omega d_{[L, M]} - (-1)^{(l+q)m} [M, \omega] d_L = d_{\omega \wedge [L, M]} + (-1)^{q+l+m-1} i_{d\omega \wedge [L, M]} - \\ \qquad - (-1)^{(l+q)m} d_{[M, \omega] \wedge L} + (-1)^{lm+qm+l+q+m} i_{d[M, \omega] \wedge L}. \end{cases} \tag{5.17}$$

Equating the subscripts of d gives (5.15); and equating the subscripts of i gives (5.15) with $d\omega$ instead of ω.

Remark. $[L, M]$ defined by (5.7) is $(l+m)!/l!m!$ times the expression denoted by the same symbol in [3] (6.1). Since the bracket $[L, M]$ was defined in terms of components [6]) we will now express the components of $[L, M]$ in terms of those of L and M. L, as a vector l-form, has components $L^{\varkappa}_{\lambda_1\ldots\lambda_l}$; M has components $M^{\varkappa}_{\mu_1\ldots\mu_m}$, and $[L, M]^{\varkappa}_{\lambda_1\ldots\lambda_l\mu_1\ldots\mu_m}$ are the components of $[L, M]$. Let $x^{\varkappa}$ be any one of the local coordinate functions, then $L^{(\varkappa)} =_{\text{def}} [L, x^{\varkappa}]$ is a scalar l-form whose components $(L^{(\varkappa)})_{\lambda_1\ldots\lambda_l}$ satisfy $(L^{(\varkappa)})_{\lambda_1\ldots\lambda_l} = L^{\varkappa}_{\lambda_1\ldots\lambda_l}$. Equation (5.8b) then gives

$$(5.18)\qquad [L, M^{(\varkappa)}] - (-1)^{lm}[M, L^{(\varkappa)}] = [L, M]^{(\varkappa)}.$$

Writing $dM^{(\varkappa)}$ out in components we find $(m+1)\,\partial_{[\varrho} M^{\varkappa}_{\mu_1\ldots\mu_m]}$, and thus $[L, M^{(\varkappa)}]$ has components

$$(5.19)\qquad \begin{cases} \frac{(l+m)!}{l!\,m!}\{(\partial_\tau M^{\varkappa}_{[\mu_1\ldots\mu_m}) L^{\tau}_{\lambda_1\ldots\lambda_l]} - m\,(\partial_{[\mu_1} M^{\varkappa}_{|\tau|\mu_2\ldots\mu_m}) L^{\tau}_{\lambda_1\ldots\lambda_l]} + \\ \qquad\qquad + (-1)^l m\, \partial_{[\lambda_1} (M^{\varkappa}_{|\tau|\mu_2\ldots\mu_m} L^{\tau}_{\lambda_2\ldots\lambda_l\mu_1]})\} = \\ = \frac{(l+m)!}{l!\,m!}\{L^{\tau}_{[\lambda_1\ldots\lambda_l}\partial_{|\tau|} M^{\varkappa}_{\mu_1\ldots\mu_m]} + m\, M^{\varkappa}_{\tau[\mu_2\ldots\mu_m}\partial_{\mu_1} L^{\tau}_{\lambda_1\ldots\lambda_l]}\}. \end{cases}$$

Hence we find

$$(5.20)\qquad \begin{cases} [L, M]^{\varkappa}_{\lambda_1\ldots\lambda_l\mu_1\ldots\mu_m} = \frac{(l+m)!}{l!\,m!}\{L^{\tau}_{[\lambda_1\ldots\lambda_l}\partial_{|\tau|} M^{\varkappa}_{\mu_1\ldots\mu_m]} - M^{\tau}_{[\mu_1\ldots\mu_m}\partial_{|\tau|} L^{\varkappa}_{\lambda_1\ldots\lambda_l]} - \\ \qquad\qquad - l\, L^{\varkappa}_{\tau[\lambda_2\ldots\lambda_l}\partial_{\lambda_1} M^{\tau}_{\mu_1\ldots\mu_m]} + m\, M^{\varkappa}_{\tau[\mu_2\ldots\mu_m}\partial_{\mu_1} L^{\tau}_{\lambda_1\ldots\lambda_l]}\}. \end{cases}$$

This establishes the relation with [3] (6.1).

