GR Lecture 2 22.02.2011

X - vector field on M

φ_t - 1-parameter group of transformations associated with it.

It is a) local ~~~~ in U around p
b) local in time t around 0.

Explicitely:

$$\boxed{X(f)\circ\varphi_t = \frac{d}{dt} f\circ\varphi_t}$$

Taking $f = x^\mu : U \longrightarrow \mathbb{R}^1$ μ-th component:

$$X^\mu\left(x^\rho(\varphi_t(p))\right) = \frac{d}{dt} x^\mu(\varphi_t(p))$$

$x^\rho(\varphi_t(p)) = x_t^\rho$ - coordinates of the map φ_t

$$U \xrightarrow{\varphi_t} U \xrightarrow{x} \mathbb{R}^n$$

$$x\circ\varphi_t = x_t$$

$$\boxed{\frac{d}{dt} x_t^\mu = X^\mu(x_t^\rho)}$$

Examples

1) $Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ in $\mathbb{R}^3$.

What is its 1-par. group of transformations?

$$\begin{cases} \frac{d}{dt} x_t = -y_t \\ \frac{d}{dt} y_t = x_t \qquad\qquad \Big| i \\ \frac{d}{dt} z_t = 0 \quad \Longrightarrow \quad z_t = z_0 \end{cases}$$

$$\frac{d}{dt}(x_t + iy_t) = i\,(x_t + iy_t) \Rightarrow x_t + iy_t = e^{it}(x_0 + iy_0)$$

$$x_t + i y_t = \cos t\, x_0 - \sin t\, y_0 + i\left(\sin t\, x_0 + \cos t\, y_0\right)$$

$$\begin{cases} x_t = \cos t\, x_0 - \sin t\, y_0 \\ y_t = \sin t\, x_0 + \cos t\, y_0 \\ z_t = z_0 \end{cases}$$

$$\varphi_t(p) = \left(\cos t\, x_0 - \sin t\, y_0,\ \sin t\, x_0 + \cos t\, y_0,\ z_0\right)$$

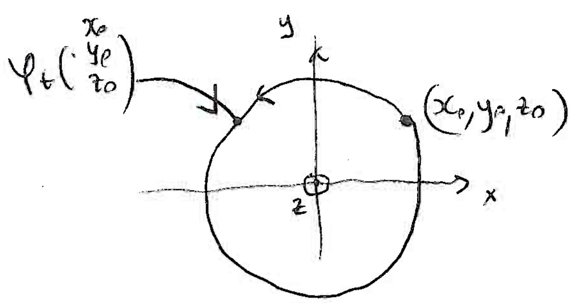
$$\varphi_0(p) = (x_0, y_0, z_0) = p$$

$$\varphi_t\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$\Rightarrow \quad \varphi_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mathbb{R}^3 \xrightarrow[\text{diff}]{} \mathbb{R}^3$$

$\gamma(t) = \varphi_t(p)$ – curve!

integral curve of a vector field X passing through p.



2) $X = \partial_x - \frac{1}{x^2}\partial_y$ on $\mathbb{R}_+ \setminus \{0\} \times \mathbb{R}$

$$\frac{d}{dt} x_t = 1 \quad \Rightarrow \quad x_t = t + x_0$$

$$\frac{d}{dt} y_t = -\frac{1}{x_t^2} \qquad \frac{d}{dt} y_t = -\frac{1}{(t+x_0)^2}$$

$$y_t = \frac{1}{t + x_0} + c \qquad y_0 = \frac{1}{x_0} + c$$

$$\varphi_t(x_0, y_0) = \left(t + x_0, \frac{1}{t+x_0} + y_0 - \frac{1}{x_0}\right)$$

we now see that $\varphi_t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is <u>not</u> linear.
also: it is only defined for t small enough; otherwise $\frac{1}{t+x_0}$ blows up.

Observe that everywhere where φ_t is defined we have:
$\varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$ as it should be!

9) <u>Local 1-parameter groups of transformations of M</u>

$\forall p \in M \quad \exists \underset{\substack{\uparrow \\ \text{open}}}{U_p}, \ \varepsilon > 0$:

$$\exists \,]-\varepsilon, \varepsilon[\times U_p \xrightarrow{\varphi_t} U_p$$
$$(t, p) \xmapsto{\text{diff}} \varphi_t(p)$$
$$\varphi_{t+t'} = \varphi_t \circ \varphi_{t'} \quad \forall\, t, t' \text{ s.t. } |t|, |t'|, |t+t'| < \varepsilon.$$

10) <u>Integral curve of X passing through p</u>

φ_t - local 1-par. group of transf associated with X

$t \longrightarrow \gamma(t) = \varphi_t(p)$ - curve passing through p
$\uparrow$
Integral curve of X passing through p

Sometimes φ_t is called a FLOW of X.

