Spherically symmetric metric:

$$\begin{cases} g = e^{2\mu} dt^2 - e^{2\nu} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) \\ \mu = \mu(r,t) \\ \nu = \nu(r,t) \end{cases}$$

coframe
$$\begin{cases} \theta^0 = e^{\mu} dt \\ \theta^1 = e^{\nu} dr \\ \theta^2 = r\, d\theta \\ \theta^3 = r \sin\theta\, d\varphi \end{cases} \qquad g = g_{\mu\nu}\theta^\mu\theta^\nu \qquad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$Ric(g) = 0$$

$$\Rightarrow \quad \boxed{g = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2(d\theta^2 + \sin^2\theta d\varphi^2)}$$

- Note that
  $(\text{Spherical symmetry} + Ric(g) = 0) \Rightarrow (\text{stationary} + \text{static})$
- m - is an integration constant.

  in the weak field aproximation:
  $$g_{00} = 1 + \frac{2\varphi}{c^2} = 1 - \frac{2m}{r} \quad \Rightarrow \quad \varphi = -\frac{mc^2}{r}$$

In Newton's theory $\varphi = -\frac{GM}{r} \Rightarrow \boxed{m = \frac{GM}{c^2}}$

- $\frac{\partial}{\partial t}$ is a Killing vector

$$g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 1 - \frac{2m}{r} \begin{cases} \xrightarrow{r > 2m} \text{timelike} \\ \xrightarrow{r < 2m} \text{space like} \end{cases}$$

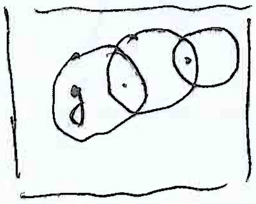
← field is NOT stationary there!

hypersurface $r = 2m$???

10) Maximal analytic extension of the Schwarzschild metric

Kruskal 1960!

$\mathbb{R}^n$



$$g = \left(1-\frac{2m}{r}\right)dt^2 - \frac{dr^2}{1-\frac{2m}{r}} - r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)$$

$$= \underbrace{\left(1-\frac{2m}{r}\right)\left(dt^2 - \frac{dr^2}{\left(1-\frac{2m}{r}\right)^2}\right)}_{\text{we are interested in this term}} - r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)$$

$$r^* = \int \frac{dr}{1-\frac{2m}{r}} = \int \frac{r-2m+2m}{r-2m}\, dr = r + 2m \log\left(\frac{r}{2m}-1\right)$$

$u = t - r^*$

$v = t + r^*$

$2m < r < \infty$

$\Downarrow$

$r^* \in \,]-\infty, +\infty[$

$$g = \left(1-\frac{2m}{r}\right) du\, dv - \dots$$

$$\begin{cases} \bar{u} = -4m\, e^{-\frac{u}{4m}} \\ \bar{v} = 4m\, e^{\frac{v}{4m}} \end{cases}$$

$$\Rightarrow \quad d\bar{u}\, d\bar{v} = e^{\frac{v-u}{4m}}\, du\, dv = e^{\frac{r^*}{2m}}\, du\, dv =$$

$$= e^{\frac{r}{2m}}\left(\frac{r}{2m}-1\right) du\, dv$$

$$\Rightarrow \quad g = \frac{2m}{r}\, e^{-\frac{r}{2m}}\, d\bar{u}\, d\bar{v} - r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)$$

We constructed a coordinate transformation

$$\begin{pmatrix} t \\ r \end{pmatrix} \to \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$$

What is the relation for $r = r(\bar{u}, \bar{v})$?

$$\boxed{\bar{u}\cdot\bar{v} = -16m^2\, e^{\frac{r^*}{2m}} = -16m^2\, e^{\frac{r}{2m}}\left(\frac{r}{2m}-1\right)}$$

Can we invert?

$$\frac{d(\bar{u}\bar{v})}{dr} = -16m^2\left[\frac{1}{2m}\left(\frac{r}{2m}\cancel{-1}\right) \cancel{+\frac{1}{2m}}\right] e^{\frac{r}{2m}} = -4r\, e^{\frac{r}{2m}} \neq 0 \text{ if } \underline{\underline{r>0}}$$

Thus implicit function theorem tells us that we can invert, getting $r = r(\bar{u}\cdot\bar{v})$, for every $r>0$ (including $r=2m$!!!)

