Lecture 11, 10.05.2011

Einstein field equations:

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

Linearization:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\|h_{\mu\nu}\| \ll 1$$

Gauge freedom: $h_{\mu\nu} \mapsto h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ (*)

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \quad , \quad h = h_{\mu\nu} \eta^{\mu\nu}$$

De Donder gauge:

$$\partial_\mu \bar{h}^\mu{}_\rho \equiv 0 ,$$

still gauge freedom (*) but with ξ_μ restricted to

$$\Box \xi_\mu = 0 \quad , \quad \Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$$

Linearized equations in DeDonder gauge:

$$-\frac{1}{2} \Box \bar{h}_{\rho\nu} + \Lambda (\eta_{\rho\nu} + h_{\rho\nu}) = \frac{8\pi G}{c^4} T_{\rho\nu}$$

Without cosmological constant:

$$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}}$$

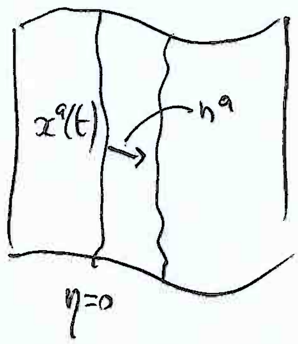
9) Tidal forces – geodesic deviation equation

a) Newton's theory

$x^a(t)$ – a trajectory of a particle in space

$x^a(t,\eta)$ – 1-parameter family of such trajectories.

s.t. $x^a(t,0) = x^a(t)$



$$n^a = \frac{\partial x^a}{\partial \eta}\Big|_{\eta=0} \quad \text{- separation vector}$$

If particles are in a free fall in gravitational field described by the Newtonian potential φ then

$\varphi = \varphi(x^a, t)$

$$\frac{d^2 x^a}{dt^2} + \frac{\partial \varphi}{\partial x^a} = 0 \quad \Big| \frac{\partial}{\partial \eta}\Big|_{\eta=0}$$

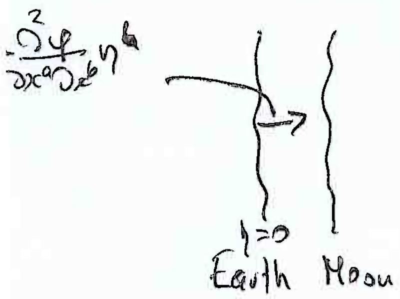
$$\frac{d^2 n^a}{dt^2} + \frac{\partial^2 \varphi}{\partial \eta \partial x^a}\Big|_{\eta=0} = 0$$

$$\frac{d^2 n^a}{dt^2} + \frac{\partial^2 \varphi}{\partial x^b \partial x^a} \cdot \frac{\partial x^b}{\partial \eta}\Big|_{\eta=0} = 0$$

$$\boxed{\frac{d^2 n^a}{dt^2} + \frac{\partial^2 \varphi}{\partial x^a \partial x^b} \cdot n^b = 0}$$

Linear equation for the separation vector n^a. Its changes measure 'tidal force' between close trajectories. This force is proportional to

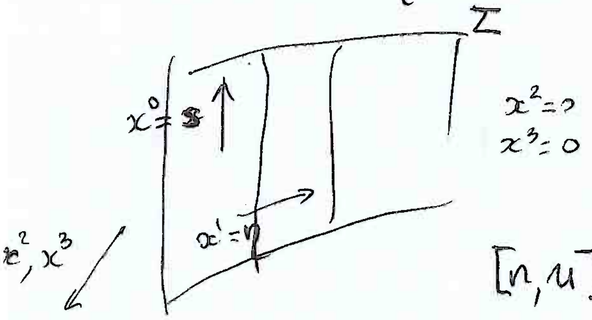
$$\frac{\partial^2 \varphi}{\partial x^a \partial x^b} \quad \text{– second derivatives of the potential.}$$



b) in GR

$x^\mu(s)$ – trajectory of a free particle in space-time

$x^\mu(s,\eta)$ – 1-par. family of such traj., $x^\mu(s,0) = x^\mu(s)$.



$$u = \frac{\partial}{\partial s}\Big|_\Sigma \qquad n = \frac{\partial}{\partial \eta}\Big|_\Sigma$$

s – affine parameter for each ~~…~~ η.

