Lecture 12, 17.05.2011

6) Stationary and static gravitational fields

Def

(M,g) is stationary iff there exists a vector field X on M s.t.

1) $\mathcal{L}_X g = 0$ (i.e. X is Killing)

2) $g(X,X) > 0$ (i.e. X is timelike)

LOCALLY: Given any ~~timelike~~ vector field on M we can introduce a local chart s.t. $X = \frac{\partial}{\partial x^0}$ and $x^\mu = (x^0; x^i)$

Then

1) $(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + \cancel{\text{terms in which } X^\mu \text{ is differentiated}}$ $= 0$ since $X^\mu = (1,0,0,0)$

$\Rightarrow (\mathcal{L}_X g)_{\mu\nu} = 0 \Leftrightarrow X^\rho \partial_\rho g_{\mu\nu} = \partial_0 g_{\mu\nu} = 0$

2) $g(X,X) = g_{00} > 0$

$$\Rightarrow \boxed{X \text{ stationary}} \overset{\text{locally}}{\Leftrightarrow} \boxed{\exists\, (x^0, x^i) \text{ s.t. } X = \frac{\partial}{\partial x^0},\ \frac{\partial g_{\mu\nu}}{\partial x^0} = 0,\ g_{00} > 0}$$

Def

(M,g) is <u>static</u> iff

1) (M,g) is stationary
2) the distribution $X^\perp = \{Y : g(X,Y)=0\}$ is integrable.

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LOCALLY:

if (M,g) is static with X being timelike Killing we can introduce a coordinate system (x^0, x^i) s.t. $X = \frac{\partial}{\partial x^0}$ and $x^0 = \text{const}$ enumerates leaves of the foliation defined by $X^\perp$.

$$\Rightarrow \quad g_{0k} = g\Big(\underset{X}{\underset{\parallel}{\frac{\partial}{\partial x^0}}}, \underset{X^\perp}{\underset{\cap}{\frac{\partial}{\partial x^k}}}\Big) = 0.$$

The other way around:

if $g_{0k} = 0 \Rightarrow \underset{g(X)}{} = g(X,\cdot) = g_{\mu\nu} X^\nu dx^\mu = g_{\mu 0}\, dx^\mu = g_{00}\, dx^0$

$$\Rightarrow \quad dg(X) \wedge g(X) = 0 \underset{\text{Fröbenius}}{\Rightarrow} \text{the anihilator of } g(X)$$

i.e. Y s.t. $Y \lrcorner\, g(X) = 0$ is integrable

but $0 = Y \lrcorner\, g(X) = g(X,Y) \Rightarrow Y \in X^\perp$.

$\Rightarrow$

Spacetime (M,g) is static iff in the neighbourhood of any point there exist a coordinate system (x^0, x^i) s.t.

$$g = g_{00}\, dt^2 + g_{ij}\, dx^i dx^j$$

$\frac{\partial g_{00}}{\partial t} = 0$, $\frac{\partial g_{ij}}{\partial t} = 0$, $g_{00} > 0$, g_{ij} has signature - - -

7) spherical symmetry

Def

(M,g) is spherically symmetric iff the isometry group is $SO(3)$ with 2-dimensional orbits which are space-like and have topology of S^2.

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$SO(3)$ is a symmetry group $\Rightarrow$
X_1, X_2, X_3 are Killing vectors satisfying commutation relations of the Lie algebra of $\underline{so}(3)$.

We can choose X_1, X_2, X_3 to satisfy:

$$[X_1, X_2] = X_3 \quad + \text{ cyclic. permutations}$$

__Locally__: we introduce coordinates

$$(\underbrace{x^0, x^1}_{\text{label orbits}}, \underbrace{x^2, x^3}_{\text{coordinates on a given orbit}})$$

__recall__:

(M,g), $\dim M = n$ $\Rightarrow$ maximal symmetry group has dimension $\frac{n(n+1)}{2}$

$\Rightarrow$ spaces of constant curvature with metric $h = \dfrac{h_{\mu\nu}\, dy^\mu dy^\nu}{\left(1 + \frac{K}{4} h_{\mu\nu} y^\mu y^\nu\right)^2}$

$n = 2 \Rightarrow \dim G = 3 \Rightarrow$
sign. $(++)$

$$\begin{cases} h = d\theta^2 + \sin^2\theta\, d\varphi^2 & SO(3) \\ h = dx^2 + dy^2 & SO(2) \ltimes \mathbb{R}^2 \\ h = d\theta^2 + \mathrm{sh}^2\theta\, d\varphi^2 & SO(1,2) \end{cases}$$

__Ex.__
Find transf from the system (y^μ) to the system as in here

We have $G = SO(3)$ of dimension 3 acting on orbits S^2 of dimension 2 and preserving the induced metric $g|_{S^2}$ there.

