$\Rightarrow$ since $[X_i, X_j] = 0 \Rightarrow$ distribution $\mathrm{Span}\{X_i\} = \mathcal{D}$ is integrable and Frobenius thm says that there exists coordinate system $\{y^A\}$ s.t.

$$X_i = A_i{}^j(y^A)\frac{\partial}{\partial y^j}$$

Matrix $A_i{}^j$ is INVERTIBLE at each point y^A on M since X_i are l. independent (and $\frac{\partial}{\partial y^i}$ is a basis at each point for $\mathcal{D}$)

$\Rightarrow \exists\ B_i{}^j = B_i{}^j(y^A)$ s.t.

$$A_i{}^j B_j{}^k = \delta_i{}^k = B_i{}^j A_j{}^k \qquad (1)$$

We are now looking for m functions

$$x^i = x^i(y^A) \quad \text{s.t.}$$

$$\frac{\partial x^i}{\partial y^j} = B_j{}^i$$

Do they exist?

Yes, provided that

$$\frac{\partial^2 x^i}{\partial y^k \partial y^j} = \frac{\partial^2 x^i}{\partial y^j \partial y^k} \quad \text{i.e. iff} \quad \boxed{B_j{}^i{}_{,k} = B_k{}^i{}_{,j}} \ (*)$$

To check that $(*)$ is satisfied we use the condition that $[X_i, X_j]$ commute:

$$0 = [A_i{}^j \partial_{y^j}, A_k{}^l \partial_{y^l}] = (A_i{}^j A_k{}^l{}_{,j} - A_k{}^j A_i{}^l{}_{,j})\partial_{y^l}$$

i.e. $\boxed{A_i{}^j A_k{}^l{}_{,j} = A_k{}^j A_i{}^l{}_{,j}}$

Now we differentiate (1)

$$0 = B_i{}^j{}_{,l} A_j{}^k - B_i{}^j A_j{}^k{}_{,l}$$

$$\Rightarrow \quad B_i{}^j{}_{,l} A_j{}^k = B_i{}^j A_j{}^k{}_{,l}$$

$$B_i{}^j{}_{,l} A_j{}^k B_k{}^m = B_i{}^j A_j{}^k{}_{,l} B_k{}^m$$

$$\Rightarrow \quad B_i{}^m{}_{,l} = B_i{}^j A_j{}^k{}_{,l} B_k{}^m$$

We need antisymmetrization of this in i, l indices:

$$B_i{}^m{}_{,l} - B_l{}^m{}_{,i} = \left(B_i{}^j A_j{}^k{}_{,l} - B_l{}^j A_j{}^k{}_{,i}\right) B_k{}^m$$

But $A_s{}^i\left(B_i{}^j A_j{}^k{}_{,l} - B_l{}^j A_j{}^k{}_{,i}\right) =$

$$= A_s{}^k{}_{,l} - B_l{}^j A_j{}^k{}_{,i} A_s{}^i =$$

$$= A_s{}^k{}_{,l} - B_l{}^j A_j{}^i A_s{}^k{}_{,i} = A_s{}^k{}_{,l} - A_s{}^k{}_{,l} = 0$$

$\Rightarrow$ because $A_s{}^i$ is invertible

$B_i{}^j A_j{}^k{}_{,l} - B_l{}^j A_j{}^k{}_{,i} = 0$ and consequently

$B_i{}^m{}_{,l} - B_l{}^m{}_{,i} = 0$.