Inserting the so inverted $r = r(\bar{u}\cdot\bar{v})$ we get

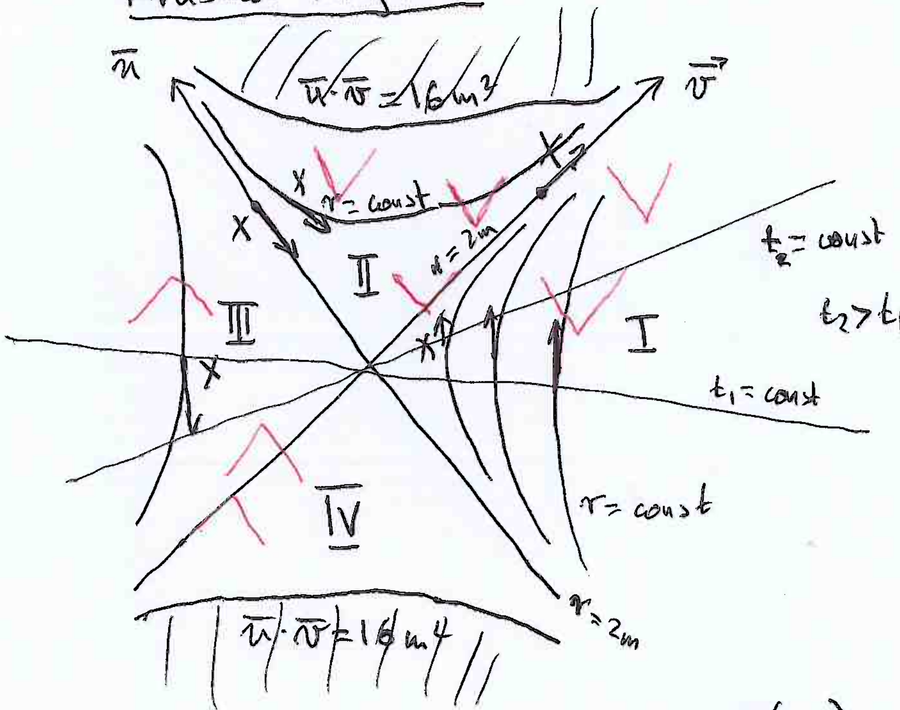
$$g = \frac{2m}{r(\bar{u}\cdot\bar{v})} e^{-\frac{r(\bar{u}\cdot\bar{v})}{2m}} d\bar{u}\, d\bar{v} - r(\bar{u}\cdot\bar{v})^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

This covers all $r>0$ in the coordinates (r,t).

Note that $r>0 \implies \bar{u}\cdot\bar{v} < +16m^2$.

$X = \frac{\partial}{\partial t}$ - Killing vector

Kruskal diagram



$\times\, S^2$

$$\frac{\bar{v}}{\bar{u}} = -e^{\frac{t}{m}}$$

$$t = const \implies \bar{v} = -e^{\frac{t_0}{m}}\bar{u}$$

$$\bar{u} = -e^{-\frac{t_0}{m}}\bar{v}$$

$$g(X,X) = -\frac{\bar{u}\bar{v}}{16m^2}\, 2g\left(\frac{\partial}{\partial\bar{u}}, \frac{\partial}{\partial\bar{v}}\right)$$

$$X = \partial_t = \frac{\partial\bar{u}}{\partial t}\partial_{\bar{u}} + \frac{\partial\bar{v}}{\partial t}\partial_{\bar{v}} =$$

$$= -\frac{\bar{u}}{4m}\partial_{\bar{u}} + \frac{\bar{v}}{4m}\partial_{\bar{v}}$$

I, III $\quad g(X,X) > 0 \quad$ Killing timelike

II, IV $\quad g(X,X) < 0 \quad$ Killing spacelike

$r = 2m \quad$ Killing null.

(I, II) — part covered by the Schwarzschild chart (r,t).

11) Radial free fall in Schwarzschild.

Equation for radial geodesic:

$$\delta \int ds = 0$$

$$g = \left(1-\frac{2m}{r}\right)dt^2 - \frac{dr^2}{1-\frac{2m}{r}} - r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

Radial motion: $\theta = \text{const}$, $\varphi = \text{const}$.

$$ds^2 = \left(1-\frac{2m}{r}\right)dt^2 - \frac{dr^2}{1-\frac{2m}{r}}$$

$ds = \mathcal{L}(\lambda)\, d\lambda$; take $\lambda = r$ as a parameter:

$$ds^2 = \left[\left(1-\frac{2m}{r}\right)t'^2 - \frac{1}{1-\frac{2m}{r}}\right]dr^2$$

$$\mathcal{L} = \sqrt{Xt'^2 - \frac{1}{X}} \qquad X = 1-\frac{2m}{r}, \quad \lambda = r$$

We are looking for $t = t(r)$.

Euler-Lagrange equations

$$\frac{d}{dr}\frac{\partial \mathcal{L}}{\partial t'} - \frac{\partial \mathcal{L}}{\partial t} = 0 \implies \frac{d}{dr}\frac{\partial \mathcal{L}}{\partial t'} = 0$$

$$\implies \frac{\partial \mathcal{L}}{\partial t'} = -\sqrt{X_0} = \text{const}$$

$$(*) \quad \boxed{\frac{2Xt'}{2\sqrt{\quad}} = -\sqrt{X_0}}$$

t' or $\frac{dr}{dt} < 0$

particles fall down into the center.

$$X^2 t'^2 = X_0\left(t'^2 X - \frac{1}{X}\right)$$

$$\left(X^2 - X_0 X\right)t'^2 = -\frac{X_0}{X}$$

$$(**) \quad \boxed{t' = -\frac{1}{X}\sqrt{\frac{X_0}{X_0 - X}}}$$

$X = X_0 \implies t' \to -\infty$

$X_0 = X(r_0)$ $\qquad \frac{dr}{dt} \to 0$

r_0 – point of the start

Time of the free fall from r_0 to $r=2m$

- in the Schwarzschild time (time of an distant observer)

$$T=-\int_{r_0}^{2m}\frac{1}{x}\sqrt{\frac{x_0}{x_0-x}}\,dr=\underbrace{\int_{2m}^{r_0}\frac{1}{x}\sqrt{\frac{x_0}{x_0-x}}\,dr}_{\text{this integral is divergent!}}=+\infty$$

- in the proper time of a particle:

$$\frac{ds}{dr}=-\mathcal{L}=-\sqrt{xt'^2-\frac{1}{x}}\underset{(*)}{=}\frac{x}{\sqrt{x_0}}t'\underset{(**)}{=}-\frac{1}{\sqrt{x_0-x}}$$

$$\Rightarrow\quad S=\int_{2m}^{r_0}\frac{dr}{\sqrt{x_0-x}}=\text{one can explicitly integrate}<+\infty\quad !!!$$

(8) Basic physical consequences of Schwarzschild.