$\Rightarrow$ $g|_{S^2}$ must be the space of constant curvature $\left(\frac{2\cdot(2+1)}{2} = 3\right)$,

$\Rightarrow$ $g|_{S^2} = \text{const}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$

$\Rightarrow$ • coordinate system

$$(x^0, x^1, \theta, \varphi)$$

• the Killing vectors

$$X_3 = \frac{\partial}{\partial\varphi}$$

$$X_1 = -\sin\varphi \frac{\partial}{\partial\theta} - \cos\varphi \,\mathrm{ctg}\,\theta \frac{\partial}{\partial\varphi}$$

$$X_2 = \cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \,\mathrm{ctg}\,\theta \frac{\partial}{\partial\varphi}$$

(check that $[X_1, X_2] = X_3$ + cycl)

• Fact the most general solution of

$$\mathcal{L}_{X_i} g = 0 \quad \text{with} \quad g|_{S^2} = g|_{\substack{x^0=\text{const}\\ x^1=\text{const}}} = \text{Const}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

is:

$$\boxed{g = g_{AB}(x^C)\, dx^A dx^B - r^2(x^C)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \qquad x^A = (x^0, x^1) \qquad A, B, C = 0, 1.}$$

<u>three cases</u>:

1) $X_\mu = \partial_\mu r$ is spacelike: $(\partial_\mu r)\cdot(\partial^\mu r) < 0$

2) $X_\mu = \partial_\mu r$ is timelike : $(\partial_\mu r)(\partial^\mu r) > 0$

3) $X_\mu = \partial_\mu r$ is ~~null~~ such that: $(\partial_\mu r)(\partial^\mu r) = 0$ — $\partial_\mu r \neq 0$ NULL; $\partial_\mu r \equiv 0$.

Ad 1) $(\partial_\mu r)(\partial^\mu r) < 0$:

since $\partial_\mu r \neq 0$ and is independent of θ and φ we cane take r as one of coordinates x^0, x^1. It is spacelike $\Rightarrow$ take $x^1 = r$.

Then

$$g = g_{00}\, dx^{0\,2} + 2\, g_{01}\, dx^0\, dr + g_{11}\, dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$\begin{pmatrix} x^1 = r \\ x^0 \end{pmatrix} \xrightarrow[\text{new coordinates}]{} \begin{pmatrix} r \\ t \end{pmatrix}$$

i.e.
$$x^1 = r$$
$$x^0 = x^0(r, t)$$

$$\Rightarrow g = g_{00} (x^0_{,r}\, dr + x^0_{,t}\, dt)^2 + 2\, g_{01} (x^0_{,r}\, dr + x^0_{,t}\, dt)\, dr + g_{11}\, dr^2 + \ldots$$

$$= x^{0\,2}_{,t}\, g_{00}\, dt^2 + 2 \underbrace{(g_{00}\, x^0_{,r} x^0_{,t} + g_{01}\, x^0_{,t})}_{=0} dr\,dt + (g_{00}\, x^{0\,2}_{,r} + 2 g_{01} x^0_{,r} + g_{11})\, dr^2 + \ldots$$

$$\Rightarrow \underbrace{x^0_{,r} = -\frac{g_{01}}{g_{00}}}_{\text{Can solve!}}$$

$g_{00} \neq 0$ because otherwise the metric is degenerate!

$$\Rightarrow g = (\;)\, dt^2 + (\;)\, dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$(\;) \to g_{00}\, x^{0\,2}_{,t}$

$$0 > g^{\mu\nu} r_{,\mu} r_{,\nu} = g^{11} = \frac{g_{00}}{\det(g_{AB})}$$

$$\left(\begin{array}{cc|c} g_{00} & g_{01} & 0 \\ g_{10} & g_{11} & \\ \hline 0 & & \ddots \end{array}\right)^{-1} = \left(\begin{array}{cc|c} & & 0 \\ \hline 0 & & \ddots \end{array}\right) \quad \frac{g_{00}}{\det(g_{AB})}$$

g_{AB} has sign $+-$ $\Rightarrow \det(g_{AB}) < 0$

$\Rightarrow g_{00} > 0$

Thus finally:

$$\boxed{g = e^{2\mu(r,t)}dt^2 - e^{2\nu(r,t)}dr^2 - r^2(d\theta^2 + \sin^2\vartheta\, d\varphi^2)} \qquad (*)$$

$\mu = \mu(r,t)$
$\nu = \nu(r,t)$ arbitrary functions.

one of 1), 2), 3) possibilities of SPHERICALLY SYMMETRI Metrics in 4-dimensional LORENTZIAN geometry.