Let D_1, D_2, D_3 be derivations of degrees r_1, r_2, r_3 respectively; then the Jacobi identity follows by simply writing out the left side of

$$(5.21)\qquad (-1)^{r_3 r_1}[D_1,[D_2,D_3]] + (-1)^{r_1 r_2}[D_2,[D_3,D_1]] + (-1)^{r_2 r_3}[D_3,[D_1,D_2]] = 0.$$

and verifying the cancellation of all terms.

Proposition (5.7). The following identities are consequences of the Jacobi identity (5.21), where $L \in \Psi_l$, $M \in \Psi_m$, $N \in \Psi_n$:

$$(5.22)\qquad \begin{cases} [L \barwedge N, M] + (-1)^{l(n-1)}[L, M \barwedge N] - [L, M] \barwedge N = \\ = (-1)^{m(l-1)} L \barwedge [N, M] + (-1)^{l-1} M \barwedge [N, L]; \end{cases}$$

$$(5.23)\qquad (-1)^{nl}[L, [M, N]] + (-1)^{lm}[M, [N, L]] + (-1)^{mn}[N, [L, M]] = 0.$$

Proof. Take in (5.21) the derivations d_L, d_M, i_N, then the first term in (5.21) becomes, using (5.7, 9):

$$(5.24)\qquad \begin{cases} (-1)^{l(n-1)}[d_L, [d_M, i_N]] = (-1)^{(l+m)(n-1)+1}[d_L, d_{M\barwedge N} + (-1)^m i_{[N,M]}] = \\ = (-1)^{(l+m)(n-1)+1} d_{[L, M\barwedge N]} + (-1)^{m(l+n)} d_{L\barwedge[N,M]} + (-1)^{m(l+n)+l} i_{[[N,M],L]}. \end{cases}$$

Similarly, the second term of (5.21) becomes

$$(5.25)\qquad \begin{cases} (-1)^{lm}[d_M, [i_N, d_L]] = (-1)^{lm} d_{[M, L\barwedge N]} + (-1)^{mn+l+m+1} d_{M\barwedge[N,L]} + \\ + (-1)^{n(l+m)+l} i_{[[L,N],M]}. \end{cases}$$

[6]) For the details of the component notation used here we refer to Schouten [11].

The third term is different:

$$(5.26)\qquad \begin{cases} (-1)^{m(n-1)}[i_N, [d_L, d_M]] = (-1)^{m(n-1)}[i_N, d_{[L,M]}] = \\ = (-1)^{m(n+1)} d_{[L,M]\barwedge N} + (-1)^{l(m+n)+l} i_{[[M,L],N]}. \end{cases}$$

The sum of the left-hand terms vanishes; hence the sum of the terms on the right with i_*, and the sum of the terms with d_*, will vanish. This gives (5.23) and (5.22) respectively.

Remark. (5.23) could have been obtained more easily by applying (5.21) to d_L, d_M, d_N; but in the present computation, (5.23) came along free with (5.22). — If one applies (5.21) to i_L, i_M, i_N one finds an obvious algebraic identity of no interest whatever. If one takes d_L, i_M, i_N, one finds simultaneously this same algebraic identity and (5.22).

The main results derived in this section are summarized below.

Theorem II. To any scalar form ω and any vector form L over X there exists a differential concomitant $[L, \omega]$, which is a scalar form; and to any two vector forms L, M there is a differential concomitant $[L, M]$ which is a vector form. The two differential concomitants satisfy identities, most of which are quite similar:

a) three $\wedge$-product rules (5.1a), (5.4), (5.15);
b) two $\barwedge$-product rules (5.10), (5.22);
c) two Jacobi identities (5.8b), (5.23);
d) two exterior differentiation formulas (5.1b), (5.8c), which are equivalent.

