Lecture 14, 31.05.2011

(10) Cosmology

Gravitation as the main force that determines the evolution of the Universe.

1) Cosmological principles

- Perfect cosmological principle:

  "apart from local inhomogenities the Universe is homogeneous at every place and at every time"

realization: Einstein universe

$$M = \mathbb{R} \times S^3 \quad , \quad g = dt^2 - R^2 \frac{dx^2+dy^2+dz^2}{\left(1+\frac{1}{4}(x^2+y^2+z^2)\right)^2}$$

satisfies Einstein's equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \varkappa T_{\mu\nu}$$

$T_{\mu\nu}$ - perfect fluid with $\rho = \frac{2}{R^2 c^2}$

$$\Lambda = -\frac{1}{R^2}$$

<u>Sign of Lambda</u>: note that if the signature changes from $+---$ to $-+++$ sign of Lambda is different

- principle of spatial homogeneity
  (Kopernican principle)

  " . . . . at every place."

- principle of spatial isotropy:

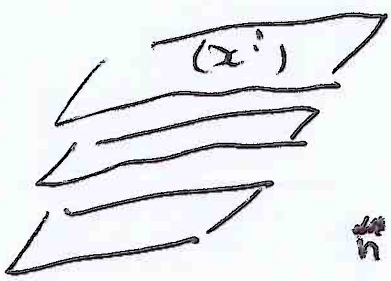
  " . . . . . in every direction"

2) <u>Mathematical formulation of spatial homogeneity:</u>

There is a symmetry group G of the gravitational field and matter fields which acting on (M, g) has SPACELIKE hypersurfaces as its orbits.

$\Rightarrow$

$t \uparrow$



foliation of space-time by 3-dimensional spatial orbits

n - normal vector to the orbits

$n = \frac{\partial}{\partial t}$, n is unit.

positive definite

$$\Rightarrow \quad g = dt^2 - g_{ij}(t, x^k)\, dx^i dx^j$$

x^i - coordinates on the orbits

t - cosmic time $\equiv$ proper time of an observer whose worldlines are $\perp$ to the orbits.

$t = \text{const}$ - orbits of G

$x^i = \text{const}$ - worldline of an observer comoving with the Universe.

g_{ij} - depend on this how G acts on the orbits and what is G.

$G \supset G_0$ - which acts simply transitively on the orbits — exist: Bianchi models; do not exist: Kantowski-Sachs.

3) <u>Bianchi models.</u>

Let $p_0 \in$ orbit $\Rightarrow$

any other point p on the orbit defines $a \in G_0$ s.t.

$$p = p_0 \cdot a$$

so

$$\text{Orbit} = \{ p_0 a : a \in G_0 \} \simeq G_0 .$$

The metric g must be preserved under the action of G_0

$\Rightarrow \theta^i$ - Left invariant forms on G_0 i.e. s.t.

if X_j are generators of G_0 then

$$\mathcal{L}_{X_j} \theta^i = 0 .$$

<u>Fact</u> if $G_0 = \{ a = a(t_i) \}$ then

$E_j \theta^j = a^{-1} da$ where E_j is a basis for the Lie algebra of G_0.

$\Rightarrow$ the metric

$$\boxed{g = dt^2 - g_{ij}(t)\, \theta^i \theta^j}$$

Obviously $\mathcal{L}_{X_i} g = 0$ for X_i gen. of G.

4) Bianchi classification of 3-dimensional Lie algebras.

$$[X_i, X_j] = c^k{}_{ij} X_k \qquad (C.R)$$

(1) $\quad c^k{}_{ij} = -c^k{}_{ji}$ — antisymmetry

(2) $\quad c^i{}_{j[k} c^j{}_{lm]} = 0$ — Jacobi identity.

Changing the basis $X_i \longmapsto a^{-1j}{}_i X_j$ results in change of the structure constants

$a \in GL(n, \mathbb{R})$: $\quad c^k{}_{ij} \longmapsto a^{-1l}{}_i a^{-1m}{}_j c^n{}_{lm} a^k{}_n = c'^k{}_{ij} \qquad (*)$

So Lie algebras are in (1) to (1) correspondance with $c^k{}_{ij}$ satisfying (1) and (2) given modulo the action of $GL(n, \mathbb{R})$

Nonequivalent Lie algebras $\equiv$ orbits of the action $(*)$ for $c^i{}_{jk}$ satisfying (1)-(2).