$\Longrightarrow$ The Equation $\dfrac{\partial x^i}{\partial y^j} = B_j{}^i(y^A)$ for $x^i = x^i(y^A)$ has solutions

$\Rightarrow$ take $\quad x^i = x^i(y^A) \qquad i = 1, \dots m$

$\qquad\qquad x^\alpha = y^\alpha \qquad \alpha = m+1, \dots, n$

The set of functions

$$x^A = (x^i(y^A), x^\alpha(y^A))$$ is a cordinate system on M

since

$$\frac{\partial x^A}{\partial y^B} = \begin{pmatrix} B^i{}_j & \text{smth} \\ 0 & \mathbb{1} \end{pmatrix}, \quad \det\left(\frac{\partial x^A}{\partial y^B}\right) \neq 0.$$

In this coordinate system:

$$X_i = A_i{}^j \frac{\partial}{\partial y^j} = A_i{}^j \frac{\partial x^A}{\partial y^j} \partial_{x^A} = A_i{}^j \frac{\partial x^k}{\partial y^j} \partial_{x^k} + \cancel{A_i{}^j \frac{\partial x^\alpha}{\partial y^j} \partial_{x^\alpha}}$$

$$= A_i{}^j B_j{}^k \partial_{x^k} = \frac{\partial}{\partial x^i}$$

$\square$.

② Tensors and tensor fields

1) Tensors i.e. multilinear maps.

V - vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\dim_{\mathbb{K}} V = n < \infty$

V^* - its dual:

$$V^* = \{ \lambda : V \xrightarrow{\text{linear}} \mathbb{K} \}$$

Given a pair $(v, \lambda) \in V \times V^*$ we have a bilinear map

$$\langle \cdot , \cdot \rangle : V \times V^* \longrightarrow \mathbb{K}$$
$$(v, \lambda) \longmapsto \langle v, \lambda \rangle := \lambda(v)$$

(obviously linear in the first argument, since λ is linear; linearity in the second argument by
$$(v, \alpha\lambda + \beta\mu) \longmapsto \langle v, \alpha\lambda + \beta\mu \rangle = (\alpha\lambda + \beta\mu)(v) = \alpha\lambda(v) + \beta\mu(v)$$
$\forall \alpha, \beta \in \mathbb{K}$).

Definition

The space V^r_s of s-times covariant and r-times contravariant tensors is the space of all maps

$$T : \underbrace{V \times V \times \ldots \times V}_{s\text{-times}} \times \underbrace{V^* \times \ldots V^*}_{r\text{-times}} \longrightarrow \mathbb{K}$$

which are linear in $\boxed{\text{each}}$ of its $(s+r)$ arguments.

Remarks

1° V^r_s denoted also by $V^r_s = \underbrace{V \otimes \ldots V}_{r} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s} = \overset{r}{\otimes} V \overset{s}{\otimes} V^*$

is a vector space over $\mathbb{K}$:

$$(\alpha T_1 + \beta T_2)(v_1, \ldots v_s, \lambda_1, \ldots \lambda_r) = \alpha T_1(v_1, \ldots v_s, \lambda_1, \ldots \lambda_r) + \beta T_2(v_1, \ldots v_s, \lambda \ldots \lambda$$

2° $\boxed{V^1_0 = V \; ; \; V^0_1 = V^*}$ indeed:

$$V^1_0 = V \ni v : V^* \xrightarrow[\text{linear}]{} \mathbb{K}$$
$$\lambda \longmapsto \langle v, \lambda \rangle$$

similarly:

$$V^0_1 = V^* \ni \lambda : V \xrightarrow[\text{linear}]{} \mathbb{K}$$
$$v \longmapsto \langle v, \lambda \rangle$$

3° $\boxed{\dim V^r_s = n(r+s)}$ indeed:

$e = \{e_\mu\}_{\mu=1,\dots n}$ basis in V.

Then: $\omega = \{\omega^\mu\}_{\mu=1,\dots n}$ defined by

$$\omega^\mu \in V^* \quad \text{and} \quad \omega^\mu(e_\nu) = \langle e_\nu, \omega^\mu \rangle = \delta^\mu_\nu$$

is a basis in V^*.