$$[n,u] = 0$$

Geodesic equation (in affine parametrization)

$$\nabla_u \cdot u = 0 \qquad \Big| \cdot \nabla_n$$

$$\nabla_n \nabla_u u = 0$$

Recall: $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$

i.e.

$$\cancel{\nabla_n \nabla_u u} - \nabla_u \nabla_n u - \cancel{\nabla_{[n,u]} u} = R(n,u)u$$

$$\nabla_u \nabla_n u + R(n,u)u = 0$$

(Geodesic deviation equation) PHYS

$\Updownarrow$

(Jacobi equation) MAT.

Recall:

$$Q(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$0 = \nabla_n u - \nabla_u n - 0$$

$$\Rightarrow \quad \nabla_u \cdot \nabla_u \cdot n + R(n,u)u = 0$$

$$\boxed{\nabla_u^2 n + R(n,u)u = 0}$$

$$\boxed{(\nabla_u^2 n)^\mu + R^\mu{}_{\nu\rho\sigma} n^\rho u^\sigma u^\nu = 0}$$

equation linear in n.

For 'tidal forces' responsable is Riemann!

10) How field equations determine the motion of matter?

$\Lambda \equiv 0$ from now on, unless stated otherwise.

$$G_{\mu\nu} = \varkappa T_{\mu\nu}$$

$$\Rightarrow \quad \nabla^{\mu} G_{\mu\nu} = 0 \quad \Rightarrow \quad \underline{\nabla^{\mu} T_{\mu\nu} = 0}$$

This last equation has consequences:

Examples

- Perfect fluid

$c=1$

$$T_{\mu\nu} = \begin{pmatrix} \varrho c^2 & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} \quad \text{in a frame where } u^{\mu} = (1,0,0,0)$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

signature: $+ - - -$

$$\Rightarrow \quad \boxed{T_{\mu\nu} = (\varrho + p) u_{\mu} u_{\nu} - p g_{\mu\nu}} \quad \boxed{u_{\mu} u^{\mu} = 1}$$

ϱ - density of matter, p - preasure

$$\nabla_{\mu} T^{\mu\nu} = 0 \Rightarrow$$

$$u^{\mu} \nabla_{\mu} \left[(\varrho + p) u^{\nu} \right] + (\varrho + p) u^{\nu} \nabla_{\mu} u^{\mu} - g^{\mu\nu} \nabla_{\nu} p = 0 \quad \Big| \, u_{\mu}$$

$$\nabla_{\nu} \left[(\varrho + p) u^{\nu} \right] - \underbrace{u^{\mu} \nabla_{\mu} p}_{\nabla_{u} p = \dot{p}} = 0$$

$$(\mathrm{I}) \quad \boxed{\dot{p} = \nabla_{\nu} \left[(\varrho + p) u^{\nu} \right]}$$

$$(\rho+p)u^\nu\nabla_\nu u^\mu = g^{\mu\nu}\nabla_\nu p - u^\mu u^\nu\nabla_\nu p$$

$$\boxed{(\rho+p)u^\nu\nabla_\nu u^\mu = \underbrace{(g^{\mu\nu} - u^\mu u^\nu)}\nabla_\nu p} \quad (\mathrm{II})$$

projector on the <u>space</u> $\perp$ to u^μ.

Relativistic Euler's equation

In case of DUST $\quad p \equiv 0$

$\Rightarrow$ (II) gives:

$$\rho\, u^\nu\nabla_\nu u^\mu = 0 \qquad \rho \neq 0$$

$\Rightarrow \boxed{u^\nu\nabla_\nu u^\mu = 0}$ particles of the fluid moves along geodesics

(I) gives

$\boxed{\nabla_\nu(\rho u^\nu) = 0}$ relativistic continuity equation.

Note that if there is a <u>spatial</u> gradient of the preasure then the particles of the fluid do NOT move along geodesics (obvious - they are not free - there is a force coming from preasure!)

Note that in electrodynamics $dF \equiv 0$, $d{*F} = 4\pi {*j}$

we have $d^2{*F} = 0 \Rightarrow d{*j} = 0 \Leftrightarrow \underline{\partial_\mu j^\mu = 0}$

So we have only equation (I).

