Lecture 10, 19.04.2011

Summary of the so far obtained formulation of GR

- space-time – (M, g) – 4-dimensional Riemannian manifold with metric g of signature $(+---)$
- gravity: manifests itself when the metric g has nonzero Riemann curvature $R_{\mu\nu\rho\sigma} \neq 0$
- newtonian concept of gravitational force: ELIMINATED
- movement of particles on which no force is exerted:

particles move along the geodesics $\gamma = \gamma(s)$ in space-time, i.e. in a coordinate charts where $\gamma = (x^\mu(s))$, $g = g_{\mu\nu} dx^\mu dx^\nu$:

$$\boxed{\frac{d^2 x^\mu}{ds^2} + \left\{ {\mu \atop \sigma\rho} \right\} \frac{dx^\sigma}{ds} \frac{dx^\rho}{ds} = 0} \quad (*)$$

Three possibilities for a sign of $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$:

(1) massive particles move along geodesics on which

$$g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 1$$

s – is an affine parameter for γ and is called PROPER TIME of a particle

(2) massless particles move along NULL geodesics i.e. s.t.

$$g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

~~they are interpr~~ These geodesics are interpreted as trajectories of photons. Thus ~~they~~ they are counterparts of light rays from Minkowski spacetime. (Also other massless particles, different than fotons move along these geodesics).

(3) if $g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} < 0$ then so far we have not observed particles with such trajectories. They move with spatial velocity > speed of light. Tachyons (?)

- Newtonian Limit of equation (*)

$$\frac{dx^\mu}{ds} \simeq u^\mu = \left(1, \frac{v^a}{c}\right), \qquad ds = c\,dt$$

$$\frac{du^\mu}{ds} + \left\{ {\mu \atop \nu\rho} \right\} u^\nu u^\rho = 0 \implies \frac{dv^a}{dt} + \frac{c^2}{2} g_{00,a} = 0$$

$$\implies \frac{d\vec{v}}{dt} = -\vec{\nabla}\left(\frac{c^2}{2} g_{00}\right)$$ Comparing this with Newtonian:

$$\frac{d\vec{v}}{dt} = -\vec{\nabla}\phi \implies \boxed{g_{00} = 1 + \frac{2\phi}{c^2}}$$

$\Rightarrow$ Metric coefficients are COUNTERPARTS of Newtonian gravitational potential.

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5) Einstein's field equations

In Newton theory the full set of equations is as follows:

$$\begin{cases} \text{(1)} \quad \frac{d\vec{v}}{dt} = -\vec{\nabla}\phi & \text{— equation of motion for particles} \\ \text{(2)} \quad \Delta\phi = 4\pi G \rho & \text{— equation governing gravitational field,} \end{cases}$$

POISSON EQUATIO

We have a counterpart of (1) i.e. $\frac{du^\mu}{ds} + \{^{\mu}_{\nu\rho}\} u^\nu u^\rho = 0$

$g_{\mu\nu} u^\mu u^\nu = \{^1_0$

What about (2) ?

It should be an equation of the same sort as the Poisson equation i.e.

L = 2nd derivatives of gravitational field

R = source of gravitational field.

Principle of covariance

All equations of the theory should have the property of being frame independent: they should look the same in any frame!

$\Longrightarrow$ should be tensorial $<$ scalar / vectorial / tensorial — what kind of a tensor?

Right hand side:

relativistic objects that are counterparts of mass density ρ are:

1) 4-momentum $p^\mu = mc\frac{dx^\mu}{ds}$, $g_{\mu\nu} p^\mu p^\nu = m^2 c^4$

2) energy-momentum tensor $T_{\mu\nu}$, e.g. for

perfect fluid with energy density ρ, and homogeneous pressure p

$$T_{\mu\nu} = \begin{pmatrix} \rho c^2 & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

(more generally:

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu}$$

c=1 because signature + - - -)

$\rightsquigarrow$ atempts of having scalar and vectorial field equations failed.

$\Rightarrow$ the equations should be for a tensor of type $\binom{0}{2}$, which is balanced on the right hand side by $T_{\mu\nu}$.