If $\underline{n=3}$ they can be easily classified:

In $\mathbb{R}^3$ we have ϵ_{ijk} - Levi-Civita symbol

This satisfies

$$\epsilon_{ijk}\epsilon^{ilm} = \delta^l_j \delta^m_k - \delta^l_k \delta^m_j$$

Take $\boxed{t^i = c^i{}_{jk} \epsilon^j{}_{lm} \epsilon^{klm}}$

This is $\underline{\text{zero}}$ because of (2).

$$\boxed{c^i{}_{jk} c^j{}_{lm} \epsilon^{klm} = 0} \Leftrightarrow (2).$$

Observe that if c^i_{jk} satisfy $c^i_{jk} = -c^i_{kj}$ they can be written as:

$$\boxed{c^i_{jk} = n^{il}\epsilon_{jkl} - \delta^i_j a_k + \delta^i_k a_j} \qquad (S.C.)$$

To prove it take

$$c^{il} = \frac{1}{2} c^i_{jk}\epsilon^{jkl}. \text{ Then}$$

$$n^{il} = \frac{1}{2}(c^{il} + c^{li})$$

$$a_m = \frac{1}{2}\epsilon_{mil} c^{il}$$

Now: we ~~have~~ replaced c^i_{jk} with $c^i_{jk} = -c^i_{kj}$ by n^{il} s.t. $n^{il} = n^{(il)}$ and a_m.

Jacobi identity:

$$t^i = 4n^{il}a_l \overset{J.I}{=} 0$$

So 3-dimensional Lie algebras can be represented by

$$(c^i_{jk}) \Longleftrightarrow (n^{il}, a_m) \quad \text{s.t.}$$

$$\begin{cases} n^{il} = n^{(il)} \\ n^{il}\cdot a_l = 0 \end{cases}$$

The action of $GL(3,\mathbb{R})$ on c^i_{jk} translates to the action of $GL(3,\mathbb{R})$ on n^{il} and a_m.

n^{il} is symmetric $\Rightarrow$ diagonalizable.

$\Rightarrow$ using $GL(3,\mathbb{R})$ we can bring n^{il} in the form

$$n^{il} = \mathrm{diag}(n^1, n^2, n^3). \quad \text{where}$$

$$\underline{\underline{n^i = \pm 1, 0}}$$

We can still use such elements of $GL(3,\mathbb{R})$ that preserve $\begin{pmatrix} n^1 & & \\ & n^2 & \\ & & n^3 \end{pmatrix}$ to bring a_m to the simplest form.

E.g. if $\begin{pmatrix} n^1 & & \\ & n^2 & \\ & & n^3 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ we have entire $SO(3)$ to act on a_m. In such a case we can always bring a_m in the form $a_m = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}$

Scalling:

$$X_1 \to \lambda_1 X_1 \quad , \quad X_2 \to \lambda_2 X_2 , \quad X_3 \to \lambda_3 X_3$$

does not changes $\begin{pmatrix} n^1 & & \\ & n^2 & \\ & & n^3 \end{pmatrix}$ iff

$$(\lambda_1\lambda_2 - \lambda_3) n_3 = 0 , \quad (\lambda_3\lambda_1 - \lambda_2) n_2 = 0 , \quad (\lambda_2\lambda_3 - \lambda_1) n_1 = 0$$

but changes a_m to

$$(a_1, a_2, a_3) \longmapsto (\lambda_1 a_1, \lambda_2 a_2, \lambda_3 a_3)$$

Writing explicitly (C.R.) using (S.C.) with $n^{il} = \begin{pmatrix} n^1 & & \\ & n^2 & \\ & & n^3 \end{pmatrix}$

we get:

$$[X_1, X_2] = n^3 X_3 - a_2 X_1 + a_1 X_2$$

$$[X_3, X_1] = n^2 X_2 - a_1 X_3 + a_3 X_1$$

$$[X_2, X_3] = n^1 X_1 - a_3 X_2 + a_2 X_3$$

All nonequivalent types split onto two basic types depending on this if $a = 0$ or not