Define:

$$e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes \omega^{\nu_1} \otimes \dots \otimes \omega^{\nu_s} \in V^r_s \quad \text{by}$$

$$(e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes \omega^{\nu_1} \otimes \dots \otimes \omega^{\nu_s})(e_{\alpha_1}, \dots e_{\alpha_s}, \omega^{\beta_1}, \dots \omega^{\beta_r}) :=$$

$$= \langle e_{\mu_1}, \omega^{\beta_1} \rangle \dots \langle e_{\mu_r}, \omega^{\beta_r} \rangle \langle e_{\alpha_1}, \omega^{\nu_1} \rangle \dots \langle e_{\alpha_s}, \omega^{\nu_s} \rangle =$$

$$\left(= \delta_{\mu_1}{}^{\beta_1} \dots \delta_{\mu_r}{}^{\beta_r} \, \delta_{\alpha_1}{}^{\nu_1} \dots \delta_{\alpha_s}{}^{\nu_s} \right)$$

+ linearity in each argument.

<u>Fakt</u>: $\{e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes \omega^{\nu_1} \otimes \dots \otimes \omega^{\nu_s}\}_{\substack{\mu_i = 1, \dots n; \; i = 1 \dots r \\ \nu_j = 1, \dots n; \; j = 1 \dots s}}$

is a <u>basis</u> in V^r_s.

Thus $V^r_s \ni T = T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \, e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes \omega^{\nu_1} \otimes \dots \otimes \omega^{\nu_s}$

e.g. $\underset{V^1_1 \ni}{\langle \cdot, \cdot \rangle} = e_\alpha \otimes \omega^\alpha$ (check!)

4°. Change of bases

$V \ni v = v^\mu e_\mu$

$V^* \ni \lambda = \lambda_\mu \omega^\mu$

Let $a \in GL(n, \mathbb{R})$; $a = (a^\mu{}_\nu)$

Suppose that e_μ is related to e'_μ by
$$e_\mu = e'_\nu a^\nu{}_\mu \quad \text{i.e. by} \quad e'_\mu = e_\nu a^{-1\nu}{}_\mu ,$$
how ω^μ is related to ω'^μ ?

Of course ω'^μ is linearly related to ω^μ so that
$$\omega'^\mu = b^\mu{}_\nu \omega^\nu . \quad \text{We have to find } b^\mu{}_\nu .$$

We have:
$$\delta_\mu{}^\nu = \langle e'_\mu, \omega'^\nu \rangle = \langle e_\alpha a^{-1\alpha}{}_\mu , b^\nu{}_\beta \omega^\beta \rangle =$$
$$= a^{-1\alpha}{}_\mu b^\nu{}_\beta \langle e_\alpha, \omega^\beta \rangle = a^{-1\alpha}{}_\mu b^\nu{}_\beta \delta_\alpha{}^\beta =$$
$$= a^{-1\alpha}{}_\mu b^\nu{}_\alpha \qquad \Big| \cdot a^\mu{}_\varrho$$
$$\Rightarrow \quad a^\nu{}_\varrho = b^\nu{}_\varrho \quad \Rightarrow \quad \boxed{b^\nu{}_\varrho = a^\nu{}_\varrho}$$

Thus
$$\boxed{e'_\mu = e_\nu a^{-1\nu}{}_\mu \iff \omega'^\mu = a^\mu{}_\nu \omega^\nu} \qquad a \in GL(n, \mathbb{R}).$$

5° Transformation of components

$$v = v^\mu e_\mu = v'^\mu e'_\mu = v^\alpha c^\mu{}_\alpha \cdot e_\varrho a^{-1\varrho}{}_\mu =$$
$$= \underbrace{c^\mu{}_\alpha a^{-1\varrho}{}_\mu}_{=\delta_\alpha{}^\varrho} v^\alpha e_\varrho$$

$$\Rightarrow \quad c^\mu{}_\alpha = a^\mu{}_\alpha$$

$$\Rightarrow \quad \boxed{e'_\mu = e_\nu a^{-1\nu}{}_\mu \Rightarrow v'^\mu = a^\mu{}_\alpha v^\alpha}$$

Similarly $\lambda = \lambda_\mu \omega^\mu$ then $\Rightarrow$ $\boxed{\lambda'_\mu = \lambda_\nu a^{-1\nu}{}_\mu}$

Components of vectors transform as bases 1-forms

Components of covectors transform as basis vectors.