1) Equations of motion of a test particle in Schwarzschild.

$$g=\left(1-\frac{2m}{r}\right)dt^2-\frac{dr^2}{1-\frac{2m}{r}}-r^2\left(d\theta^2+\sin^2\theta\,d\varphi^2\right)$$

Remark

We know that geodesics are extremals of the arc length functional:

$$\delta\int\sqrt{|g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu|}\,d\lambda=0$$

This is independent of parametrization, but is not good for null geodesics

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}-\frac{\partial\mathcal{L}}{\partial x^\mu}=0\qquad\frac{\partial}{\partial\dot{x}^\mu}\mathcal{L}\sim\frac{1}{\mathcal{L}\equiv0}\,!$$

To deal with null geodesics one considers ENERGY functional:

$$\frac{1}{2}\,\delta \int g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu \, d\lambda = 0. \qquad E = \int \frac{1}{2} g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu \, d\lambda$$

This DOES depend on parametrization. But: now

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu \quad \text{and}$$

E-L equations:

$$\frac{1}{2}\frac{d}{d\lambda}\frac{\partial}{\partial \dot{x}^\rho}\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right) - \frac{1}{2}\frac{\partial}{\partial x^\rho}\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right) = 0$$

$$\frac{d}{d\lambda}\left(g_{\mu\rho}\dot{x}^\mu\right) - \frac{1}{2} g_{\mu\nu,\rho}\,\dot{x}^\mu\dot{x}^\nu = 0$$

$$g_{\mu\rho}\,\ddot{x}^\mu + g_{\mu\rho,\nu}\,\dot{x}^\nu\dot{x}^\mu - \frac{1}{2} g_{\mu\nu,\rho}\,\dot{x}^\mu\dot{x}^\nu = 0$$

$$g_{\mu\rho}\,\ddot{x}^\mu + \frac{1}{2}\left(g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho}\right)\dot{x}^\mu\dot{x}^\nu = 0$$

$$\boxed{\ddot{x}^\mu + \left\{ {\mu \atop \nu\rho} \right\} \dot{x}^\nu \dot{x}^\rho = 0}$$

geodesic equation in which λ is an affine parameter.

Interpretation of the affine parameter λ:

$$ds = \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\, d\lambda \implies \left(\frac{ds}{d\lambda}\right)^2 = g_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu$$

calculate using E-L. eqs.

$$\frac{d}{d\lambda}\left(\frac{ds}{d\lambda}\right)^2 = \underbrace{\frac{d}{d\lambda}\left(g_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu\right) \stackrel{\downarrow}{=} 0}$$

This in particular means that $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ is a FIRST INTEGRAL for the E-L eqs.

$$\| \qquad\qquad$$

$$2\,\frac{ds}{d\lambda}\frac{d^2 s}{d\lambda^2} \implies \lambda = as + b$$

$\implies$ λ is essentially a proper time.

unless $s \equiv 0$ as for null geodesics and then

λ is $\underline{a}$ parameter along a null geod.

- Energy functional for Schwarzschild:

$$\boxed{E = \frac{1}{2}\int\left[\left(1-\frac{2m}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{2m}{r}} - r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)\right]ds.}$$

If we consider timelike geodesics, we have:

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 1 \qquad \underline{c=1}$$

$$2\mathcal{L} = \boxed{\left(1-\frac{2m}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{2m}{r}} - r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right) = 1} \quad (*)$$

(first integral - r.h.s is always constant for geodesics).

- Euler-Lagrange' equation for θ:

$$\frac{d}{ds}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\boxed{-\left(r^2\dot{\theta}\right)^{\cdot} + r^2 \sin\theta\cos\theta\,\dot{\varphi}^2 = 0}$$

this is automatically satisfied at $\theta = \frac{\pi}{2}$

From the spherical symmetry we can restrict to $\boxed{\theta = \frac{\pi}{2}}$ and forget about $\delta\theta$.

- $\mathcal{L}$ does not depend on t and φ $\Rightarrow$

$$\frac{d}{dt}\left[\left(1-\frac{2m}{r}\right)\dot{t}\right] = 0$$

$$\frac{d}{d\varphi}\left(-r^2\dot{\varphi}\right) = 0$$

Thus:

$$\begin{cases} \left(1-\frac{2m}{r}\right)\dot{t} = \text{const} = E & (1) \\ -r^2\dot{\varphi} = \text{const} = -L & (2) \end{cases}$$

- We need the last E-L equation corresponding to δr. Instead of going for it, we use the ~~constraint~~ first integral (*) in which we insert (1) and (2):

$$\frac{E^2}{1-\frac{2m}{r}} - \frac{\dot{r}^2}{1-\frac{2m}{r}} - \frac{L^2}{r^2} = 1$$