<u>Exercise</u>
Analyze the cases 2), 3) in the same way!

8) <u>Schwarzschild metric</u>

<u>short story</u>: imposing $Ric(g) \equiv 0$ on $(*)$
one gets:

$$\boxed{g = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2(d\theta^2 + \sin^2\vartheta\, d\varphi^2)} \qquad (S)$$

K. Schwarzschild 1916 (he assumed in addition that $\mu = \mu(r)$, $\nu = \nu(r)$ i.e. <u>no</u> time dependence in these functions)

Birkhoff 1923 has shown that his assumption is <u>not</u> needed to get to (S)

9) INTEGRATION of the spherically symmetric $Ric(g)=0$ condition

$$g = e^{2\mu(r,t)}dt^2 - e^{2\nu(r,t)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

orthonormal frame:

$$\theta^0 = e^{\mu}dt \qquad\qquad g = \theta^{0\,2} - \theta^{1\,2} - \theta^{2\,2} - \theta^{3\,2} =$$

$$\theta^1 = e^{\nu}dr \qquad\qquad = g_{\mu\nu}\theta^{\mu}\theta^{\nu}$$

$$\theta^2 = r\,d\theta \qquad\qquad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\theta^3 = r\sin\theta\, d\varphi$$

$$\begin{cases} d\theta^0 = e^{\mu}\mu' \, dr\wedge dt = -e^{-\nu}\mu'\,\theta^0\wedge\theta^1 & = -\Gamma^0{}_{\mu}\wedge\theta^{\mu} \\ d\theta^1 = e^{\nu}\dot{\nu}\, dt\wedge dr = e^{-\mu}\dot{\nu}\,\theta^0\wedge\theta^1 & = -\Gamma^1{}_{\mu}\wedge\theta^{\mu} \\ d\theta^2 = dr\wedge d\theta = \dfrac{e^{-\nu}}{r}\theta^1\wedge\theta^2 & = -\Gamma^2{}_{\mu}\wedge\theta^{\mu} \\ d\theta^3 = \sin\theta\, dr\wedge d\varphi + r\cos\theta\, d\theta\wedge d\varphi = \dfrac{e^{-\nu}}{r}\theta^1\wedge\theta^3 + \dfrac{\operatorname{ctg}\theta}{r}\theta^2\wedge\theta^3 & = -\Gamma^3{}_{\mu}\wedge\theta^{\mu} \end{cases}$$

Connection

$$\begin{cases} \Gamma^0{}_1 = e^{-\nu}\mu'\,\theta^0 + e^{-\mu}\dot{\nu}\,\theta^1 \\ \Gamma^1{}_0 = -\Gamma_{10} = \Gamma_{01} = \Gamma^0{}_1 = e^{-\nu}\mu'\theta^0 + e^{-\mu}\dot{\nu}\,\theta^1 \\ \Gamma^2{}_1 = \dfrac{e^{-\nu}}{r}\theta^2 \\ \Gamma^3{}_1 = \dfrac{e^{-\nu}}{r}\theta^3 \\ \Gamma^3{}_2 = \dfrac{\operatorname{ctg}\theta}{r}\theta^3 \end{cases}$$

all other zero
modulo symmetry

$$\Gamma_{\mu\nu} = -\Gamma_{\nu\mu}$$

Curvature $\quad \Omega^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu$

$$\Omega^0{}_1 = d\Gamma^0{}_1 + \cancel{\Gamma^0{}_1 \wedge \Gamma^1{}_1} =$$

$$= \underline{(e^{-\nu}\mu')' e^{-\nu}}\, \theta^1 \wedge \theta^0 + \underline{(e^{-\mu}\dot\nu)^{\cdot} e^{-\mu}}\, \theta^0 \wedge \theta^1 +$$

$$+ e^{-\nu}\mu' (-e^{-\nu}\mu')\theta^0 \wedge \theta^1 + e^{-\mu}\dot\nu (e^{-\mu}\dot\nu\, \theta^0 \wedge \theta^1)$$