6° <u>More generally:</u> $T \in V^r_s$

$$T = T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\, e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes \omega^{\nu_1} \otimes \dots \otimes \omega^{\nu_s}$$

· If we change a basis:

$\{e_\mu\} \longmapsto \{e'_\mu\}$ s.t.

(*) $\boxed{e'_\mu = e_\nu a^{-1\nu}{}_\mu}$ then the compnents of T change as

$$(**)\quad \boxed{\begin{aligned} &T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \longmapsto \\ &T'^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s} = a^{\alpha_1}{}_{\mu_1} \dots a^{\alpha_r}{}_{\mu_r} T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} a^{-1\nu_1}{}_{\beta_1} \dots a^{-1\nu_s}{}_{\beta_s} \end{aligned}}$$

This is a classical PHYSICISTS definition of $\binom{r}{s}$-type tensor:

It is an equivalence class of pairs:

$(e_\mu, T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s})$ s.t.

$(e_\mu, T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}) \sim (e'_\mu, T'^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}) \Leftrightarrow$

$\Leftrightarrow \exists a \in GL(n, \mathbb{R})$ s.t.

(e', T') is related to (e, T) via (*) & (**).

<u>Let us formalize this a bit more:</u>

(3) Cartan's formalism of vector valued forms

1) Action of a group G in a set S.

$$G \times S \ni (g, s) \xmapsto{\varphi} \varphi_g(s) \in S \quad \text{s.t.}$$

1° $\varphi_e = id_S$

2° $\varphi_{a\cdot b} = \varphi_a \circ \varphi_b$ ||| 2° $\varphi_{a\cdot b} = \varphi_b \circ \varphi_a$

left action — right action

~~Remarks~~ Ex 1 1-par. group of transformations of M: $G = \mathbb{R}$, $S = M$:

$$\mathbb{R} \times M \ni (t, s) \xmapsto{\varphi} \varphi_t(s) \in M$$

$$\varphi_0 = id_M$$

$$\varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$$

~~Remarks~~ Ex 2 $G = SO(3)$, $S = \mathbb{R}^3$

$$SO(3) \times \mathbb{R}^3 \ni (a, \vec{v}) \Rightarrow a \cdot \vec{v} = \varphi_a(\vec{v}) \in \mathbb{R}^3$$

$$\varphi_{\mathbb{1}}(\vec{v}) = \mathbb{1}\vec{v} = \vec{v} \quad \Rightarrow \quad \varphi_{\mathbb{1}} = id_{\mathbb{R}^3}$$

$$\varphi_{a\cdot b}(\vec{v}) = \cancel{\varphi}(a\cdot b)(\vec{v}) = a(b(\vec{v})) = \varphi_a(\varphi_b(\vec{v}))$$

$$\varphi_{ab} = \varphi_a \circ \varphi_b.$$

Remarks

1) φ_a is a bijection

$$\varphi_a \circ \varphi_{a^{-1}} = \varphi_{a\cdot a^{-1}} = \varphi_e = id_S$$

$$\Rightarrow (\varphi_a)^{-1} = \varphi_{a^{-1}}$$

2) if φ is a right action ($\varphi_{ab} = \varphi_b \circ \varphi_a$)

$\Rightarrow$ We consider ψ:

$$\psi_a = \varphi_{a^{-1}} \quad \Rightarrow \quad \psi \text{ is a left action}$$

Indeed:

$$\psi_{ab} = \varphi_{(ab)^{-1}} = \varphi_{b^{-1}a^{-1}} = \cancel{\ldots}$$

$$= \varphi_{a^{-1}} \circ \varphi_{b^{-1}} = \psi_a \circ \psi_b.$$

2) G <u>acts transitively</u> on $S \Leftrightarrow \forall\, p,q \in S \;\exists\, a \in G$ s.t. $q = \varphi_a(p)$

<u>Ex 3</u> take $S = G$

$$G \times S \ni (a,b) \longmapsto \varphi_a(b) = a \cdot b \in S = G$$

It is a good left action: $\varphi_{ab}(c) = (ab)(c) = a \cdot b \cdot c = (\varphi_a \circ \varphi_b)(c)$

if b and $c \in G \Rightarrow a$ s.t. $a \cdot b = c$ is equal to

$$a = c \cdot b^{-1}.$$

G acts <u>transitively</u> on itself (right action: $\varphi_a(b) = b \cdot a$).