Thus we have

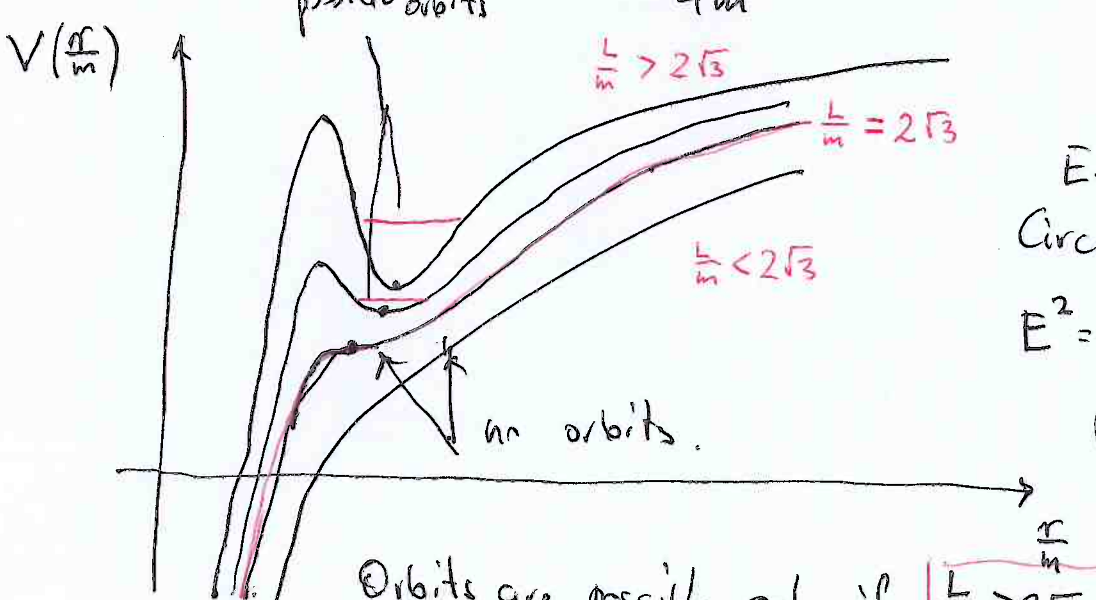
$$\boxed{E^2 = \underbrace{\dot{r}^2}_{\text{kinetic energy}} + \underbrace{\left(1-\frac{2m}{r}\right)\left(1+\frac{L^2}{r^2}\right)}_{\text{effective potential}}} \qquad (E)$$

Effective potential

$$V(r) = \left(1-\frac{2m}{r}\right)\left(1+\frac{L^2}{r^2}\right)$$

$$V'(r) = 0 \iff r = \frac{L^2 \pm L\sqrt{L^2-12m^2}}{2m}$$

$$V''(r) = 0 \iff r = \frac{3L^2 \pm \sqrt{3}\sqrt{3L^4-32L^2m^2}}{4m}$$



E.g. Circular orbits:

$$E^2 = V(r_+) = \left(1-\frac{2m}{r_+}\right)\left(1+\frac{L^2}{r_+^2}\right)$$

Orbits are possible only if $\boxed{\frac{L}{2m} > 2\sqrt{3}}$

- Dependence on φ:

$$\frac{dr}{d\varphi} = r' = \frac{\frac{dr}{ds}}{\frac{d\varphi}{ds}}$$

$$\dot{r} = \frac{dr}{d\varphi}\dot{\varphi} \underset{(2)}{=} r'\frac{L}{r^2}$$

and thus, the radial equation is:

$$E^2 = r'^2\frac{L^2}{r^4} + V(r)$$

- Those who remember the integration of the Kepler problem appreciate the change of variables:

$$u = \frac{1}{r} \quad \Rightarrow \quad r' = -\frac{u'}{u^2}$$

$$\Rightarrow \quad L^2 u'^2 = E^2 - (1-2mu)(1+L^2u^2)$$

$$u'^2 + u^2 = \frac{E^2-1}{L^2} + \frac{2m}{L^2}u + 2mu^3 \qquad \Big| \frac{d}{d\varphi}:$$

$$2u'u'' + 2uu' = \frac{2m}{L^2}u' + 6mu^2u'$$

If $u' \neq 0$ i.e. if we exclude CIRCULAR orbits we have

$$\boxed{u'' + u = \frac{m}{L^2} + 3mu^2} \qquad (\text{REL})$$

- Comparison with Newtonian theory:

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right] - \tilde{V}(r)$$

φ - is cyclic variable:

$$r^2\dot{\varphi} = L = \text{const}$$

radial equation:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 + \frac{\partial\tilde{V}}{\partial r} = 0$$

$$\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = r'\frac{L}{r^2} = -Lu' \, ; \quad \frac{d^2r}{dt^2} = -Lu''\frac{d\varphi}{dt} = -L^2u''u^2$$

$$\Rightarrow \; -L^2 u'' u^2 - L^2 u^3 + \frac{\partial \tilde{V}}{\partial r} = 0$$

$$\boxed{u'' + u = \frac{1}{L^2}\frac{\partial \tilde{V}}{\partial r}\frac{1}{u^2}} \qquad \text{(POT)}$$

and if the Newtonian potential is $\tilde{V} = -\frac{M}{r}$ we get

$$\boxed{u'' + u = \frac{M}{L^2}} \quad \text{Newton}$$

$$\boxed{u'' + u = \frac{m}{L^2} + 3mu^2} \quad \text{Einstein.}$$

- How big is $3mu^2$ when compared to $\frac{m}{L^2}$:

$$\frac{3mu^2}{\frac{m}{L^2}} = 3u^2L^2 = 3\frac{1}{r^2}(r^2\dot{\varphi})^2 = 3\left(r\frac{d\varphi}{dt}\right)^2 = 3\frac{v_\perp^2}{c^2}$$

in Solar system $3\frac{v_\perp^2}{c^2}$ is Highest for MERCURY

$$3\frac{v_\perp^2}{c^2} \simeq \underline{\underline{7.7 \cdot 10^{-8}}}$$

- <u>MIRACOLO</u>:

take $\tilde{V} = -\frac{m}{r} - m\frac{L^2}{r^3}$ and insert in (POT):

Then
$$\frac{1}{L^2}\frac{\partial \tilde{V}}{\partial r}\frac{1}{u^2} = \frac{1}{L^2}\left(\frac{m}{r^2} + 3\frac{mL^2}{r^4}\right)\frac{1}{u^2} =$$

$$= \frac{m}{L^2} + 3mu^2$$

Thus the Newtonian equation (POT) coincides with the relativitic one (REL) when we replace the Newtonian potential by

$$\tilde{V} = -\frac{m}{r} - m\frac{L^2}{r^3}$$

So relativistic effects in Schwarzschild $\equiv$ perturbation of Newt. pot. by a term $-\frac{mL^2}{r^3}$.