So the postulated form of the equation is.

$$E_{\mu\nu} = \varkappa T_{\mu\nu}$$

This should be built in terms of something which is of second differentiability order in terms of a counterpart of gravitational potential.

$$\Rightarrow \quad E_{\mu\nu} = E_{\mu\nu}\left(g_{\alpha\beta}, \partial_\gamma g_{\alpha\beta}, \partial^2_{\gamma\tau} g_{\alpha\beta}\right)$$

<u>But</u>: in special relativity, as well as in any reasonable theory we have PRINCIPLE OF MASS CONSERVATION.

IN SR it is depicted by

$$\partial^\mu T_{\mu\nu} = 0.$$

Einstein in 1913, 1914 postulated the field equations in the form

$$\Box g_{\mu\nu} = \varkappa T_{\mu\nu} + \left(\frac{\partial g}{\partial x}\right)^2_{\mu\nu}$$

obvious relativistic generalization of the Laplacian

this term keeps track of the energy-momentum of gravitational field itself

these equations are NOT tensorial

$\Rightarrow$ $$R_{\mu\nu} = \varkappa T_{\mu\nu}$$ (Einstein 1915)

Ricci tensor.

$\Rightarrow$ these equations are tensorial & have the leading terms as above.

But are WRONG because, by a principle of minimal coupling we want also

$$\nabla^\mu T_{\mu\nu} = 0.$$

This gives further conditions

$$\nabla^\mu R_{\mu\nu} = 0$$

which makes the system too complicated.

In the end of 1915 Einstein found a tensor

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ s.t. $\nabla^\mu G_{\mu\nu} = 0$ is an identity, and postulated the equations in the form

$$\boxed{G_{\mu\nu} = \varkappa T_{\mu\nu}}, \quad G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}$$

This is very similar to Maxwell's theory, where Maxwell's equations read

$$\underbrace{\partial^\mu F_{\mu\nu}}_{G_\nu} = -4\pi j_\nu$$

and the simple identity $\partial^\nu \partial^\mu F_{\mu\nu} = 0$,

i.e. $\partial^\nu G_\nu = 0$ implies $\partial^\nu j_\nu = 0$ i.e. the conservation of the current.

Interestingly:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}$$

$\Lambda = \text{const}$ still satisfies

$$\nabla^\mu (G_{\mu\nu} + \Lambda g_{\mu\nu}) = 0 .$$

Einstein in <u>1917</u> modified his field equations to the more general form

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \varkappa T_{\mu\nu}}$$

Einstein's equations with COSMOLOGICAL CONSTANT $\Leftrightarrow \Lambda$.

His reason for Λ was estaethic: the metric

$$g = c^2 dt^2 - R^2 g_{S^3}$$

did not satisfied Einstein's equations $G_{\mu\nu} = \varkappa T_{\mu\nu}$.

6) Exercise - The Einstein Universe

Einstein tensor for the Einstein Universe.

$$g = (c\,dt)^2 \mp R^2 g_{S^3}, \qquad M = \mathbb{R} \times S^3$$

for Lorentz signature!

$$S^3 = \{ \mathbb{R}^4 : (x^1, x^2, x^3, x^4) \text{ s.t. } x^{1^2} + x^{2^2} + x^{3^2} + x^{4^2} = 1 \}$$

$$x^1 = \sin\chi \sin\theta \sin\varphi$$
$$x^2 = \sin\chi \sin\theta \cos\varphi$$
$$x^3 = \sin\chi \cos\theta$$
$$x^4 = \cos\chi$$

$$dx^{1^2} + dx^{2^2} + dx^{3^2} + dx^{4^2} \Big|_{S^3} = g_{S^3}$$

$$dx^1 = \ldots$$
$$dx^2 = \ldots$$
$$dx^3 = \ldots$$
$$dx^4 = \ldots$$

$$g_{S^3} = dx^{1^2} + dx^{2^2} + dx^{3^2} + dx^{4^2} =$$
$$= d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right)$$

$\Rightarrow$ Metric for the Einstein Universe

$$g = (c\,dt)^2 - R^2 \left(d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right)$$

Orthonormal frame:

$$\begin{cases} e^0 = c\,dt \\ e^1 = R\,d\chi \\ e^2 = R \sin\chi \, d\theta \\ e^3 = R \sin\chi \sin\theta \, d\varphi \end{cases}$$

$$g = e^{0^2} - e^{1^2} - e^{2^2} - e^{3^2} =$$
$$= g_{\mu\nu} e^\mu e^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Structure equations:

(SE)
$$\begin{cases} de^\mu + \Gamma^\mu{}_\nu \wedge e^\nu = 0 & \text{(no torsion)} \\ \cancel{dg_{\mu\nu}} - \Gamma_{\mu\nu} - \Gamma_{\nu\mu} = 0 & \text{(metricity)} \\ \Gamma_{\mu\nu} = g_{\mu\alpha}\Gamma^\alpha{}_\nu . \end{cases}$$

We need to ~~ba~~calculate de^μ.

$$\begin{cases} de^0 = 0 \\ de^1 = 0 \\ de^2 = R\cos\chi \, d\chi \wedge d\theta = e^1 \wedge e^2 \, \frac{\operatorname{ctg}\chi}{R} = \frac{\operatorname{ctg}\chi}{R} e^1 \wedge e^2 \\ de^3 = R\cos\chi \sin\theta \, d\chi \wedge d\varphi + R\sin\chi\cos\theta \, d\theta \wedge d\varphi \\ \quad = e^1 \wedge e^3 \, \frac{\operatorname{ctg}\chi}{R} + \frac{\operatorname{ctg}\theta}{R\sin\chi} e^2 \wedge e^3 = \\ \quad = \frac{\operatorname{ctg}\chi}{R} e^1 \wedge e^3 + \frac{\operatorname{ctg}\theta}{R\sin\chi} e^2 \wedge e^3 \end{cases}$$

$$A = \frac{\operatorname{ctg}\chi}{R} \qquad B = \frac{\operatorname{ctg}\theta}{R\sin\chi}$$

$$de^0 = 0 \qquad = -\Gamma^0{}_\mu \wedge e^\mu = -\Gamma_{0\mu} \wedge e^\mu \qquad 1$$
$$de^1 = 0 \qquad = -\Gamma^1{}_\mu \wedge e^\mu = \Gamma_{1\mu} \wedge e^\mu \qquad 2$$
$$de^2 = Ae^1 \wedge e^2 \qquad = -\Gamma^2{}_\mu \wedge e^\mu = \Gamma_{2\mu} \wedge e^\mu \qquad 3$$
$$de^3 = Ae^1 \wedge e^3 + Be^2 \wedge e^3 \qquad = -\Gamma^3{}_\mu \wedge e^\mu = \Gamma_{3\mu} \wedge e^\mu \qquad 4$$

$$\Gamma_{0i} = 0$$
$$\Gamma_{21} = -Ae^2 \quad \Rightarrow \Gamma_{12} = Ae^2$$
$$\Gamma_{31} = -Ae^3 \quad \Rightarrow \Gamma_{13} = Ae^3$$
$$\Gamma_{32} = -Be^3 \quad \Rightarrow \Gamma_{23} = Be^3$$

and all other $\Gamma_{\mu\nu}$'s are zero!

ALL (SE) equations are satisfied $\Rightarrow$ we found Γ and it is LC by the uniqueness of LC con.

$$\begin{cases} \Gamma_{0\mu} = 0 = \Gamma_{\mu 0} \\ \Gamma_{12} = -\Gamma_{21} = Ae^2 \\ \Gamma_{13} = -\Gamma_{31} = Ae^3 \\ \Gamma_{23} = -\Gamma_{32} = Be^3 \end{cases}$$

Curvature:

$$\Omega_{\mu\nu} = d\Gamma_{\mu\nu} + \Gamma_{\mu\rho} \wedge \Gamma^{\rho}{}_{\nu} =$$

$$= d\Gamma_{\mu\nu} + \Gamma_{\mu\kappa} \wedge \Gamma^{\kappa}{}_{\nu} = d\Gamma_{\mu\nu} - \Gamma_{\mu\kappa} \wedge \Gamma_{\kappa\nu} =$$

$$= d\Gamma_{\mu\nu} + \Gamma_{\mu\kappa} \wedge \Gamma_{\nu\kappa}$$

$$\Omega_{0\mu} = d\Gamma_{0\mu} + \Gamma_{0\kappa} \wedge \Gamma_{\kappa\mu} = 0$$

$$\Omega_{ij} = d\Gamma_{ij} + \sum_{k=1}^{3} \Gamma_{ik} \wedge \Gamma_{jk}$$

We need to know $d\Gamma_{ij}$.