$a = 0$

		n^1	n^2	n^3
	I	0	0	0
Heisenberg	II	1	0	0
	VI_0	1	-1	0
	VII_0	1	1	0
$SL(2,\mathbb{R})$	VIII	1	1	-1
$SO(3)$	IX	1	1	1

$a \neq 0 \Rightarrow a = (0,0,a)$

	n^1	n^2	n^3	a
V	0	0	0	1
IV	1	0	0	1
VI_h	1	-1	0	$h>0$
VII_h	1	1	0	$h>0$

$(VI_1 = III)$

Left inv. forms in local frame

I $\quad \theta^1 = dx, \ \theta^2 = dy, \ \theta^3 = dz$

II $\quad \theta^1 = dx - z\,dy, \ \theta^2 = dy, \ \theta^3 = dz$

IV $\quad \theta^1 = dx, \ \theta^2 = e^{x} dy, \ \theta^3 = e^{x}(dz + x\,dy)$

V $\quad \theta^1 = dx, \ \theta^2 = e^{x} dy, \ \theta^3 = e^{x} dz$

VI $\quad \theta^1 = dx, \ \theta^2 = e^{Ax}(\cosh x\,dy - \sinh x\,dz)$
$\qquad \theta^3 = e^{Ax}(-\sinh x\,dy + \cosh x\,dz)$

VII $\quad \theta^1 = dx \quad \theta^2 = e^{Ax}(\cos x\,dy - \sin x\,dz)$
$\qquad \theta^3 = e^{Ax}(\sin x\,dy + \cos x\,dz)$

VIII $\quad \theta^1 = \cosh y \cos z\,dx - \sin z\,dy$
$\qquad \theta^2 = \cosh y \sin z\,dx + \cos z\,dy$
$\qquad \theta^3 = \sinh y\,dx + dz$

IX $\quad \theta^1 = \cos y \cos z\,dx - \sin z\,dy$
$\qquad \theta^2 = \cos y \sin z\,dx + \cos z\,dy$
$\qquad \theta^3 = -\sin y\,dx + dz$

$\Rightarrow \quad g = dt^2 - g_{ij}(t)\,\theta^i\theta^j$ with θ^i as above exhausts all possible metrics which are spatially homogeneous and which admit a simply transitive ~~group~~ symmetry group G_3. G_3 is a group of the Bianchi type determined by θ^i.

g as above + Einstein equations

$\equiv$ Bianchi models

5) Mathematical formulation of spatial isotropy

G - the symmetry group of g and matter fields acts on spacelike hypersurfaces (which are orbits of G) and at every point has $SO(3)$ as its isotropy group.

6) Tensors invariant with respect to the isotropy group.

$$T_p(M) = \mathbb{R}n \oplus T_p\underbrace{\Sigma}_{\text{orbit}}$$

invariant vectors:

$$v^\mu \begin{cases} v^0 - \text{along } n \\ v^i \end{cases}$$

v^i must be zero; otherwise it would distinguish a direction, and isotropy would be broken!

v^0 must be constant on the orbits; otherwise spatial homogeneity would be broken $\Rightarrow$ $v^0 = v^0(t)$.

$$T^{\mu\nu} \begin{cases} T^{00} = \rho(t) \\ T^{0i} = 0 \\ T^{i0} = 0 \\ T^{ij} = T^{(ij)} + T^{[ij]} \end{cases}$$

$T_k = \epsilon_{kij} T^{[ij]} = 0$

must be proportional to δ^{ij}. $T^{(ij)} = p(t)\,\delta^{ij}$

Thus any G-invariant second rank tensor $T^{\mu\nu}$ has to have $T^{00} = \rho(t)$, $T^{0i} = T^{i0} \equiv 0$, $T^{ij} = p(t)\delta^{ij}$, where $\rho = \rho(t)$ and $p = p(t)$ are only functions of time.

This, when written invariantly, i.e. only using n and g means that

$$T^{\mu\nu} = \rho(t) n^\mu n^\nu - p(t)(g^{\mu\nu} - n^\mu n^\nu). \quad (*)$$

<u>Now</u>: $\underset{n}{\mathcal{L}} g_{\mu\nu}$ is a second rank tensor, so it also must be of the form $(*)$.