- Perihelion advance

$3mu^2$ – small perturbation

$$u(\varphi) = \frac{1}{p}(1 + e\cos\varphi) \quad \text{Keplerian motion.}$$

$$p = a(1-e^2) = \frac{L^2}{m}$$

(because $u'' + u = \frac{1}{p}$)

So now we start with a Keplerian solution

$$u^{(0)} = \frac{1}{p}(1 + e\cos\varphi)$$

and solve for $u^{(1)}$ as

$$u^{(1)\,''} + u^{(1)} = \frac{m}{L^2} + 3m\, u^{(0)\,2}$$

i.e. we have to solve:

$$u^{(1)\,''} + u^{(1)} = \frac{m}{L^2} + 3\frac{m^3}{L^4}(1 + e\cos\varphi)^2 \qquad (*)$$

Consider three equations

$$v'' + v = \begin{cases} A \\ A\cos\varphi \\ A\cos^2\varphi \end{cases}$$

Their particular solutions are:

$$v = \begin{cases} A \\ \frac{1}{2}A\varphi\sin\varphi \\ \frac{1}{2}A - \frac{1}{6}A\cos 2\varphi \end{cases}$$

Thus a general solution to $(*)$ is:

$$\boxed{u^{(1)} = \frac{m}{L^2}(1 + e\cos\varphi) + \frac{3m^3}{L^4}\left(1 + e\varphi\sin\varphi + \frac{e^2}{2} - \frac{e^2}{6}\cos 2\varphi\right)}$$

Let us put perihelion at $\varphi = 0$.

Next perihelion is in

$$\varphi = 2\pi + \Delta\varphi$$

when $u' = 0$.

$$0 = u' = -\frac{me}{L^2}\sin\varphi + \frac{3m^3}{L^4}\left(e\sin\varphi + e\varphi\cos\varphi + \frac{e^2}{3}\sin 2\varphi\right)\Big|_{2\pi+\Delta\varphi}$$

$$0 = -\sin\Delta\varphi + \frac{3m^2}{L^2}(2\pi + \cancel{\Delta\varphi})\cos\Delta\varphi + \cancel{\underbrace{\frac{3m^2}{L^2}\left(\sin\Delta\varphi + \frac{e}{3}\sin 2\Delta\varphi\right)}_{\text{very small!}}}$$

$$\Rightarrow \quad \Delta\varphi \simeq \operatorname{tg}\Delta\varphi \hat{\simeq} \frac{6m^2\pi}{L^2}$$

$$\boxed{\Delta\varphi \hat{\simeq} 3\pi \frac{\frac{2GM}{c^2}}{a(1-e^2)}}$$

43 seconds / 100 years

Einstein 1915.
he said that he had heart palpitations when he has got this result.

- **Deflection of light**

we replace equation (E) by the constraint $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$, i.e. by the requirement that our geodesic is null.

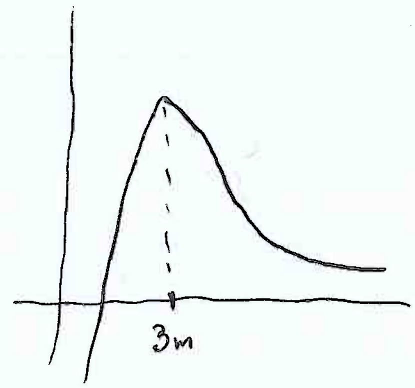
$$\Rightarrow \quad \frac{E^2}{1-\frac{2m}{r}} - \frac{\dot{r}^2}{1-\frac{2m}{r}} - \frac{L^2}{r^2} = 0$$

$$\Rightarrow \quad \boxed{E^2 = \dot{r}^2 + \left(1 - \frac{2m}{r}\right)\frac{L^2}{r^2}} \quad (EN).$$

Now $V(r) = \left(1 - \frac{2m}{r}\right)\frac{L^2}{r^2}$

$V'(r) = 0 \Leftrightarrow r = 3m$

unstable CIRCULAR foton orbit.