Equation II $m\frac{du^\mu}{d\tau} = \frac{q}{c}F^{\mu\nu}u_\nu$ is independent of the field equations!

11) Continuity equation in GR

SR

$$\partial_\mu j^\mu = 0 \quad \xrightarrow[\text{coupling}]{\text{minimal}} \quad \nabla_\mu j^\mu = 0$$

associated to j^μ there is a pseudoscalar 3-form of the current

$$*j = j^\mu *\theta_\mu$$

$$*\theta_\mu = \frac{1}{3!}\eta_{\mu\nu\rho\sigma}\,\theta^\nu\wedge\theta^\rho\wedge\theta^\sigma \qquad \eta_{\mu\nu\rho\sigma} = \sqrt{|\det g|}\,\epsilon_{\mu\nu\rho\sigma}$$

$$D\eta_{\mu\nu\rho\sigma} = 0 \quad (\text{because } Dg \equiv 0)$$

$D\theta^\mu \equiv 0$

$$d*j \underset{\substack{\uparrow\\ \text{pseudoscalar}}}{=} D*j = D\left(j^\mu *\theta_\mu\right) = Dj^\mu \wedge *\theta_\mu =$$

$$= \nabla_\nu j^\mu\, \theta^\nu \wedge *\theta_\mu$$

$$\theta^\tau \wedge *\theta_\mu = \frac{1}{3!}\eta_{\mu\nu\rho\sigma}\,\theta^\tau\wedge\theta^\nu\wedge\theta^\rho\wedge\theta^\sigma =$$

$$= \frac{1}{3!}\eta_{\mu\nu\rho\sigma}\,\epsilon^{\tau\nu\rho\sigma}\,\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3$$

$$\epsilon_{\mu\nu\rho\sigma}\,\epsilon^{\tau\nu\rho\sigma} = 3!\,\delta^\tau_\mu$$

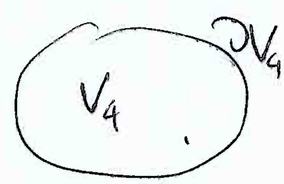
$$= \delta^\tau_\mu \underbrace{\sqrt{|\det g|}\,\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3}_{\mathrm{vol}(g) = *1}$$

$$\Rightarrow \quad d*j = \nabla_\mu j^\mu \cdot \mathrm{vol}(g)$$

$$\underbrace{d*j = 0}_{\substack{\uparrow\\ \text{continuity equation}}} \Leftrightarrow \nabla_\mu j^\mu = 0.$$

$$d*j = 0$$

$$0 = \int_{V_4} d*j = \int_{\partial V_4} *j$$



$$\int_{\partial V_4} *j = 0.$$

12) Gravitational waves in the weak field aproximation.

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

Vacuum: $T_{\mu\nu} = 0$

$$\boxed{\Box \bar{h}_{\mu\nu} = 0}$$

Solutions : monochromatic plane wave:

$$\bar{h}_{\mu\nu} = \mathrm{Re}\left(a_{\mu\nu} e^{-ik_\delta x^\delta}\right)$$

$$\Box \bar{h}_{\mu\nu} = 0 \iff \boxed{k_\delta k^\delta = 0}$$

k s.t. $g(k,k)=0$ null vector field - propagation vector of the wave.

De Donder condition

$$\partial_\mu \bar{h}^{\mu\nu} = 0 \iff \boxed{a_{\mu\nu} k^\nu = 0}$$

What is the freedom in choosing $a_{\mu\nu}$?

$a_{\mu\nu} = a_{\nu\mu} \Rightarrow$ 10 components

$a_{\mu\nu} k^\nu = 0 \Rightarrow -4$ conditions

Gauge freedom: $\Box \xi_\nu = 0 \iff \xi_\nu = \mathrm{Re}\left(i C_\nu e^{ik_\alpha x^\alpha}\right)$

$$a_{\mu\nu} \mapsto a_{\mu\nu} + k_\mu C_\nu + k_\nu C_\mu - \eta_{\mu\nu} k_\delta C^\delta$$

$\Rightarrow -4$ components that could be killed by this gauge;

$$10 - 4 - 4 = 2$$

$\Rightarrow$ $a_{\mu\nu}$ has TWO relevant components.