Thus we have

$$\underset{n}{\mathcal{L}} g_{\mu\nu} = \alpha(t) n_\mu n_\nu - \beta(t)(g_{\mu\nu} - n_\mu n_\nu)$$

If we take X_i as vector fields dual to θ^i, we get $g_{ij} = g(X_i, X_j)$, and:

$$\underset{n}{\mathcal{L}} g_{ij} = -\beta(t) g_{ij} \qquad (\text{because } n_i \equiv 0)$$

but $n = \frac{\partial}{\partial t}$ so $\boxed{\frac{\partial g_{ij}}{\partial t} = -\beta(t) g_{ij}}$

This solves to:

$$g_{ij}(t) = g_{ij}(0) R^2(t)$$

And thus

$$\underline{g = dt^2 - R^2(t) g_{ij}(0) \theta^i \theta^j}$$

$$g = dt^2 - R^2(t) h \quad \text{where}$$

$\underline{h = g_{ij}(0)\theta^i\theta^j}$ is a Riemannian metric on each Σ.

7) Robertson - Walker metric

note that the orbits Σ have dimension $\dim\Sigma = 3$

To calculate the dimension of the full symmetry group G we use the fact that

$$\Sigma = G/H \quad , \text{ where } H \text{ is the isotropy subgroup of a point.}$$

But our isotropy assumption is that

$$H = SO(3).$$

So $\dim H = 3$

Thus

$$\underbrace{\dim\Sigma}_{3} = \dim G - \underbrace{\dim H}_{3}$$

$\Rightarrow$ $\underline{\dim G = 6}$

The group G is a symmetry group of the metric g, and since t - the cosmic time is a geometric quantity, it also preserves t.

~~Thus G preserves also~~

Since $g = dt^2 - R^2(t)\, h$ and g and t is preserved by G, also the metric h must be G-invariant.

But h - is a 3-dimensional metric and G is its symmetry group and has $\dim G = 6$, which is MAXIMAL for dimension 3.

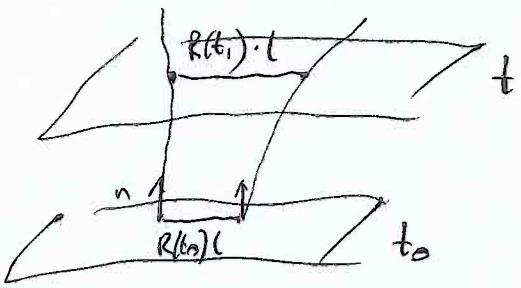
Thus $\underline{h \text{ must be}}$ a metric of constant $\underline{\text{curvature}}$!!

We have $h = g_{ij}(0)\theta^i\theta^j = \dfrac{dx^2 + dy^2 + dz^2}{\left(1 + \frac{k}{4}(x^2+y^2+z^2)\right)^2}$ and finally:

$$\Rightarrow \quad \boxed{g = dt^2 - R^2(t)\,\frac{dx^2+dy^2+dz^2}{\left(1+\frac{k}{4}(x^2+y^2+z^2)\right)^2}} \qquad k = \begin{cases} 0 \\ 1 \\ -1 \end{cases}$$

the most general metric which satisfy principle of spatial homogeneity and isotropy.

<u>Robertson-Walker</u> metric.



$\frac{R(t_1)}{R(t_0)}$ measures changes in spatial distance of points of fixed coordinates x, y, z from time t_0 to t_1

<u>$R(t)$ - scale factor.</u>

8) <u>Friedman models</u> <u>FLRW</u> The energy momentum tensor must be G invariant $\Rightarrow T^{\mu\nu} = \rho(t) n^\mu n^\nu - (g^{\mu\nu} - n^\mu n^\nu) p(t)$,

i.e. we fill the Universe with perfect fluid whose particles move with the Universe $u^\mu = n^\mu$.

