GR lecture 1, 17.02.2011.

(0) Models of space-time:

1) $\boxed{\text{Special Relativity}}$

M - 4-dimensional manifold
equipped with g - metric of Lorentz signature
which is flat

(M, g) - Minkowski spacetime

2) $\boxed{\text{General Relativity}}$:

M - 4-dimensional manifold
equipped with g - metric of Lorentz signature
which satisfy certain curvature conditions.

The conditions are called Einstein equations
in the regions not containing matter:

$$Ric(g) = \Lambda g \quad (*)$$

(M, g) - Einstein manifold
+ $(*)$

More generally $\underset{\substack{\uparrow \\ \text{Einstein tensor}}}{Ein(g)} + \Lambda g = \underset{\substack{\uparrow \\ \text{energy momentum tensor of matter fields.}}}{T}$

e.g. Minkowski space-time is Einstein manifold
which is flat.
In particular it has $\Lambda \equiv 0$.

There exists Einstein manifolds which are
different from the Minkowski spacetime.

(1) Notations and basic notions from diff. geo.

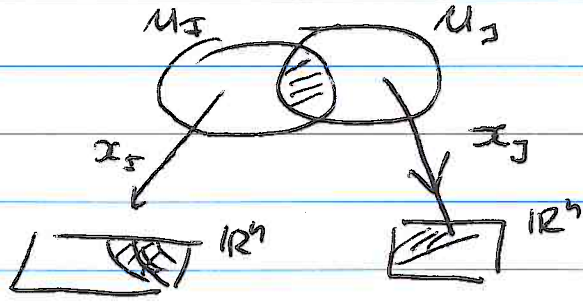
1) M – differentiable manifold of class C^k

M – topological Hausdorff space equipped with a family of pairs $(\mathcal{U}_I, x_I)$ – called coordinate charts, such that

a) $\mathcal{U}_I \subset M$ (open), $\quad x_I : \mathcal{U}_I \longrightarrow \mathbb{R}^{\boxed{n}}$ local ~~hom~~ homeomorphism

b) $(\mathcal{U}_I, x_I)$ and $(\mathcal{U}_J, x_J)$ are compatible of class C^k

i.e.:



$$\begin{cases} x_I \circ x_J^{-1}\big|_{x_J(\mathcal{U}_I \cap \mathcal{U}_J)} : \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ x_J \circ x_I^{-1}\big|_{x_I(\mathcal{U}_I \cap \mathcal{U}_J)} : \mathbb{R}^n \longrightarrow \mathbb{R}^n \end{cases}$$

both these maps are of differentiability class C^k for each $I, J \in$ Indices

c) $\bigcup_{I \in \text{Indices}} \mathcal{U}_I = M$

$(M, (\mathcal{U}_I, x_I))$ – M is equipped with atlas of class C^k

Atlas is maximal iff one can not add more coordinate charts without violating C^k compatibility.

$p \in \mathcal{U}, x$

$x(p) \in \mathbb{R}^n$

$\shortparallel$

$(x^\mu(p))$ ← coordinates

$\boxed{M \text{ with maximal atlas of class } C^k \equiv \text{differentiable manifold of class } k.}$

$k = 0, 1, \ldots, \infty, \omega$

$k=0$: topological manifold; $k=\infty$: smooth manifold; $k=\omega$: analytic manifold.

$\boxed{n} \equiv$ dimension of M

In the following: ONLY SMOOTH manifolds

2) Differentiable maps; diffeomorphisms.

1) M_1, M_2 manifolds

$F: M_1 \to M_2$ is <u>differentiable</u> if

$\forall (U_I, x_I) \in \mathcal{A}_1, \; \forall (V_J, y_J) \in \mathcal{A}_2$

$$\mathbb{R}^{n_1} \xleftarrow{x_I} U_I \xrightarrow{F} V_J \xrightarrow{y_J^{-1}} \mathbb{R}^{n_2}$$

$x_I \circ F \circ y_J^{-1}$ is smooth.