How to choose them?

Choose a 4-velocity of an observer: u^μ

$$\begin{cases} a_{\mu\nu}k^\nu = 0 & \text{4-eqs} \\ u^\mu a_{\mu\nu} = 0 & \text{3-eqs} \quad \text{since } u^\mu u_\mu = 1 \\ a^\mu{}_\mu = 0 & \text{1-eq} \end{cases}$$

TT-gauge — transverse-traceless gauge

$$u^\mu = (1,0,0,0)$$
$$k^\mu = (\omega, 0, 0, \omega) \quad \Rightarrow \quad \begin{array}{l} a_{\mu 0} = 0 \\ a_{\mu 3} = 0 \end{array}$$

$$a_{\mu\nu} = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left(\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & a_+ & a_\times & 0 \\ 0 & a_\times & -a_+ & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}$$

$\{e_\mu\}$ - frame of vector fields

$$e_+ = e_1 \otimes e_1 - e_2 \otimes e_2$$
$$e_\times = e_1 \otimes e_2 + e_2 \otimes e_1$$

TWO independent solutions:

$$\bar{h}_{+\mu\nu} = \mathrm{Re}\left[a_+ e_{+\mu\nu} e^{-i\omega(t-z)}\right]$$
$$\bar{h}_{\times\mu\nu} = \mathrm{Re}\left[a_\times e_{\times\mu\nu} e^{-i\omega(t-z)}\right]$$

two different polarizations of a gravitational wave

13) <u>Evolution of a ball of dust in the field of monochromatic gravitational wave.</u>

Let h^{TT} be a solution for $h^{\mu\nu}$ in TT-gauge

Insert it in the linearized formula for the Riemann tensor for $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ obtained in the previous lecture:

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\nu \partial_{[\sigma} h^{\mu}{}_{\rho]} - \partial^{\mu} \partial_{[\sigma} h_{\rho]\nu}$$

$u = (1,0,0,0)$, $R^{\mu}{}_{0\nu0} = -\frac{1}{2} \ddot{h}^{TT\mu}{}_{\nu} = -\frac{1}{2} h^{TT\mu}{}_{\nu,00}$

Jacobi equation for the vector n^{μ} connecting two particles in the dust through which the gravitational wave passes:

$$(\nabla^2_u n)^{\mu} + R^{\mu}{}_{\nu\rho\sigma} u^{\nu} n^{\rho} u^{\sigma} = 0$$

$$\ddot{n}^{\mu} + R^{\mu}{}_{0\rho0} n^{\rho} = 0$$

$$\boxed{\ddot{n}^{\mu} - \frac{1}{2} \ddot{h}^{TT\mu}{}_{\nu} n^{\nu} = 0}$$

$h^{TT0}{}_{\mu} = h^{TT3}{}_{\mu} = 0 \implies$

$$\left.\begin{array}{l} \ddot{n}^0 = 0 \\ \ddot{n}^3 = 0 \end{array}\right\} \text{no longitudinal relative acceleration}$$

relative acceleration only in x-y direction

Solving by the method of succesive aproximations:

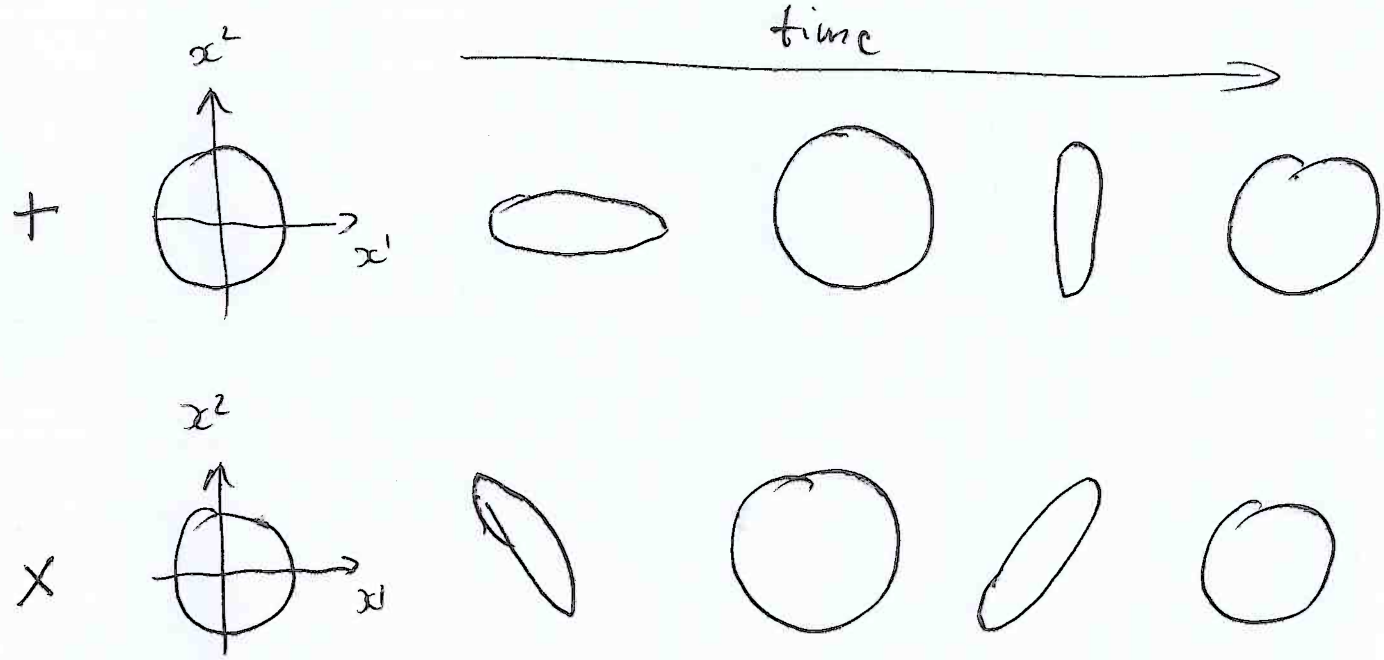
Start : $n^\mu{}_{(0)} = \text{const}$

I^{st} step : insert $n^\mu{}_{(0)}$ in the second term in the equation:

$$\ddot{n}^\mu_{(1)} = \frac{1}{2}\, \ddot{h}^{TT\mu}{}_\nu\, n^\nu_{(0)}$$

$$\Rightarrow \quad n^\mu{}_{(1)} = \frac{1}{2}\, h^{TT\mu}{}_\nu\, n^\nu_{(0)} + (\text{const})^\mu$$

time dependence is here



Radiation in full theory is a quite difficult and subtle subject.

Read section XIV of Space-time and gravitation of Kopczyński + Trautman to have an idea!

(7) Symmetries

1) Transformation groups.

M - manifold

G - Lie group

Action: $M \times G \ni (p, a) \xrightarrow{\text{smooth}} pa = R_a(p) \in M$

$$(pa)b = p(a \cdot b) \,, \qquad p \cdot e = p$$

G acts effectively on M if $\forall\, e \neq a \in G \;\exists p \in M \; pa \neq p$.
(every element $\neq e$ moves at least one point)

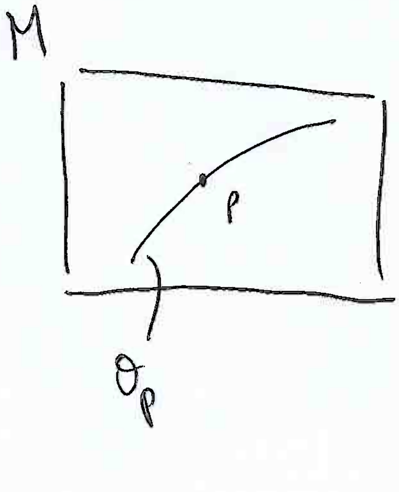
Orbit: $\mathcal{O}_p = \{ p' \in M : \; p' = p \cdot a \quad a \in G \}$

If $\mathcal{O}_p \equiv M \implies G$ acts transitively on M.

we say that M is homogeneous w.r.t. the action of G.