- Equation for the trajectories parametrized by φ:

$$E^2 = r'^2 \frac{L^2}{r^4} + \left(1 - \frac{2m}{r}\right)\frac{L^2}{r^2}$$

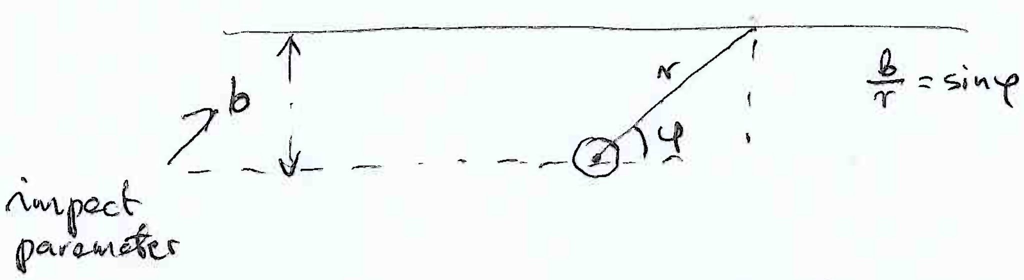
$u = \frac{1}{r}$

$$u'^2 + u^2 = \frac{E^2}{L^2} + 2mu^3 \qquad \frac{d}{d\varphi}$$

$$\boxed{u'' + u' = 3mu^2}$$

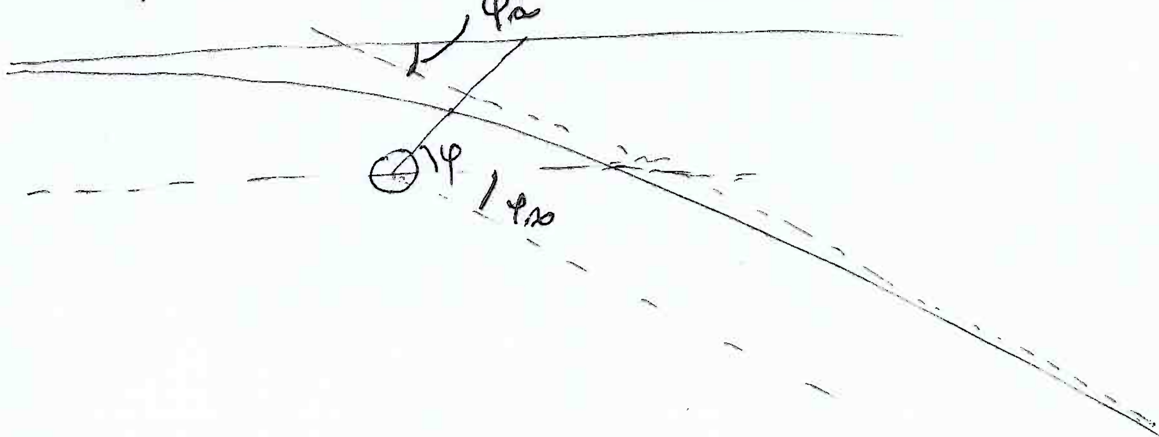
If $\underline{m = 0}$ the solution is given by:

$$u = \frac{1}{b}\sin\varphi$$ — equation of a straight line



$\frac{b}{r} = \sin\varphi$

In the presence of mass situation changes:



Making the same aproximation as before we have

$$u^0 = \frac{1}{b}\sin\varphi$$

$$u^{(1)''} + u^{(1)} = 3m\,u^{(0)2} \qquad \text{i.e.}$$

$$u'' + u = \frac{3m}{b^2}\left(1-\cos^2\varphi\right)$$

A particular solution here is:

$$u_1 = \frac{3m}{2b^2}\left(1+\frac{1}{3}\cos 2\varphi\right)$$

So:

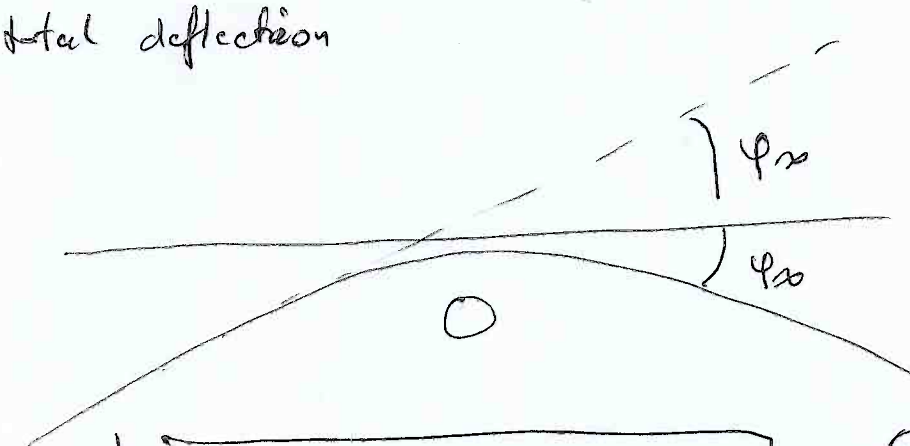
$$u = \frac{1}{b}\sin\varphi + \frac{3m}{2b^2}\left(1+\frac{1}{3}\cos 2\varphi\right)$$

When $r \to \infty$ ~~then~~ $u \to 0$, and for small φs we have

$$0 = \frac{1}{b}\varphi_\infty + \frac{3m}{2b^2}\left(1+\frac{1}{3}\right)$$

$$\varphi_\infty = -\frac{2m}{b}$$

The total deflection



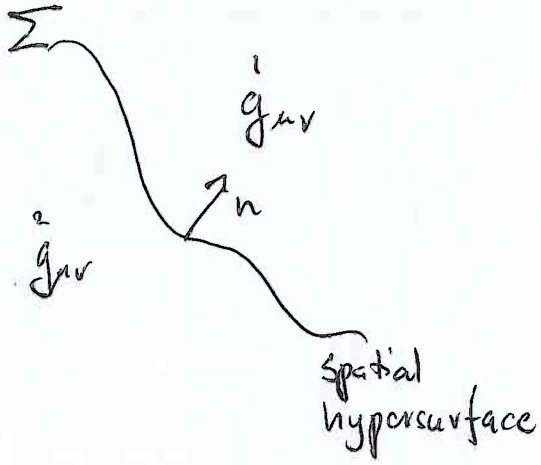
$$\boxed{\delta = 2|\varphi_\infty| = \frac{4m}{b} = \frac{4GM}{c^2 b}} = \frac{2R_s}{b}$$

Schwarzschild radius $\boxed{R_s = \frac{2GM}{c^2}}$

For the Sun : $\boxed{\delta = 1.75'' \cdot \frac{R_\odot}{b}}$

(9) Spherically symmetric stars

1) Matching conditions in GR.