<u>Examples</u>

- ~~$M_1 \subset \mathbb{R}$~~ • $M_1 \subset \mathbb{R}^1 \Rightarrow F$ is called <u>a curve</u> in M_2
  open interval
- $M_2 \subset \mathbb{R}^1 \Rightarrow F$ is called a function; $\mathcal{F}(M)$ set of ~~set~~ functions
- F is bijection + F, F^{-1} are differentiable $\Rightarrow$

$$\boxed{F \equiv \text{diffeomorphism}}$$

- M_1, M_2 are diffeomorphic if there exists diffeomorphism F $F: M_1 \to M_2$.

3) <u>Cartesian product of Manifolds</u>

$(M_1, \mathcal{A}_1) \quad , \quad (M_2, \mathcal{A}_2)$

$M_1 \times M_2$ and $\mathcal{A} = \left\{ (U_I \times V_J, x_I \times y_J), \; \begin{matrix} (U_I, x_I) \in \mathcal{A}_1 \\ (V_J, y_J) \in \mathcal{A}_2 \end{matrix} \right\}$

$x_I \times y_J (p_1, p_2) = (x_I(p_1), y_J(p_2)) \in \mathbb{R}^{n_1+n_2}$

$(M_1 \times M_2, \mathcal{A})$

manifold of dim n_1+n_2
and class $k = \min(k_1, k_2)$.

4) Submanifold ~~N~~ of M of dimension m

$N \subset M$:

$\forall p \in N \ \exists (\mathcal{U}, x) \in \mathcal{A}_M$:
1° $p \in \mathcal{U}$
2° $x(N \cap \mathcal{U}) \subset \mathbb{R}^m \times \{0\}$
3° $pr_{\mathbb{R}^m} x(N \cap \mathcal{U})$ open

$\mathcal{A}_N = \left\{ \left(N \cap \mathcal{U},\ pr_{\mathbb{R}^m} \circ x|_{N \cap \mathcal{U}} \right) \right\}$ atlas on N.

5) Vector fields:

$\mathcal{F}(M)$ - set of smooth functions on M

algebra:

$(f_1 + f_2)(p) = f_1(p) + f_2(p)$

$(c \cdot f)(p) = c \cdot f(p)$

$(f_1 \cdot f_2)(p) = f_1(p) \cdot f_2(p)$.

$X : \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$ is a vector field on M iff:

1). X is $\mathbb{R}$-linear

$X(c_1 f_1 + c_2 f_2) = c_1 X(f_1) + c_2 X(f_2)$ $\quad \forall c_1, c_2 \in \mathbb{R}$ $\ \forall f_1, f_2 \in \mathcal{F}(M)$)

2) $X(f \cdot g) = X(f) \cdot g + f \cdot X(g)$.

6) Commutator of vector fields: ~~a new vector f~~

$[X,Y](f) = X(Y(f)) - Y(X(f))$

it is made in such a way that 1) and 2) is satisfied (check!)

$\Rightarrow [X,Y]$ is a vector field.

$\mathfrak{X}(M)$ - set of all vector fields on M → Lie algebra over $\mathbb{R}$; → module over $\mathcal{F}(M)$

7) Values of X at p. : Given X – a vector field on M and $p \in M$ we have $X_p : \mathcal{F}(M) \to \mathbb{R}$

$$X_p(f) = (X(f))(p).$$

~~The~~ X_p is 1) linear

2) $X_p(f \cdot g) = X_p(f)g(p) + f(p)X_p(g)$

~~This defines~~

Conditions 1) and 2) define a tangent vector at p

Space of tangent vectors at p is denoted $T_p(M)$ and called tangent space at p. (Obviously it is a vector space over $\mathbb{R}$)

Example

$]-\varepsilon, \varepsilon[\ni t \xrightarrow{\gamma} \gamma(t)$ – a curve in M s.t. $\gamma(0) = p$

Take $X_p(f) = \frac{d}{dt} f \circ \gamma(t)\big|_{t=0}$.

it obviously satisfies 1) and 2) so X_p is a tangent vector at p.