Einstein equations:

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

reduce to only two equations <u>EXERCISE !</u>

$$\begin{cases} \frac{8\pi G}{3}\rho = \frac{\dot{R}^2 + k}{R^2} & (1) \\ \frac{8\pi G}{3}(\rho + 3p) = -2\,\frac{\ddot{R}}{R} & (2) \end{cases}$$

<u>Friedman 1924</u>

To close the system we have to add equation of state

$$p = p(\rho)$$

If cosmological constant

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \varkappa T_{\mu\nu}$$

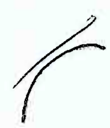
Exercise

$$\Rightarrow \begin{cases} \dfrac{8\pi G}{3}\rho = \dfrac{\dot{R}^2 + k}{R^2} + \Lambda \\ \dfrac{8\pi G}{3}(\rho + 3p) = -\dfrac{2\ddot{R}}{R} - \Lambda \end{cases}$$

Lemaître

$\Lambda \equiv 0$

$$\rho + 3p > 0 \Rightarrow \ddot{R} < 0 \Rightarrow$$



$\exists\, t_0$ s.t. $R(t_0) = 0 \Rightarrow$ initial singularity.

For dust: $p \equiv 0$

$$\ddot{R} = -\frac{1}{2}\frac{\dot{R}^2 + k}{R}$$

$$\frac{8\pi G}{3}\dot{\rho} = \left(\frac{\dot{R}^2 + k}{R^2}\right)^{\cdot} = \frac{2\dot{R}\ddot{R}}{R^2} - 2\frac{\dot{R}^2 + k}{R^3}\dot{R} =$$

$$= \frac{2\dot{R}}{R^2}\left(-\frac{1}{2}\frac{\dot{R}^2 + k}{R} - \frac{\dot{R}^2 + k}{R}\right) = -3\frac{\dot{R}}{R^2}\,\frac{\dot{R}^2 + k}{R} =$$

$$= -3\frac{\dot{R}}{R}\,\frac{8\pi G}{3}\rho$$

$$\Rightarrow \dot{\rho} + 3\frac{\dot{R}}{R}\rho = 0 \Rightarrow \dot{\rho}R + 3\dot{R}\rho = 0$$

$$(\rho R^3)^{\cdot} = 0$$

$$\Rightarrow \rho R^3 = \text{const}$$

$$\Rightarrow \frac{4}{3}\pi \rho R^3 = \text{const} := \underline{M}$$

inserting this in (1) we get

$$\frac{2MG}{R} = \dot{R}^2 + k \qquad \text{or}$$

$$\boxed{\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = -\frac{1}{2}k} \qquad \text{Friedman equation!}$$

(A) $k = 0 \Rightarrow \frac{dR}{dt} = \sqrt{\frac{2MG}{R}}$

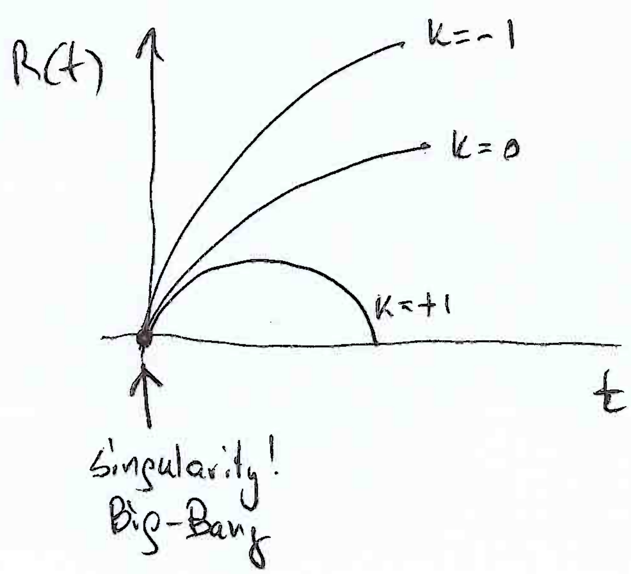
$$\Rightarrow R = \sqrt[3]{\frac{9MG}{2}}\, t^{2/3}$$

(B) $k = 1 \Rightarrow$

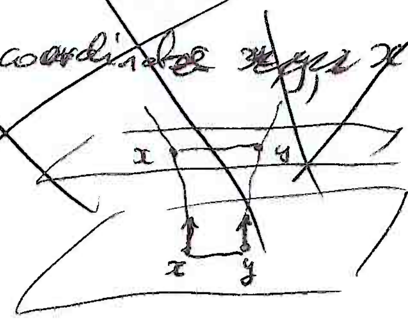
$$t = MG(\eta - \sin\eta)$$
$$R = MG(1 - \cos\eta)$$