Isotropy group of p

$$H_p = \{ b \in G : \; p \cdot b = p \}$$



$$\mathcal{O}_p \simeq G / H_p$$

$$\boxed{\dim \mathcal{O}_p = \dim G - \dim H_p}$$

Let $a(t)$ be a curve in G passing through e, s.t.

$a(0)=e$. Given a point p in M

this defines a curve $p(t)$ in M by

$$p(t) = p \cdot a(t)$$

In such a case ~~we have~~ at each point p in M

we have

$$X^*_p = \dot{p}(0)$$

which in turn defines a vector field X^* on M

associated with the curve $a(t)$.

Thus we have a map:

$$\mathfrak{g} \ni \dot{a}(0) = X \longmapsto X^* \in \mathfrak{X}(M)$$

it is

a) linear

b) $[X,Y]^* = [X^*,Y^*]$ isomorphism of Lie algebras.

2) <u>G-invariant tensors</u>

K - a tensor field on M is G invariant

iff $\forall a \in G$

$$\tilde{R}_a K = K$$

$$R_a : M \longrightarrow M$$

$$p \longrightarrow R_a(p) = pa$$

In particular G connected

$$\tilde{R}_{a(t)} K = K \iff \underset{X^*}{\mathcal{L}} K = 0 \quad \forall X \in \mathfrak{g}.$$

4) Maximal number of symmetries.

$$R^{\mu}{}_{\nu\rho\sigma} \equiv 0 \quad \Longleftrightarrow \quad T^{\mu}{}_{\nu\rho} \equiv 0$$

Killing equation $\nabla_{(\mu} X_{\nu)} = 0$ becomes

$$(*) \qquad \partial_{(\mu} X_{\nu)} = 0 \qquad \Big| \, \partial_\rho$$

$$\Downarrow$$

$$\partial_\rho \partial_{(\mu} X_{\nu)} = 0$$

we also, obviously, have:

$$\partial_{[\rho} \partial_{\mu]} X_\nu = 0$$

$$\Longrightarrow \quad \partial_\rho \partial_\mu X_\nu = 0$$

$$\Longrightarrow \quad \partial_\mu X_\nu = a_{\mu\nu} = \text{const.} \qquad (**)$$

$$(*) \Longrightarrow \quad a_{\mu\nu} + a_{\nu\mu} = 0$$

This and $(**)$ $\Longrightarrow$

$$\boxed{\begin{gathered} X_\mu = a_{\mu\nu} x^\nu + a_\mu \\ a_\mu = \text{const} \\ a_{\mu\nu} = -a_{\nu\mu} = \text{const} \end{gathered}}$$

$\dim M = n$

Dimension of the symmetry algebra:

$$\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}.$$

$$X_\mu = \underbrace{a_{\mu\nu} x^\nu}_{\text{generators of Lorentz transf.}} + \underbrace{a_\mu}_{\text{generators of translations}}$$

3) Killing fields and Killing equation

(M, g) - Riemannian manifold

$\varphi: M \xrightarrow{\text{diff}} M$ is an isometry

$\Longleftrightarrow \varphi^* g = g \Longleftrightarrow g(\varphi_* X, \varphi_* Y) = g(X, Y)$

φ_t - 1-parameter group of local isometries

$$\varphi_t^* g = g \Longleftrightarrow \mathcal{L}_X g = 0$$

X - a vector field associated with φ_t

Since $[\mathcal{L}_{X_1}, \mathcal{L}_{X_2}] = \mathcal{L}_{[X_1, X_2]}$ then

the vector fields X form a Lie algebra.

They are called <u>infinitesimal symmetries</u> of g or <u>Killing fields.</u>

The algebra of Killing fields is a Lie algebra of symmetries of g.

$$(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + \partial_\mu X^\rho g_{\rho\nu} + \partial_\nu X^\rho g_{\mu\rho} =$$

$$= \nabla_\mu X^\rho g_{\rho\nu} + \nabla_\nu X^\rho g_{\mu\rho} = \nabla_\mu X_\nu + \nabla_\nu X_\mu$$

in an orthonormal frame in which $\Gamma \equiv 0$

Killing equation:

$$\boxed{\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0}$$

Riemann tensor:

$$R^{\mu\nu}{}_{\rho\sigma} = C^{\mu\nu}{}_{\rho\sigma} + \frac{4}{n-2} \check{R}^{[\nu}{}_{[\sigma} \delta^{\mu]}{}_{\rho]} + \frac{2}{n(n-1)} R\, \delta^{[\nu}{}_{[\sigma} \delta^{\mu]}{}_{\rho]}$$

The symmetry group preserves Riemann, and its irreducible components.