In particular, we need

$$dA = -\frac{e^1}{R^2 \sin^2 \chi}$$

$$dB = -\frac{\operatorname{ctg}\chi \operatorname{ctg}\theta}{R^2 \sin\chi} e^1 - \frac{e^2}{R^2 \sin^2\chi \sin^2\theta}$$

We only need to calculate $\Omega_{12}, \Omega_{13}, \Omega_{23}$

$$\Rightarrow \Omega_{12} = -\frac{e^1 \wedge e^2}{R^2}, \quad \Omega_{13} = -\frac{e^1 \wedge e^3}{R^2}, \quad \Omega_{23} = \frac{-e^2 \wedge e^3}{R^2}$$

$$\Rightarrow R_{1212} = -\frac{1}{R^2}, \quad R_{1313} = -\frac{1}{R^2}, \quad R_{2323} = -\frac{1}{R^2}$$

$$R^1{}_{212} = \frac{1}{R^2}, \quad R^1{}_{313} = \frac{1}{R^2}, \quad R^2{}_{323} = \frac{1}{R^2}$$

Ricci: $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \Rightarrow R_{0\mu} = 0$

$R_{11} = R^i{}_{1i1} = \frac{2}{R^2} = R_{22} = R_{33}$ $\quad R_{ij} = \frac{2}{R^2}\delta_{ij}$

Ricci scalar:

$$R = g^{\mu\nu} R_{\mu\nu} = -R_{11} + R_{22} + R_{33} = \frac{-6}{R^2}$$

Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} =$$

$$= \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \frac{2}{R^2}\delta_{ij} \end{array}\right) + \frac{1}{2}\frac{6}{R^2}\left(\begin{array}{c|c} 1 & \\ \hline & -\delta_{ij} \end{array}\right) =$$

$$= \frac{1}{R^2}\left(\begin{array}{c|c} +3 & \\ \hline & -\delta_{ij} \end{array}\right) =$$

$$= \frac{1}{R^2}\left[\left(\begin{array}{c|c} 1 & \\ \hline & -\delta_{ij} \end{array}\right) + \left(\begin{array}{c|c} 2 & \\ \hline & \end{array}\right)\right] =$$

$$= +\frac{1}{R^2} g_{\mu\nu} + \left(\begin{array}{c|c} \frac{2}{R^2} & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \left(\begin{array}{c|c} \rho & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$\Lambda = \frac{-1}{R^2} \qquad \kappa\rho = \frac{2}{R^2 c^2}$$

↑

Reason for introduction of Λ.

If there is a cosmological constant $\mathbb{R}\times S^3$ is a solution to Einstein's equations with no pressure and en-density $\kappa\rho = \frac{2}{R^2 c^2}$

7) Linearization of ~~Newtonian limit of~~ Einstein's equations.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

and now we work in COORDINATE frame; i.e.

$$g = g_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu + h_{\mu\nu} dx^\mu dx^\nu =$$

$$= \underbrace{(dx^{0\,2} - dx^{1\,2} - dx^{2\,2} - dx^{3\,2})}_{\text{Minkowski}} + \underbrace{h_{\mu\nu} dx^\mu dx^\nu}_{\text{small perturbation}}$$

$$|h_{\mu\nu}| \ll 1$$

$h_{\mu\nu}$ can be made simpler, since we have still possibility to change a coordinate system ~~xxxx~~

$$x^\mu \longrightarrow x^\mu = x'^\mu + \xi^\mu$$

We assume small perturbation i.e. $|\xi^\mu| \ll 1$

$$g'_{\mu\nu} = g_{\rho\sigma} a^{-1\rho}{}_\mu \, a^{-1\sigma}{}_\nu$$

$$(a^{-1})^\rho{}_\mu = \frac{\partial x^\rho}{\partial x'^\mu}$$

$$\Rightarrow \quad g'_{\mu\nu} = g_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} = g_{\rho\sigma} \left(\delta^\rho_\mu + \frac{\partial \xi^\rho}{\partial x'^\mu}\right)\left(\delta^\sigma_\nu + \frac{\partial \xi^\sigma}{\partial x'^\nu}\right) \simeq$$

$$\parallel \qquad = g_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \eta_{\mu\nu} + h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$\eta_{\mu\nu} + h'_{\mu\nu}$

$$\Rightarrow \quad \boxed{h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu}$$

gauge transformation for the weak (LINEARIZED) gravitational field.