It is a solution of the equation

$$\frac{d\gamma}{dt} = X_{\gamma(t)} \quad \text{s.t. } \gamma(0) = p.$$

<u>Fact</u>
Given X - a smooth vector field on M there exists $\varepsilon > 0$ and a unique curve $\gamma(t) = \varphi_t(p)$ s.t. $\frac{d\gamma}{dt} = X_{\gamma(t)}$, $\gamma(0) = p$.

Example

1) Check that commutator of 2 vector fields is a vector field

$$[X,Y](\alpha f_1+\beta f_2) \underset{\substack{\uparrow \\ \text{obviously}}}{=} \alpha [X,Y](f_1) + \beta [X,Y](f_2) \quad \forall \alpha,\beta\in\mathbb{R}$$

$$[X,Y](f\cdot g) = X(Y(f\cdot g)) - Y(X(f\cdot g)) =$$
$$= X\left[Y(f)\cdot g + f\cdot Y(g)\right] - Y(X(f)g + f\cdot X(g)) =$$
$$= X(Y(f))\,g + \cancel{Y(f)\cdot X(g)} + \cancel{X(f)Y(g)} + f\,X(Y(g)) +$$
$$- Y(X(f))\,g - \cancel{X(f)Y(g)} - \cancel{Y(f)X(g)} - f\,Y(X(g))$$

$$= [X,Y](f)\cdot g + f\cdot [X,Y](g) \qquad \text{ok.}$$

2) In local coordinates:

$$X = X^\mu \partial_\mu \quad , \qquad Y = Y^\nu \partial_\nu$$

$$[X,Y](f) = [X^\mu\partial_\mu, Y^\nu \partial_\nu](f) =$$
$$= X^\mu \partial_\mu (Y^\nu f_{,\nu}) - Y^\nu\partial_\nu (X^\mu f_{,\mu}) =$$
$$= X^\mu Y^\nu_{,\mu} f_{,\nu} + \cancel{X^\mu Y^\nu f_{,\nu\mu}} - Y^\nu X^\mu_{,\nu} f_{,\mu} - \cancel{Y^\nu X^\mu f_{,\mu\nu}} =$$
$$= X^\mu Y^\nu_{,\mu} \frac{\partial f}{\partial x^\nu} - Y^\mu X^\nu_{,\mu} \frac{\partial f}{\partial x^\nu} =$$
$$= (X^\mu Y^\nu_{,\mu} - Y^\mu X^\nu_{,\mu})\,\partial_\nu(f)$$

$$\Rightarrow \boxed{[X,Y] = \underbrace{(X^\mu Y^\nu_{,\mu} - Y^\mu X^\nu_{,\mu})}\,\partial_\nu}$$

components ~~coordinates~~ of $[X,Y]$ if components ~~coordinates~~ of X and Y are respectively X^μ, Y^ν.

11) Properties of the Commutator:

1° $[\cdot,\cdot]$ – bilinearity

2° $[X,Y] = -[Y,X]$ – antisymmetry

3° $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$ Jacobi identity

$(\mathfrak{X}(M), [\cdot,\cdot])$ Lie algebra of smooth vector fields on M

$[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$
is considered as a multiplication

Lie algebra over $\mathbb{R}$ – INFINITE dimensional.

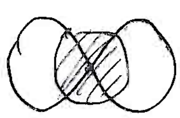
np. $\partial_x, x\partial_x, x^2\partial_x, \ldots$ are linearly indep. over $\mathbb{R}$.

4° $[fX, gY] = f\cdot g[X,Y] + f\cdot X(g)Y - gY(f)X$

Usefullness of commutator: $\boxed{\text{Frobenius theorem.}}$

Recall: N is a submanifold of M (of dim n) iff

$\forall p \in N \ \exists (\mathcal{U}, x) \in \mathcal{A}_M$ s.t.
1) $p \in \mathcal{U}$
2) $x(N \cap \mathcal{U}) \subset \mathbb{R}^m \times \{\underbrace{0,0,\ldots,0}_{n-m}\}$
3) $pr_{\mathbb{R}^m} x(N \cap \mathcal{U})$ is open in $\mathbb{R}^m \times \{0,\ldots,0\}$

e.g.  is NOT a submanifold of $\mathbb{R}^2$

 $\notin \mathbb{R}^1 \times \{0\}$

12) <u>Distribution S of dimension m on M</u>

$$M \ni p \xmapsto{S} S_p \subset T_p(M)$$

↖ vector subspace of dim m.

if

$\forall p \in N \; \exists \, \mathcal{U}$ open of p $\exists \{X_i\}_{i=1,\dots m}$ smooth v. fields on $\mathcal{U}$ s.t.