For the group to be maximal it should not be restricted by preservations of C and $\check{R}$.

$\Rightarrow$ necessary condition for the symmetry group to be maximal is

$$C^{\mu\nu}{}_{\rho\sigma} \equiv 0 \quad \text{and} \quad \check{R}^{\mu}{}_{\nu} \equiv 0.$$

$$\Rightarrow \quad R^{\mu\nu}{}_{\rho\sigma} = \frac{2}{n(n-1)} R\, \delta^{[\nu}{}_{[\sigma} \delta^{\mu]}{}_{\rho]}$$

$$\Rightarrow \quad \Omega^{\mu}{}_{\nu} = \frac{1}{2} R^{\mu}{}_{\nu\rho\sigma} \vartheta^{\rho} \wedge \vartheta^{\sigma} = \frac{R}{n(n-1)} \vartheta^{\mu} \wedge \vartheta^{\nu}$$

Bianchi identity:

$$0 = D\Omega^{\mu}{}_{\nu} = \frac{DR}{n(n-1)} \wedge \vartheta^{\mu} \wedge \vartheta^{\nu} = \frac{dR}{n(n-1)} \wedge \vartheta^{\mu} \wedge \vartheta^{\nu}$$

$n = 2 \Rightarrow$ no conditions

$n \geq 3 \Rightarrow R = \text{const}$

Spaces of constant curvature

Thm

Every (M,g) having group of isometries of maximal dimension can be locally represented by a coordinate system (x^μ) in which the metric g reads:

$$g = \frac{\eta_{\mu\nu}\, dx^\mu dx^\nu}{\left(1 + \frac{K}{4}\eta_{\mu\nu} x^\mu x^\nu\right)^2}$$

$$\eta_{\mu\nu} = \mathrm{diag}(\underbrace{1, \dots 1}_{p}, \underbrace{-1, \dots -1}_{q})$$

Exercise

The curvature of this metric

$$\Omega^{\mu\nu} = K\, \theta^\mu \wedge \theta^\nu \qquad \left(R = n(n-1)K\right)$$

where

$$\theta^\mu = \frac{dx^\mu}{1 + \frac{K}{4}\eta_{\alpha\beta} x^\alpha x^\beta}$$

Every (M,g) as in the theorem can be obtained as

$M = \{ y \in \mathbb{R}^{n+1}$ s.t. $k y^{0\,2} + g_{\mu\nu} y^\mu y^\nu = k r^2 \}$

$K = \frac{1}{k r^2}$ and the metric

$$g = k\, dy^{0\,2} + g_{\mu\nu}\, dy^\mu dy^\nu \Big|_M$$

5) (Anti) De Sitter space

If $p = 1$, $q = 3$

$$\Rightarrow \quad g = \frac{dt^2 - dx^2 - dy^2 - dz^2}{\left(1 + \frac{K}{4}(t^2 - x^2 - y^2 - z^2)\right)^2}$$

$$0 = \mathring{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \implies R_{\mu\nu} = \frac{1}{4} g_{\mu\nu} R = 3K g_{\mu\nu}$$

$$\implies R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{1}{4} g_{\mu\nu} R = 0 \implies G_{\mu\nu} + 3K g_{\mu\nu} = 0$$

This is a solution of Einstein's vacuum equation with cosmological constant

$$\Lambda = 3K.$$

If $K > 0$ it is called De Sitter space

$K < 0$ Anti De Sitter space

Equivalent form of the (Anti) de Sitter metric (after suitable change of coordinates):

$$g = (1 - Kr^2)dt^2 - \frac{dr}{1 - Kr^2} - r^2(d\theta^2 + \sin^2\vartheta \, d\varphi^2)$$