More precisely one defines X_p as an equivalence class $[\gamma]$ of curves passing through p by:

$$\gamma_1 \sim \gamma_2 \iff \forall f \quad \frac{d}{dt} f \circ \gamma_1(t)\big|_{t=0} = \frac{d}{dt} f \circ \gamma_2(t)\big|_{t=0}$$

Fact $\dim T_p(M) = n$

Basis in $T_p(M)$:

$p \in \mathcal{U}, x$ $\qquad x^\mu$ – coordinates in $\mathcal{U}$

$x^\mu(p) \ \forall p \in \mathcal{U}$.

$$\left(\frac{\partial}{\partial x^\mu}\Big|_p \right) \text{ is a basis in } T_p(\mathcal{U}) = T_p(M).$$

$$\left(\frac{\partial}{\partial x^\mu}\right)(x^\nu) = \delta^\nu_\mu$$

Locally in $(\mathcal{U}, x)$ a vector field X can be written as

~~Locally around p~~

$$X = X^\mu \frac{\partial}{\partial x^\mu} \qquad x^\mu : p \xrightarrow{x} x^\mu(p)$$

~~with~~ $X^\mu = X(x^\mu)$ $\qquad$ μ-th component

One can reverse the construction:

FIRST define X_p as $X_p : \mathcal{F}(M) \xrightarrow{\text{linear}} \mathbb{R}$ + 1) and 2)

and SECOND define vector fields X to be

$$M \ni p \xrightarrow{X} X_p \in T_p(M)$$

such that this map is smooth.

2) 1-parameter groups of transformations

$t \in \mathbb{R}$, $\varphi_t : M \xrightarrow{\text{diffeomorphism}} M$ s.t.

$$\varphi_{t+s} = \varphi_t \circ \varphi_s \Rightarrow \begin{cases} \varphi_0 = id_M \\ \varphi_{-t} = (\varphi_t)^{-1} \end{cases}$$

Given a 1-par. group of transformations one can define a vector field X by

~~$X(f) = \frac{d}{dt} f \circ \varphi_t \big|_{t=0}$~~

~~even better more precisely and here (~~

$$X_p(f) = \frac{d}{dt} f \circ \varphi_t(p) \Big|_{t=0}$$ and varying p we get X, smooth map

~~$X: M \to$~~

$p \xrightarrow{X} X_p$

~~The other way around?~~

~~Given X where we got φ_t?~~

~~$X(f) = \frac{d}{dt}$~~

~~More precisely we take $p(s) = \varphi_s(p)$~~

~~$X_{p(s)}(f) = \frac{d}{dt} f \circ \varphi_t(\varphi_s(p)) \big|_{t=0} = \frac{d}{dt} f \circ \varphi_{t+s}(p) \big|_{t=0} = \frac{d}{dt'} f \circ \varphi_{t'}(p) \big|_{t'=s} = \frac{d}{ds} f \circ \varphi_s(p)$~~

~~$t+s = t'$~~

Thus, $\boxed{\text{every 1-par group of transformations defines its } X.}$

The other way around?

Suppose that we have X. Then we look for φ_t s.t.

$$X(f) = \frac{d}{dt} f \circ \varphi_t \Big|_{t=0} \qquad \circ \varphi_s$$

$$X(f) \circ \varphi_s = \frac{d}{dt} f \circ \varphi_t \circ \varphi_s \Big|_{t=0} =$$

$$= \frac{d}{dt} f \circ \varphi_{t+s} \Big|_{t=0} \underset{t'=t+s}{=} \frac{d}{dt'} f \circ \varphi_{t'} \Big|_{t'=s} =$$

$$= \frac{d}{ds} f \circ \varphi_s$$

We look for φ_s s.t.

$$X(f) \circ \varphi_s = \frac{d}{ds} f \circ \varphi_s$$

Locally Take $f = x^\mu$

$$X(x^\mu) = X^\mu \qquad X^\mu(x^\rho(\varphi_t(p))) = \frac{d}{dt} x^\mu(\varphi_t(p))$$

$$\boxed{\cancel{X^\mu(x^\rho(\varphi_s(p))) = \frac{d}{ds} x^\mu(\varphi_s(p))}}$$

n - ordinary differential equations on n components of $\varphi_t(p)$

$\Rightarrow$ always have solutions but LOCALLY:

1° around p

2° around 0 in $\mathbb{R} \ni t, s$

$\Rightarrow$ Locally $\boxed{\text{every smooth vector field generates a 1-parameter group of transformations.}}$