$$= \left[-(e^{-\nu}\mu')' e^{-\nu} + (e^{-\mu}\dot\nu)^{\cdot} e^{-\mu} - e^{-2\nu}\mu'^2 + e^{-2\mu}\dot\nu^2\right]\theta^0 \wedge \theta^1$$

$$\boxed{R_{0101} = -(e^{-\nu}\mu')' e^{-\nu} - e^{-2\nu}\mu'^2 + (e^{-\mu}\dot\nu)^{\cdot} e^{-\mu} + e^{-2\mu}\dot\nu^2}$$

$$\Omega^2{}_1 = d\Gamma^2{}_1 + \cancel{\Gamma^2{}_3 \wedge \Gamma^3{}_1} = \left(\frac{e^{-\nu}}{r}\right)' e^{-\nu}\theta^1 \wedge \theta^2 - \frac{e^{-\nu}}{r}\dot\nu e^{-\mu}\, \theta^0 \wedge \theta^2 +$$

$$+ \frac{e^{-\nu}}{r}\frac{e^{-\nu}}{r}\theta^1 \wedge \theta^2 =$$

$$= -\frac{e^{-\nu-\mu}}{r}\dot\nu\, \theta^0 \wedge \theta^2 + \left[\left(\frac{e^{-\nu}}{r}\right)' + \frac{e^{-\nu}}{r^2}\right] e^{-\nu}\, \theta^1 \wedge \theta^2$$

$$\boxed{R_{2102} = \frac{e^{-\nu-\mu}}{r}\dot\nu}$$

$$\boxed{R_{1212} = e^{-\nu}\left[\left(\frac{e^{-\nu}}{r}\right)' + \frac{e^{-\nu}}{r^2}\right] = -\frac{e^{-2\nu}}{r}\nu'}$$

$$\Omega^3{}_2 = d\Gamma^3{}_2 + \Gamma^3{}_1 \wedge \Gamma^1{}_2 =$$

$$= -\frac{1}{r^2 \sin^2\vartheta}\theta^2 \wedge \theta^3 - \cancel{\frac{\operatorname{ctg}\vartheta}{r^2} e^{-\nu}\theta^1 \wedge \theta^3} + \frac{\operatorname{ctg}\vartheta}{r}\left(\cancel{\frac{e^{-\nu}}{r}\theta^1 \wedge \theta^3} + \frac{\operatorname{ctg}\vartheta}{r}\theta^2 \wedge \theta^3\right) +$$

$$+ \frac{e^{-\nu}}{r}\theta^3\ \frac{e^{-\nu}}{r}\theta^2 = \left(-\frac{1}{r^2} + \frac{e^{-2\nu}}{r^2}\right)\theta^2 \wedge \theta^3$$

$$\boxed{R_{2323} = \frac{e^{-2\nu} - 1}{r^2}}$$

$$\Omega^3{}_1 = d\Gamma^3{}_1 + \Gamma^3{}_2 \wedge \Gamma^2{}_1 = \ldots$$

$$R_{1313} = \left(\left(\frac{e^{-\nu}}{r}\right)' + \frac{e^{-\nu}}{r^2}\right)e^{-\nu} = -\frac{e^{-2\nu}}{r}\nu'$$

$$\boxed{R_{1303} = -\frac{e^{-\nu-\mu}}{r}\dot\nu}$$

$$\Omega^0{}_2 = d\cancel{\Gamma^0}{}_2 + \Gamma^0{}_1 \wedge \Gamma^1{}_2 = -\left(e^{-\nu}\mu'\theta^0 + e^{-\mu}\dot{\nu}\theta^1\right)\wedge \frac{e^{-\nu}}{r}\theta^2$$

$$= -\frac{e^{-2\nu}\mu'}{r}\theta^0\wedge\theta^2 - \frac{e^{-\mu-\nu}\dot{\nu}}{r}\theta^1\wedge\theta^2$$

$$\boxed{R_{0202} = -\frac{e^{-2\nu}}{r}\mu'} \qquad \boxed{R_{0212} = -\frac{e^{-\mu-\nu}}{r}\dot{\nu}}$$

$$\Omega^0{}_3 = \Gamma^0{}_1 \wedge \Gamma^1{}_3 = -\left(e^{-\nu}\mu'\theta^0 + e^{-\mu}\dot{\nu}\theta^1\right)\wedge \frac{e^{-\nu}}{r}\theta^3$$

$$\boxed{R_{0303} = -\frac{e^{-2\nu}}{r}\mu'} \qquad \boxed{R_{0313} = -\frac{e^{-\mu-\nu}}{r}\dot{\nu}}$$