(C) $k = -1 \Rightarrow$

$$t = MG(\sinh\eta - \eta)$$
$$R = MG(\cosh\eta - 1)$$



~~$\frac{R(t_1)}{R(t_0)}$ measures changes in spatial distance of points with fixed coordinates x, y from t_0 to t_1.~~



Exercise Derivation of the Friedman equations

$R(t)$ - scale factor.

$$T^{\mu\nu} = \rho(t) n^\mu n^\nu - (g^{\mu\nu} - n^\mu n^\nu) p(t)$$

$$u^\mu = n^\mu.$$

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\theta^0 = dt$$

$$\theta^i = R(t) \frac{dx^i}{1 + \frac{k}{4}(x^2+y^2+z^2)}$$

$$g = g_{\mu\nu} \theta^\mu \theta^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\cancel{d\theta^i = \frac{\dot R}{R} \theta^0 \wedge \theta^i}$$

$$d\theta^0 = 0$$

$$d\theta^1 = \frac{\dot R}{R} \theta^0 \theta^1 + \frac{k}{2R} \theta^1 (y\theta^2 + z\theta^3)$$

$$d\theta^2 = \frac{\dot R}{R} \theta^0 \theta^2 + \frac{k}{2R} \theta^2 (x\theta^1 + z\theta^3)$$

$$d\theta^3 = \frac{\dot R}{R} \theta^0 \theta^3 + \frac{k}{2R} \theta^3 (x\theta^1 + y\theta^2)$$

$$d\theta^i + \Gamma^i{}_j \wedge \theta^j = 0$$

$$\cancel{dg_{ij}} - \Gamma^k{}_i\, g_{kj} - \Gamma^k{}_j\, g_{ki} = 0$$

$$\Rightarrow \quad \Gamma^k{}_i = \begin{pmatrix} 0 & \frac{\dot R}{R}\theta^1 & \frac{\dot R}{R}\theta^2 & \frac{\dot R}{R}\theta^3 \\ \frac{\dot R}{R}\theta^1 & 0 & \frac{k}{2R}(x\theta^2 - y\theta^1) & \frac{k}{2R}(x\theta^3 - z\theta^1) \\ \frac{\dot R}{R}\theta^2 & & 0 & \frac{k}{2R}(y\theta^3 - z\theta^2) \\ \frac{\dot R}{R}\theta^3 & - & & 0 \end{pmatrix}$$

$$\Rightarrow \quad \Omega^i{}_j = d\Gamma^i{}_j + \Gamma^i{}_k \Gamma^k{}_j =$$

$$= \begin{pmatrix} 0 & \frac{\ddot R}{R}\theta^0\theta^1 & \frac{\ddot R}{R}\theta^0\theta^2 & \frac{\ddot R}{R}\theta^0\theta^3 \\ + & 0 & \frac{(k+\dot R^2)}{R^2}\theta^1\theta^2 & \frac{k+\dot R^2}{R^2}\theta^1\theta^3 \\ & & 0 & \frac{k+\dot R^2}{R^2}\theta^2\theta^3 \\ & - & & 0 \end{pmatrix}$$

$$\Omega^i{}_j = \tfrac{1}{2} R^i{}_{jkl}\,\theta^k\theta^l$$

$$R_{jl} = R^i{}_{jil}$$

$$R_{ij} = \begin{pmatrix} -\frac{3\ddot R}{R} & & & 0 \\ & \frac{2(k+\dot R^2)+R\ddot R}{R^2} & & \\ & & \frac{2(k+\dot R^2)+R\ddot R}{R^2} & \\ 0 & & & \frac{2(k+\dot R^2)+R\ddot R}{R^2} \end{pmatrix}$$

$$G_{\mu\nu} = \begin{pmatrix} \frac{3(k+\dot R^2)}{R^2} & & & \\ & -\frac{2R\ddot R+\dot R^2+k}{R^2} & & \\ & & \ddots & \\ & & & \end{pmatrix}$$

$$G_{00} + \Lambda = 8\pi G \rho \qquad T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