So far no mention has been made of the mappings from Ψ into Φ and conversely. The mapping $\beta\colon \Phi \to \Psi$ sends Φ_q into Ψ_{q+1}, by $\omega \to I \wedge \omega$, and the mapping $\gamma\colon \Psi \to \Phi$ sends Ψ_{q+1} back into Φ_q, by *contraction*. Let $\omega^1, \ldots, \omega^n$ and $u_1, \ldots, u_n$ be dual bases of $T^*_x(X)$ and $T_x(X)$, then $\gamma(L) = \sum_{i=1}^{n} \omega^i \circ L \barwedge u_i$, which does not depend on the particular choice of ω^i, u_i. For brevity, $\gamma(L)$ will be denoted by $\bar{L}$. In index notation, $\bar{L}_{\lambda_1\ldots\lambda_q} = L^{\varkappa}_{\varkappa\lambda_1\ldots\lambda_q}$. One easily sees that $\gamma \circ \beta = (n-q)$ times the identity on Φ_q. Hence, for $q < n$, $P = \frac{1}{n-q}\beta \circ \gamma$ is a projection operator in Ψ_{q+1}: i.e. $P \circ P = P$ on Ψ_{q+1}. This gives a direct sum decomposition of Ψ_{q+1}; one summand consisting of all vector forms of the form $I \wedge \omega$; the other summand consisting of the vector forms L with $\bar{L} = 0$.

Identities relating e.g. $[L, \omega]$ to $[L, I \wedge \omega]$ can easily be derived from (5.4), etc. The deduction of an identity relating $\overline{[L, M]}$ to $L, M, \bar{L}, M$:

$$(5.27)\qquad \overline{[L, M]} = (-1)^l [L, \bar{M}] - (-1)^{m(l-1)}(d\bar{L} \barwedge M - d\,\overline{(L \barwedge M)})$$

from (5.10) and (5.22) involves much more work. The simplest proof of (5.27) comes from (5.20). It is left to the reader.

§ 6. Vector 1-forms; Haantjes' Theorem.

This section gives a "direct" approach to the simplest non-trivial case of vector forms. It can be read inpendently by any reader who is familiar with the basic properties of Lie derivatives [7]). Formula (6.4) below is very suitable as an independent definition [8]) of $[h, k]$. We first derive (6.4) from the identities of § 5, and then proceed without further reference to § 5.

Let h, k be two vector forms of degree one over X; then $h \cdot u$ and $k \cdot u$ are tangent vectors for any u that is a tangent vector. The differential concomitant $[h, k]$ is then determined by (5.10), with $L = h$, $M = k$, where ω is a scalar one-form:

$$(6.1)\qquad \left\{\begin{aligned} \omega \circ [h, k] = {} & d(\omega \circ h) \barwedge k - (d\omega \barwedge h) \barwedge k - d(\omega \circ k \circ h) + \\ & + d(\omega \circ k) \barwedge h - d(\omega \circ h \circ k) + d\omega \barwedge (h \circ k). \end{aligned}\right.$$

Evaluation of this expression on two vector fields u, v gives

$$(6.2)\qquad \left\{\begin{aligned} \omega([h, k](u, v)) = {} & d(\omega \circ h)(ku, v) + d(\omega \circ h)(u, kv) - d\omega(hu, kv) + \\ & + d(\omega \circ k)(hu, v) + d(\omega \circ k)(u, hv) - d\omega(ku, hv) - \\ & - d(\omega \circ k \circ h)(u, v) - d(\omega \circ h \circ k)(u, v). \end{aligned}\right.$$

The following formula is also a special case of (5.10), now for $L = u \in \Psi_0$, $M = v \in \Psi_0$, where $df(u)$ has been replaced by $u \cdot f$:

$$(6.3)\qquad d\omega(u, v) = u \cdot \omega(v) - v \cdot \omega(u) - \omega([u, v]).$$