$\forall q \in \mathcal{U} \quad \left(X_i|_q\right)_{i=1,\dots m}$ is a basis for S_q

then S is a <u>smooth distribution</u> of dim m on M

13) Def <u>X belongs to S</u> iff $\forall p \; X_p \in S_p$

14) Def <u>S is involutive</u> $\iff \left(X, Y \in S \implies [X,Y] \in S\right)$

15) <u>Integral manifold N_S of distribution S.</u>

$N_S \subset M$ s.t. $\forall p \in N_S \quad T_p(N_S) = S_p$.

↑ submanifold

<u>Do they exist?</u>

Ex 1) X - vector field. Defines a distribution $S = \{f \cdot X, \text{ s.t. } f \in \mathcal{F}(M)\}$

i.e. $S_p = \mathbb{R} \cdot X_p$ Distribution of dim 1.

Integral manifolds? $\implies$ integral curves of X.

$$\frac{d\gamma}{dt} = X_\gamma.$$

Ex 2)

$M = \mathbb{R}^3$ $\qquad Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$S = \mathrm{Span}_{\mathcal{F}(M)}\{Y, X\}.$$

Note that
$\mathbb{R}^3 \supset S^2_r = \{(x,y,z) : x^2+y^2+z^2 = r^2\}$ are integral manifolds of S.

Indeed:

normal vector to S^2_r: $\mathrm{grad}(x^2+y^2+z^2-r^2) = (2x, 2y, 2z)$

$N = x\partial_x + y\partial_y + z\partial_z$ and

$\underset{zx-xz}{(N|X)} = \underset{yz-zy}{(N|Y)} = 0 \implies X_p$ and Y_p are tangent to S^2_r

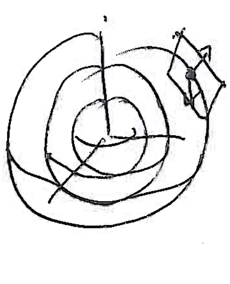
p s.t. $x^2+y^2+z^2 = r^2$

$\dim(T_p(S^2_r)) = 2$, X_p, Y_p linearly indep.

$\implies$

$N_S = S^2_r \implies$ foliation of $\mathbb{R}^3$ by 2-dimensional concentric spheres,

which are integral manifolds of S.



Note that $X, Y, Z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ are:

1) such that $Z \in S$ because:

$$zZ + yY + xX = 0.$$

2) X, Y, Z are linearly indep at each point over $\mathbb{R}$.

16) Fröbenius theorem

$$\left(S \text{ involutive} \right) \Longleftrightarrow \left(\forall p \in M \text{ we have a unique maximal integral manifold of } S \right)$$

Proof is difficult in one direction, $(\Rightarrow)$
In direction $\Leftarrow$ it is obvious.

Corollary

if S is involutive of dim m one can introduce a coordinate system (x^A) around each $p \in M$ s.t.

$(x^i) = (x^1, \dots x^m)$ – coordinates on integral manifolds

$(x^\alpha) = (x^{m+1}, \dots x^n)$ – coordinates enumerating integral manifolds.

$A = 1, \dots m, \dots n$
$i = 1, \dots m$
$\alpha = m+1, \dots n$

$\Rightarrow$ every vector field X belonging to the distribution S is of the form

$$X_i = A_i{}^j(x^A)\frac{\partial}{\partial x^j} \qquad (*)$$

In other words in this coordinate system a local frame for S can be written in the form $(*)$ with $A_i{}^j(x^A)$, certain smooth functions.

17) Corollary $\{X_i\}$ m linearly independent vector fields in $\mathcal{U}$ (i.e. at each point $p \in \mathcal{U}$ $\{X_i|_p\}$ are linearly indep)

$$[X_i, X_j] = 0 \Longleftrightarrow \exists\, (x^A) \text{ coordinate system in } \mathcal{U} \text{ s.t. } X_i = \frac{\partial}{\partial x^i}$$

Proof

$\Leftarrow$ obvious.