Let $\boxed{h = g^{\mu\nu} h_{\mu\nu} \simeq \eta^{\mu\nu} h_{\mu\nu}}$

and let us introduce

$$\boxed{\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}}$$

Then $\bar{h} = \eta^{\mu\nu}\left(h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}\right) = h - \frac{1}{2} h \cdot 4 = -h$

Thus

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{2} h \eta_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \bar{h} \eta_{\mu\nu}$$

$$\boxed{h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \bar{h} \eta_{\mu\nu}}$$

We want to calculate Einstein tensor for

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

So:

$$\Gamma^\mu{}_\nu = \left\{ {\mu \atop \nu\rho} \right\} dx^\rho = \frac{1}{2} g^{\mu\sigma} \left(g_{\sigma\rho,\nu} + g_{\nu\sigma,\rho} - g_{\rho\nu,\sigma} \right) dx^\rho$$

$$\simeq \frac{1}{2} \eta^{\mu\sigma} \left(\partial_\nu h_{\sigma\rho} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\rho\nu} \right) dx^\rho =$$

$$= \frac{1}{2} \left(\partial_\nu h^\mu{}_\rho + \partial_\rho h^\mu{}_\nu - \partial^\mu h_{\rho\nu} \right) dx^\rho$$

$\Rightarrow$ curvature:

$$\Omega^\mu{}_\nu = d\Gamma^\mu{}_\nu + \overbrace{\cancel{\Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu}}^{O(h^2)} = \frac{1}{2} \left(\partial_\sigma \partial_\nu h^\mu{}_\rho + \cancel{\partial_\sigma \partial_\rho h^\mu{}_\nu} - \partial_\sigma \partial^\mu h_{\rho\nu} \right) dx^\sigma \wedge dx^\rho$$

$$R^\mu{}_{\nu\sigma\rho} = e_\sigma \lrcorner\, e_\rho \lrcorner\, \Omega^\mu{}_\nu =$$
$$= \frac{1}{2} \left(\partial_\sigma \partial_\nu h^\mu{}_\rho - \partial_\rho \partial_\nu h^\mu{}_\sigma - \partial_\sigma \partial^\mu h_{\rho\nu} + \partial_\rho \partial^\mu h_{\sigma\nu} \right)$$

$\Rightarrow$ Ricci:

$$R_{\nu\rho} = \frac{1}{2}\left(\partial_\mu \partial_\nu h^\mu{}_\rho - \partial_\rho \partial_\nu h - \Box h_{\rho\nu} + \partial_\rho \partial^\mu h_{\mu\nu}\right) =$$

$$\Box := \partial_\mu \partial^\mu = \partial^\mu \partial_\mu$$

$$= \frac{1}{2}\left(\partial_\mu \partial_\nu \left(\bar{h}^\mu{}_\rho - \frac{1}{2}\bar{h}\,\delta^\mu{}_\rho\right) + \cancel{\partial_\rho \partial_\nu \bar{h}} - \Box\left(\bar{h}_{\rho\nu} - \frac{1}{2}\bar{h}\,\eta_{\rho\nu}\right) + \partial_\rho \partial^\mu\left(\bar{h}_{\mu\nu} - \cancel{\frac{1}{2}\bar{h}\,\eta_{\mu\nu}}\right)\right)$$

$$= \frac{1}{2}\partial_\nu \partial_\mu \bar{h}^\mu{}_\rho + \frac{1}{2}\partial_\rho \partial^\mu \bar{h}_{\mu\nu} - \frac{1}{2}\Box \bar{h}_{\rho\nu} + \frac{1}{4}\eta_{\rho\nu}\Box \bar{h}$$

$$\boxed{R_{\nu\rho} = \frac{1}{2}\partial_\nu \partial_\mu \bar{h}^\mu{}_\rho + \partial_\rho \partial_\mu \bar{h}^\mu{}_\nu - \frac{1}{2}\Box \bar{h}_{\rho\nu} + \frac{1}{4}\eta_{\rho\nu}\Box \bar{h}}$$

We have a freedom in choosing $h_{\mu\nu}$. We can modify it by

$$h \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu .$$

We use this gauge to kill terms like

$$\partial_\mu \bar{h}^\mu{}_\rho .$$

<u>DeDonder gauge:</u>

$$\boxed{\partial_\mu \bar{h}^\mu{}_\rho \equiv 0} \quad (DG)$$

Note that if $\bar{h}^\mu{}_\rho$ satisfies (DG) then we can still make

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Indeed, $\bar{h}_{\mu\nu} \mapsto \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial^\rho \xi_\rho$