This formula is applied to (6.2) by taking for ω in (6.3) successively $\omega \circ h$, $\omega \circ k$, etc. A simple computation gives then

$$(6.4)\qquad \left\{\begin{aligned} [h, k](u, v) = {} & [hu, kv] + [ku, hv] - k[hu, v] - h[ku, v] - \\ & - k[u, hv] - h[u, kv] + kh[u, v] + hk[u, v]. \end{aligned}\right.$$

This formula is now taken as a definition of $[h, k]$ for this section. $[h, k](u, v)$ is obviously a functional of the fields h, k, u, v, and satisfies

$$(6.5)\qquad \left\{\begin{aligned} &\text{a)}\quad [h, k](v, u) = -[h, k](u, v) \\ &\text{b)}\quad [k, h](u, v] = [h, k](u, v). \end{aligned}\right.$$

The algebraic dependence of $[h, k](u, v)$ on v follows from the subsequent computation; the statement for u follows by interchange of variables (6.5a). Let $[v, h]$ be the Lie derivative of h with respect to v, then we find from (6.4):

$$(6.6)\qquad [h, k](u, v) = [hu, k]v + [ku, h]v - k[u, h]v - h\,[u, k]v,$$

and this contains v in undifferentiated form.

Let $k \circ l$ be the vector 1-form defined by $(k \circ l)u = klu$, then the following identity holds:

$$(6.7)\qquad \left\{\begin{aligned} & [h, k \circ l](u, v) + [h \circ l, k](u, v) - [h, k](lu, v) - [h, k](u, lv) = \\ & \qquad = h[k, l](u, v] + k[h, l](u, v). \end{aligned}\right.$$

[7]) Cf. e.g.[11], Ch. II § 10; Ch. VIII. – The only coordinate-free treatment of Lie derivatives known to the authors is Palais [12].

[8]) In fact, this is the most practical definition for $[h, k]$ as it is used in e.g. the theory of homogeneous almost-complex structures; cf. [13].

Although it is a special case of (5.22) the proof here is so much more elementary that we reproduce it separately. – We first rearrange the terms in (6.4) with the product rule for Lie derivatives, and find

$$(6.8)\qquad [h, k](u, v) = [hu, k]v - [hv, k]u - h[u, k]v + h\,[v, k]u.$$

Then we have

$$(6.9)\qquad \left\{\begin{aligned} [h, k \circ l](u, v) = {} & [hu, k]lv + k[hu, l]v - [hv, k]lu - k[hv, l]u - \\ & - h[u, k]lv - hk[u, l]v + h[v, k]lu + hk[v, l]u, \\ [k, h \circ l](u, v) = {} & [ku, h]lv + h[ku, l]v - [kv, h]lu - h[kv, l]\,u - \\ & - k\,[u, h]lv - kh[u, l]v + k[v, h]lu + kh[v, l]u, \end{aligned}\right.$$

and therefore

$$(6.10)\qquad \left\{\begin{aligned} & [h, k \circ l](u, v) + [h \circ l, k](u, v) - [h, k](lu, v) - [h, k](u, lv) = \\ & = k([hu, l]v - [hv, l]u - h[u, l]v + h[v, l]u) + \\ & + h(-k[u, l]v + k[v, l]u + [ku, l]v - [kv, l]u) = \\ & = k[h, l](u, v) + h[k, l](u, v); \end{aligned}\right.$$

which completes the proof.

The special case $h = k$ is of importance. We define $H = \frac{1}{2}[h, h]$; then

$$(6.11)\qquad H(u, v) = [hu, hv] + hh[u, v] - h[hu, v] - h[u, hv].$$

This expression is needed in the following theorem, which was brought into its present form by Haantjes [4]. It was first formulated by Tonolo [5] for Riemannian spaces V_3 of dimension 3, and then generalized by Schouten [6] for V_n. The metric was eliminated, and H of (6.11) was introduced by Nijenhuis [1]. Haantjes' improvement [4] consists of the introduction of a neater condition (6.12) and the inclusion of one case of multiple eigenvalues of h. Here we present a shorter and simpler proof of Haantjes' version:

Theorem III. Let h take the diagonal form on a set of real eigenvectors at every point; and let the multiplicity of each eigenvalue λ be constant. Let S_λ denote the field of eigenplanes of h belonging to λ (the eigenplane of λ is the vector space of eigenvectors belonging to λ); then every S_λ and every direct sum $S_\lambda + S_\mu + \ldots + S_\nu$ is locally integrable (i.e. tangent to a family of surfaces of the same dimension) if and only if for all u and v

$$(6.12)\qquad \mathscr{H}(u, v) \equiv hhH(u, v) + H(hu, hv) - hH(hu, v) - hH(u, hv) = 0.$$

Our new proof goes as follows. Let u belong to λ, v to μ, then

$$(6.13)\qquad \left\{\begin{aligned} H(u, v) & = [\lambda u, \mu v] + hh[u, v] - h[\lambda u, v] - h[u, \mu v] = \\ & = (h \circ h - \lambda h - \mu h + \lambda\mu)[u, v] + (\lambda - \mu)v(\lambda) \cdot u + (\lambda - \mu)u(\mu) \cdot v = \\ & = (h - \lambda)(h - \mu)[u, v] + (\lambda - \mu)(v(\lambda) \cdot u + u(\mu) \cdot v). \end{aligned}\right.$$

Since $(h-\lambda)u=0$, $(h-\mu)v=0$, one easily finds by a similar computation

(6.14) $$\mathscr{H}(u, v)=(h-\lambda)^2(h-\mu)^2[u, v].$$

Because of the diagonalizability of h we know, for every vector w, that $(h-\lambda)^k w=0$, $k>0$ implies $(h-\lambda)w=0$, $w \in S_\lambda$. Therefore, for $\lambda=\mu$, $u, v \in S_\lambda$, it follows that $[u, v] \in S_\lambda$ if and only if $\mathscr{H}(u, v)=0$. And for $\lambda \neq \mu$, $u \in S_\lambda$, $v \in S_\mu$, we know that $[u, v] \in S_\lambda + S_\mu$ if and only if $\mathscr{H}(u,v)=0$. But:

$$[u, v] \in S_\lambda \text{ for all } u \in S_\lambda,\ v \in S_\lambda$$

is precisely the integrability condition for the planes S_λ; and

$$[u, v] \in S_\lambda + S_\mu \text{ for all } u \in S_\lambda,\ v \in S_\mu$$

in addition to the previous condition, is precisely the necessary and sufficient condition that also the planes $S_\lambda + S_\mu$ are integrable.

Furthermore, $S_{\lambda_1} + \ldots + S_{\lambda_k}$ is integrable for all $\lambda_1, \ldots, \lambda_k$ if and only if $S_\lambda + S_\mu$, for all λ, μ is integrable. Finally, $\mathscr{H}(u, v)$ vanishes for all u, v if and only if $\mathscr{H}(u, v)$ vanishes for all $u \in S_\lambda$, $v \in S_\mu$, where λ and μ run independently through all eigenvalues of h. This completes the proof.

Let p be a field of projection operators in the tangent bundle, and $q=I-p$ the complementary projection. Then p and q define a decomposition of the tangent bundle, which is called an X_n^m in X_n by SCHOUTEN [11], or an almost-product structure by SPENCER [9]. The differential concomitant $[p, p]=[q, q]$ is called the *torsion*. Using (6.8) and $p \circ q=0$ one finds

(6.15) $$\begin{cases} \tfrac{1}{2}[p \cdot p](u,v) = [pu, p]v - p[u, p]v = \\ \qquad = p[pu, p]v + q[pu, p]v - p[pu, p]v - p[qu, p]v = \\ \qquad = q[pu, pv] - qp[pu, v] + p[qu, q]v = \\ \qquad = q[pu, pv] + p[qu, qv]. \end{cases}$$

This relation will be used later on. The integrability of the invariant planes of p and q can immediately be expressed as $q \circ [p, p]=0$ and $p \circ [p, p]=0$ respectively [3].

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