$$G_{ij} - \Lambda \delta_{ij} = 8\pi G p \,\delta_{ij}$$

$$\Rightarrow \quad \frac{8\pi G \rho}{3} = \frac{k+\dot R^2}{R^2} + \Lambda$$

$$8\pi G p = -\frac{k+\dot R^2}{R^2} - \frac{2R\ddot R}{R^2} - \Lambda$$

$$\boxed{\begin{aligned} \frac{8\pi G\rho}{3} &= \frac{k+\dot R^2}{R^2} + \Lambda \quad (1) \\ \frac{8\pi G}{3}(\rho+3p) &= -\frac{2\ddot R}{R} - \Lambda \quad (2) \end{aligned}}$$

Friedman-Lemaitre

should be suplemented by $p = p(\rho)$

Dust $p = 0$

$$-\frac{2R\ddot R}{R^2} - \Lambda = \frac{k+\dot R^2}{R^2} + \Lambda$$

$$\boxed{\ddot R = -\frac{1}{2}\frac{\dot R^2+k}{R} - \Lambda R}$$

$$\frac{8\pi G\dot\rho}{3} = \left(\frac{\dot R^2+k}{R^2}\right)^{\cdot} = \frac{2\dot R\ddot R R^2 - 2R\dot R(\dot R^2+k)}{R^4} =$$

$$= \frac{2\dot R\left(-\frac{1}{2}\frac{\dot R^2+k}{R} - \Lambda R\right)R^2 - 2R\dot R(\dot R^2+k)}{R^4} =$$

$$= \frac{-3(R\dot R^3 + \dot R R k) - 2\Lambda R^3\dot R}{R^4} = -\frac{3\dot R}{R}\left(\frac{8\pi G\rho}{3} - \Lambda\right)$$

Dust $\begin{cases} p = 0 \\ \Lambda = 0 \end{cases}$

$$\Rightarrow \quad \ddot{R} = -\frac{1}{2}\,\frac{\dot{R}+k}{R}$$

$$\frac{8\pi G\dot{\rho}}{3} = \left(\frac{\dot{R}^2+k}{R^2}\right)^{\cdot} = \quad \overset{(1)}{=} -\frac{3\dot{R}}{R}\,\frac{8\pi G}{3}\rho$$

$$\Rightarrow \quad \dot{\rho}R + 3\rho\dot{R} = 0$$

$$\Rightarrow \quad (\rho R^3)^{\cdot} = 0$$

$$\Rightarrow \quad \boxed{\rho R^3 = \text{const}}$$

$$\Rightarrow \quad \frac{4}{3}\pi\rho R^3 = \underline{M} = \text{const.}$$

Back to (1):

$$\boxed{\frac{1}{2}\dot{R}^2 - \frac{MG}{R} = -\frac{1}{2}k} \quad \text{Friedmann equation.}$$

(A) $k = 0 \quad \Rightarrow \quad \frac{dR}{dt} = \sqrt{\frac{2M}{R}}$

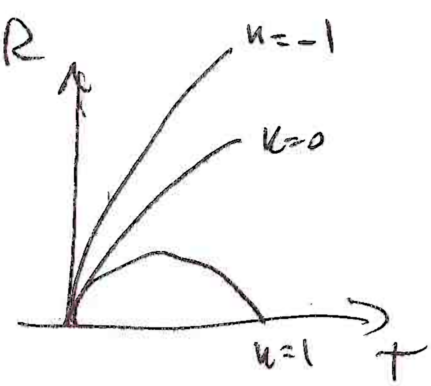
$$R = \sqrt[3]{\frac{9MG}{2}}\; t^{2/3}$$

(B) $k = 1 \quad \Rightarrow \quad t = MG(\eta - \sin\eta)$

$$R = MG(1-\cos\eta)$$

(C) $k = -1 \quad \Rightarrow \quad t = MG(\operatorname{sh}\eta - \eta)$

$$R = MG(\operatorname{ch}\eta - 1)$$



9) Friedman models and topology

$$g = dt^2 - R^2(t)\,\frac{dx^2+dy^2+dz^2}{\left(1+\frac{k}{4}(x^2+y^2+z^2)\right)^2}$$

	Space		
$k=0$	$\mathbb{R}^3$		$G = E(3) = \mathbb{R}^3 \rtimes SO(3)$
$k=1$	$\mathbb{S}^3$		$G = O(4)$
$k=-1$	$\mathbb{H}^3$		$G = O(1,3)^{\uparrow}$

Def

Riemannian manifold is geodesically complete if one can prolong the values of an affine parameter on each geodesic up to $+\infty$.