So if $\partial_\mu \bar{h}^\mu{}_\rho = 0 \Rightarrow \quad \Box \xi_\nu + \cancel{\partial_\nu \partial^\mu \xi_\mu} - \cancel{\partial_\nu \partial^\rho \xi_\rho} = 0$

$$\left\{\begin{array}{l} \text{Thus if } \bar{h}^{\mu}{}_{\rho} \text{ satisfies } \partial_\nu \bar{h}^{\nu}{}_{\rho} = 0 \quad \text{then} \\ h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \text{ also satisfies } \partial_\mu \bar{h}'^{\mu}{}_{\rho} = 0 \\ \text{iff } \quad \Box \xi_\nu = 0 \end{array}\right.$$

Returning to our $R_{\nu\rho}$ we see that in De Donder gauge we have

$$R_{\nu\rho} = -\frac{1}{2}\Box\left(\bar{h}_{\rho\nu} - \frac{1}{2}\bar{h}\eta_{\rho\nu}\right) = -\frac{1}{2}\Box h_{\rho\nu}$$

Einstein tensor

$$G_{\nu\rho} = R_{\nu\rho} - \frac{1}{2}g_{\nu\rho}R \hat{=}$$

$$= R_{\nu\rho} - \frac{1}{2}\eta_{\nu\rho}\left(-\frac{1}{2}\Box\left(\underbrace{\bar{h} - 2\bar{h}}_{-\bar{h}}\right)\right) =$$

$$= -\frac{1}{2}\Box\left(\bar{h}_{\rho\nu} - \frac{1}{2}\bar{h}\eta_{\rho\nu} + \frac{1}{2}\bar{h}\eta_{\rho\nu}\right) = -\frac{1}{2}\Box\bar{h}_{\rho\nu}$$

So the linearized Einstein's equations would be:

$$\boxed{-\frac{1}{2}\Box\bar{h}_{\rho\nu} + \Lambda(\eta_{\rho\nu} + h_{\rho\nu}) = \varkappa T_{\mu\nu}}$$

8) Newtonian limit of Einstein's equations

First — linearize

Second — $\Lambda \equiv 0$ $\rightsquigarrow$ no way of getting Poisson's equations

Third — forget about time derivatives in the Dalambertzian:

$$\Box = \frac{1}{c^2}\cancel{\partial_t^2} - \partial_x^2 - \partial_y^2 - \partial_z^2 \;)$$

$$\boxed{\frac{1}{2}\Delta\bar{h}_{\rho\nu} = \varkappa T_{\mu\nu}} \qquad (*)$$

Fourth: $\begin{pmatrix} T_{00} = \rho c^2 \, , T_{0i} = 0 \\ T_{ij} = 0 \end{pmatrix}$

$$* \Rightarrow \left\{ \begin{array}{l} \boxed{\Delta\bar{h}_{00} = 2\varkappa\rho c^2} \\ \Delta\bar{h}_{\text{inne}} = 0 \quad \Rightarrow \quad \bar{h}_{\text{inne}} = 0 \end{array} \right\} \Rightarrow \bar{h} = \bar{h}_{00}$$

now:

$$h_{00} = \bar{h}_{00} - \frac{1}{2}\bar{h}\eta_{00} = \bar{h}_{00} - \frac{1}{2}\bar{h}_{00} = \frac{1}{2}\bar{h}_{00}$$

$$\Rightarrow \bar{h}_{00} = 2h_{00} = 2\left(g_{00} - \eta_{00}\right) = 2\left(1 + \frac{2\phi}{c^2} - 1\right) = \frac{4\phi}{c^2}$$

thus:

$$\Delta\left(\frac{4\phi}{c^2}\right) = 2\varkappa\rho c^2 .$$

$$\Delta\phi = 2\frac{\varkappa c^4}{4}\rho$$

This when compared with Poisson equation $\Delta\phi = 4\pi G\rho$ gives

$$\frac{\varkappa c^4}{2} = 4\pi G \Rightarrow \varkappa = \frac{8\pi G}{c^4}$$

Hence Einstein's equations:

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}}$$