A spatial hypersurface is characterized by two tensors:

1) I^{st} fundamental form:

by definition it is the metric restricted to Σ

$$I = g|_\Sigma$$

2) II^{nd} fundamental form:

by definition it is the projection of the covariant derivative of the unit normal vector onto the surface:

$$II_{\mu\nu} = \nabla_\alpha n_\beta \, h^\alpha{}_\mu h^\beta{}_\nu \qquad \text{where}$$

$$h^\alpha{}_\beta = \delta^\alpha{}_\beta - n^\alpha n_\beta \qquad (n^2 = 1.)$$

The matching conditions for two gravitational fields along a spatial hypersurface Σ are that

1) I^{st}

and

2) II^{nd}

fundamental forms

must be continuous along Σ.

2) Spherically symmetric, static distribution of mass

$$g = e^{2\mu(r,t)}dt^2 - e^{2\nu(r,t)}dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$$

$\Rightarrow$ Einstein tensor:

$$\begin{cases} G_{00} = \frac{1}{r^2}\left(1 - e^{-2\nu} + 2r e^{-2\nu}\nu'\right) \\ G_{01} = \frac{2}{r} e^{-\mu-\nu}\dot{\nu} \\ G_{02} = G_{03} = 0 = G_{12} = G_{13} = G_{23} \\ G_{11} = \frac{1}{r^2}\left(-1 + e^{-2\nu} + 2r e^{-2\nu}\mu'\right) \\ G_{22} = G_{33} = e^{-2\nu}\left(\mu'' + \mu'^2 - \nu'\mu' + \frac{\mu'-\nu'}{r}\right) + e^{-2\mu}\left(\dot{\mu}\dot{\nu} - \ddot{\nu} - \dot{\nu}^2\right) \end{cases}$$

Go from $\vartheta^0 = e^{\mu}dt$, $\vartheta^1 = e^{\nu}dr$, $\vartheta^2 = r\,d\vartheta$, $\vartheta^3 = r\sin\vartheta\, d\varphi$

We assume that the metric is <u>static</u>; i.e in particular:

$$\boxed{\dot{\mu} = \dot{\nu} = 0.}$$

Einstein equations inside the star:

$$\begin{cases} G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi T_{\mu\nu} \\ T_{\mu\nu} = (p+\rho)u_\mu u_\nu - p g_{\mu\nu} \end{cases}$$

i.e. we assume that the matter of the star is a perfect fluid with energy density ρ, isotropic pressure p and 4-velocity of particles u_μ $u_\mu u^\mu = 1$.

$u_\mu = (u_t, u_r, u_\theta, u_\varphi)$

Staticity requires that $u_r = u_\theta = u_\varphi \equiv 0$
(because the fluid in the star should not move).

$\Rightarrow 1 = u_t^2 \cdot g^{tt}$, and since $g^{tt} = e^{-2\mu}$

we have

$u_t = e^{\mu}$, $u_r = 0$, $u_\theta = 0$, $u_\varphi = 0$

Thus $u = u_t \, dt = e^{\mu} dt = \theta^0$

Hence we have

$u_0 = 1$, $u_1 = u_2 = u_3 = 0$ in our orthonormal frame.

Einstein equations

$$\boxed{\begin{cases} \frac{1}{r^2}\left(1 - e^{-2\nu} + 2r e^{-2\nu} \nu'\right) = 8\pi \rho & 1 \\ \frac{1}{r^2}\left(-1 + e^{-2\nu} + 2r e^{-2\nu} \mu'\right) = 8\pi p & 2 \\ e^{-2\nu}\left(\mu'' + \mu'^2 - \nu'\mu' + \frac{\mu' - \nu'}{r}\right) = 8\pi p & 3 \end{cases}}$$

Recall as a consequence of $\nabla^\mu T_{\mu\nu} = 0$ we have Relativistic Euler's equation:

$$\boxed{(\rho + p) u^\nu \nabla_\nu u^\mu = (g^{\mu\nu} - u^\mu u^\nu) \nabla_\nu p}$$

Spherical symmetry requires:

$p = p(r)$
$\rho = \rho(r)$

$$u^\nu \nabla_\nu u^\mu = u^t \nabla_t u^\mu =$$

$$= u^t \left(\cancel{u^\mu{}_{,t}} + \Gamma^\mu{}_{\nu t} u^\nu \right) = u^t \Gamma^\mu{}_{tt} u^t = e^{-2\mu} \Gamma^\mu{}_{tt}$$

$$\Gamma^\mu{}_{tt} = \frac{1}{2} g^{\mu s} \left(\cancel{g_{st,t}} + \cancel{g_{ts,t}} - g_{tt,s} \right) = -\frac{1}{2} g^{\mu r} g_{tt,r}$$

$\Rightarrow$ only $\Gamma^r{}_{tt} \neq 0$ and

$$\Gamma^r{}_{tt} = \frac{1}{2} e^{-2\nu} e^{2\mu} 2\mu' = \mu' e^{2(\mu-\nu)}$$

Hence Euler's equation: for $\underline{\mu = r}$

$$(\varrho + p) \mu' e^{-2\nu} = -e^{-2\nu} p'$$

$$\Rightarrow \boxed{\mu' = -\frac{p'}{\varrho + p}}$$

other components of Euler's equation brings nothing!