Thm Hopf - Rinow

In Riemannian signature

$$\left(\begin{array}{c}\text{geodesic}\\ \text{completeness}\end{array}\right) \Leftrightarrow \left(\text{completeness}\right)$$

Thm Killing-Hopf

Every connected, geodesically complete Riemannian manifold (positive definite metric!) of constant curvature is one of the following:

$k=0 \implies \mathbb{R}^n/\Gamma \qquad \Gamma \subset E(n)$

$k=1 \implies \mathbb{S}^n/\Gamma \qquad \Gamma \subset O(n+1)$

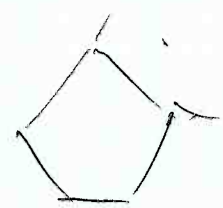
$k=-1 \implies \mathbb{H}^n/\Gamma \qquad \Gamma \subset O(1,n)^{\uparrow}$

where Γ acts freely (each $a \neq e$ moves every point) and discretely $\leftarrow$ $\forall_p \exists \mathcal{U}: \{\gamma \in \Gamma \;\; \gamma(\mathcal{U}) \cap \mathcal{U} \neq \phi\}$ is finite

If $k=1 \Rightarrow S^n/\Gamma$ is compact

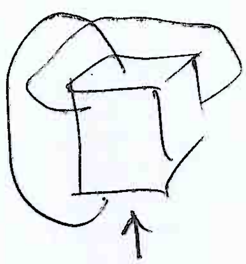
If $k=0,-1 \Rightarrow E^n/\Gamma,\ H^n/\Gamma$ may be noncompact
BUT may also be COMPACT!

E.g. Dodecahedron



with opposite faces identified and rotated by different multiplicities of 72° can be either S^n/Γ or H^n/Γ.

In both cases these spaces are compact.
Poincare-model
Seifert-Weber model



flat $k=0$, compact model.

10) Measurable ~~par~~ cosmological parameters

$$\left.\begin{aligned} \frac{8\pi G}{3}\rho &= \frac{\dot R^2 + k}{R^2} \\ 8\pi G p &= -\frac{2\ddot R}{R} - \frac{\dot R^2 + k}{R^2} \end{aligned}\right\} \Rightarrow \underline{\underline{\frac{8\pi G}{3}\rho}} = -\frac{2\ddot R}{R} = \underline{\underline{2qH^2}}$$

$8\pi G p$ $\parallel$ 0 now!

1) Hubble "constant" $H = \frac{\dot R}{R}$

2) deceleration parameter $q = -\frac{\ddot R R}{\dot R^2} \Rightarrow -\frac{\ddot R}{R} = qH^2$

3) Critical density

$$\frac{8\pi G}{3}\rho_c = H^2 \quad \Rightarrow \quad \frac{8\pi G}{3}\rho = 2q\,\frac{8\pi G}{3}\rho_c$$

$$\rightarrow \quad \frac{k}{R^2} = \frac{8\pi G}{3}\rho - H^2 = \frac{8\pi G}{3}(\rho - \rho_c) = \frac{8\pi G}{3}\rho_c(2q-1)$$

$$\mathrm{sgn}(k) = \mathrm{sgn}(\rho-\rho_c) = \mathrm{sgn}(2q-1)$$

<u>Observations</u>

Density parameter

$$\Omega = \frac{\rho}{\rho_c} = \cancel{\;\;} = 2q$$

$$\frac{1}{H} \simeq 13.7 \cdot 10^9 \text{ years}$$

$$\rho_c \sim 5 \cdot 10^{-30}\ g/cm^3$$

$$\rho \sim 2 \div 6 \cdot 10^{-31}\ g/cm^3$$

- barion 4%
- 23% dark matter
  - mater made of particles unknown to physics
- 74% cosmological constant!
  expansion is quicker now than it was!