Ricci

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = R_{0\mu0\nu} - R_{1\mu1\nu} - R_{2\mu2\nu} - R_{3\mu3\nu}$$

$\cancel{R_{00} = -R_{0101} - R_{0202}}$

$$\boxed{e^{-\nu}\left((e^{-\nu}\mu')' + e^{-2\nu}\mu'^2 - (e^{-\mu}\dot{\nu})^{\cdot} e^{-\mu} - e^{-2\mu}\dot{\nu}^2 + 2\frac{e^{-2\nu}}{r}\mu' = 0\right.}$$

$$R_{00} = -R_{1010} - R_{2020} - R_{3030} = \ldots = 0$$

$$R_{01} = -R_{2021} - R_{3031} = 2\frac{e^{-\nu-\mu}}{r}\dot{\nu} \Rightarrow \boxed{\dot{\nu} = 0}$$

$$R_{02} = \ldots = 0$$

$$R_{03} = \ldots = 0$$

$$\boxed{e^{-\nu}\left((e^{-\nu}\mu')' + e^{-2\nu}\mu'^2 + 2\frac{e^{-2\nu}}{r}\mu' = 0\right.}$$

$$R_{11} = R_{0101} - R_{2121} - R_{3131} =$$

$$= -(e^{-\nu}\mu')'e^{-\nu} - e^{-2\nu}\mu'^2 + 2\frac{e^{-2\nu}}{r}\nu' = 0$$

$$\Rightarrow \boxed{\nu' + \mu' = 0}$$

$$R_{12} = \ldots = 0$$

$$R_{13} = \ldots = 0$$

$$R_{22} = R_{0202} - R_{1212} - R_{3232} = -\frac{e^{-2\nu}}{r}\mu' + \frac{e^{-2\nu}}{r}\nu' + \frac{1-e^{-2\nu}}{r^2} = 0$$

$$\boxed{\frac{e^{-2\nu}}{r}\left(-2\mu' - \frac{1}{r}\right) + \frac{1}{r^2} = 0}$$

$R_{23} = \dots = 0$

$R_{33} = R_{0303} - R_{1313} - R_{2323} =$

$= -\frac{e^{-2\nu}}{r}\mu' + \frac{e^{-2\nu}}{r}\nu' - \frac{e^{-2\nu}-1}{r^2} = 0$

$\frac{e^{-2\nu}}{r}\left(-2\mu' - \frac{1}{r}\right) + \frac{1}{r^2} = 0$ ✓

$\Rightarrow \quad \dot{\nu} = 0$

$\mu' + \nu' = 0 \qquad \boxed{\mu + \nu = f(t)}$

$2\mu' + \frac{1}{r} = \frac{e^{2\nu}}{r}$

$2\nu' = \frac{1 - e^{2\nu}}{r}$

$e^{2\nu} = y^{-1}$

$-2\nu = \log y$

$-2\nu' = \frac{y'}{y}$

$\frac{y'}{y} = \frac{y^{-1} - 1}{r}$

$y' = \frac{1-y}{r}$

$y' = \frac{1}{r} - \frac{y}{r}$

$\frac{dy}{dr} = -\frac{y}{r} \Rightarrow \log y = -\log r + C$

$y = \frac{c}{r}$

$y' = \frac{c'}{r} - \frac{c}{r^2} = \frac{1}{r} - \frac{c}{r^2}$

$\Rightarrow c' = 1 \Rightarrow c = r + \text{const}$

$y = 1 + \frac{\text{const}}{r}$

$e^{2\nu} = \frac{1}{1 + \frac{\text{const}}{r}}$

$$\boxed{e^{2\nu} = \frac{1}{1 - \frac{2m}{r}}}$$

~~Fix~~ $e^{2\mu} = e^{-2\nu} e^{2f(t)} = e^{2ft}\left(1-\frac{2m}{r}\right)$

$r = r(t)$

$dr = \dot{r}\, dt = e^{f(t)} dt$

$\dot{r} = e^{f(t)}$

$r = \int e^{f(t)} dt$

$$g = \left(1-\frac{2m}{r}\right)\underbrace{e^{2f(t)} dt^2}_{dr^2} - \frac{dr^2}{1-\frac{2m}{r}} - r^2\left(d\theta^2 + \sin^2\vartheta\, d\varphi^2\right)$$