<u>Now</u>

(1) gives:

$$e^{-2\nu} \left(\frac{1}{r^2} - \frac{2\nu'}{r} \right) - \frac{1}{r^2} = -8\pi\varrho$$

insert $e^{-2\nu} = X$

$$\frac{X}{r^2} + \frac{X'}{r} = -8\pi\varrho + \frac{1}{r^2}$$

$$\Rightarrow \boxed{e^{-2\nu} = 1 - \frac{8\pi}{r} \int_0^r \varrho(r') r'^2 dr'}$$

Introducing $\boxed{m(r) \stackrel{\text{def}}{=} 4\pi \int_0^r r'^2 \varrho(r') dr'}$

we get:

$$\boxed{e^{-2\nu} = 1 - \frac{2m(r)}{r}}$$

If $r = R$ the radius of the star, then we have to have

$$\boxed{4\pi \int_0^R r'^2 \rho(r')\, dr' = m = \text{const}}$$

where m – Schwarzschild mass of the exterior Schwarzschild solution.

Note that outside $Ric(g) \equiv 0$, and because of spherical symmetry the solution must be Schwarzschild

Now equation (2):

$$\frac{1}{r^2}\left(-\cancel{1}+\cancel{1}-\frac{2m(r)}{r} + 2r\left(1-\frac{2m(r)}{r}\right)\left(-\frac{p'}{\rho+p}\right)\right) = 8\pi p$$

$$8\pi p + \frac{2m(r)}{r^3} = \frac{2}{r}\left(1-\frac{2m(r)}{r}\right)\left(-\frac{p'}{\rho+p}\right)$$

$$\boxed{p' = -(\rho+p)\,\frac{m(r)+4\pi p r^3}{r^2\left(1-\frac{2m(r)}{r}\right)}}$$

Oppenheimer–Volkoff equation.

Summary Spherically symmetric stars:

$$g=\left(1-\frac{2m}{r}\right)dt^2-\frac{dr^2}{1-\frac{2m}{r}}-r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)$$

for $r \geq R$

for $r < R$

(1)
$$\begin{cases} g= e^{2\mu(r)}dt^2-\frac{dr^2}{1-\frac{2m(r)}{r}}-r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right) \\ m(r)=\int_0^r 4\pi r'^2 \rho(r')dr' \\ m(R)=m \\ p'=-(\rho+p)\frac{m(r)+4\pi p r^3}{r^2\left(1-\frac{2m(r)}{r}\right)} \\ \mu'=\frac{m(r)+4\pi p r^3}{r^2\left(1-\frac{2m(r)}{r}\right)} \\ e^{2\mu(R)}=1-\frac{2m}{r} \end{cases}$$

We did not check, but this ~~au~~tomatically satisfies einstein equation (3).

We need to suplement the system by the equation of state (2) $\boxed{p=p(\rho)}$, which we add

To have a unique solution we need ~~initial~~ boundary values

(3) $\boxed{p(R)=0\,; \quad \nu_c=\nu(0)\,, \quad m_c=m(0)=0}$

3) Example – model with a constant density.

$$\rho = \rho_0 = \text{const.}$$

$$m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & r \leq R \\ M = \frac{4}{3}\pi R^3 \rho_0 & r > R \end{cases}$$

$$\rightsquigarrow \quad m(r) = M \frac{r^3}{R^3}$$

$$p' = -(\rho_0 + p)\, r \, \frac{\frac{M}{R^3} + 4\pi p}{1 - \frac{2Mr^2}{R^3}}$$

One can integrate this

$$p = \rho_0 \frac{\sqrt{1 - \frac{2Mr^2}{R^3}} - \sqrt{1 - \frac{2M}{R}}}{3\sqrt{1 - \frac{2M}{R}} - \sqrt{1 - \frac{2Mr^2}{R^3}}}$$

$$\Rightarrow \quad \mu' = \frac{M \frac{r^3}{R^3} + 4\pi p r^3}{r^2 \left(1 - \frac{2Mr^2}{R^3}\right)}$$

$$\Rightarrow \quad e^{\mu} = \frac{3}{2}\left(1 - \frac{2M}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2Mr^2}{R^3}\right)^{1/2}$$

Discussion

$$p_c = \rho_0 \frac{1-\sqrt{1-\frac{2M}{R}}}{3\sqrt{1-\frac{2M}{R}}-1}$$

① p_c – monotonically increases with R and with M

② if $9\left(1-\frac{2M}{R}\right)=1$ i.e. when

$$\frac{2M}{R} = \frac{2^3}{3^2}$$

then the preasure in the center is infinite.

$$\Rightarrow \frac{2M}{R} \leq \frac{2^3}{3^2}$$ for a star with constant density

③ if $t = const$

$$g\big|_{t=const} = -\frac{dr^2}{1-\frac{2Mr^2}{R^3}} - r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)$$

and this is – metric on a 3-dimensional sphere of radius $a = \sqrt{\frac{R^3}{2M}}$

Indeed: $r = a\sin\chi$

$dr = a\cos\chi\, d\chi$

$$\frac{dr^2}{1-\frac{2Mr^2}{R^3}} = \dots = \frac{R^3}{2M}\, d\chi^2$$

$$\Rightarrow g\big|_{t=const} = -\frac{R^3}{2M}\left(d\chi^2+\sin^2\chi\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)\right)$$