<u>BUT:</u> There are <u>non</u>-transitive actions, e.g. in <u>Ex 2</u>

two vectors in $\mathbb{R}^3$ of <u>un</u>equal length can <u>not</u> be ~~connected~~ transformed to each other by an el. of $SO(3)$!

$\Downarrow$

3) An orbit of the action of G in S

Orbit of an element $p \in S \equiv$ set of all points in S that are connected to p by an action of G.

Precisely:

$$\mathcal{O}_p = \{ p' \in S : \exists\, a \in G \quad \varphi_a(p) = p' \}$$

In particular:

G acts transitively in S iff there is only ONE orbit equal to S.

In general:

Given an action of G in S, the set S splits onto disjoint orbits — equivalence classe of the relation

$$p_1 \sim p_2 \Leftrightarrow \exists\, a \in G \text{ s.t. } \varphi_a(p) = p'$$

4) G <u>acts effectively</u> in $S \iff \forall\, e \neq a \in G\ \exists\, p \in S : \varphi_a(p) \neq p$

5) G <u>acts freely</u> in $S \iff \forall\, e \neq a \in G\ \forall\, p \in S : \varphi_a(p) \neq p$

e.g. $SO(3)$ acts effectively in $\mathbb{R}^3$ but NOT freely!

$\mathbb{R}^3 \ni \vec{0}$ when acted by each element fr. $SO(3)$ is NOT moved.

6) <u>Important example of an action of $GL(n,\mathbb{R})$.</u>

- Let: V - be an n-dimensional vector space over $\mathbb{R}$

$P(V)$ = set of all bases in V

$P(V) \ni e = (e_\mu) = (e_1, e_2, \dots, e_n)$ - basis

Take $G = GL(n,\mathbb{R}) \ni a = (a^\mu{}_\nu)$ matrices $n \times n$ invertible

$S = P(V)$

$$GL(n,\mathbb{R}) \times P(V) \ni (a, e) \longmapsto \varphi_a(e) = e \cdot a^{-1} = (e_\mu \cdot a^{-1\,\mu}{}_\nu) \in P(V)$$

$$\varphi_{ab}(e) = e\,(a \cdot b)^{-1} = e\, b^{-1} a^{-1} = \varphi_a(e b^{-1}) = \varphi_a(\varphi_b(e))$$

left action

it is

1) free

2) transitive (even <u>simply</u> transitive $\forall\, e, e'\ \exists!\, a$ s.t. $e' = \varphi_a(e)$)

7) <u>Tensors, tensor densities, pseudotensors (</u>

<u>Scheme:</u>

V- vector space of dim n, $P(V)$ - set of all bases,

$\varphi_a(e) = e\,a^{-1}$ $\qquad a \in G = GL(n,\mathbb{R})$

ρ - representation of $GL(n,\mathbb{R})$ in a vector space W.

$$\rho : GL(n,\mathbb{R}) \xrightarrow{\text{homo}} GL(W)$$

$$\rho(a \cdot b) = \rho(a) \cdot \rho(b).$$

- We define an action of $G = GL(n,\mathbb{R})$ in $S = P(V) \times W$

$$G \times S \ni (a, (e,w)) \xmapsto{\Psi_a} (\varphi_a(e), \rho(a)w) \in S$$

- We check that Ψ_a is a left action:

$$\Psi_a(e,\omega) = (\varphi_a(e), \rho(a)w)$$

$$\Psi_{ab}(e,\omega) = (\varphi_{ab}(e), \rho(ab)\omega) = ((\varphi_a \circ \varphi_b)(e), \rho(a)\cdot(\rho(b)w)) =$$

$$= (\Psi_a \circ \Psi_b)(e,\omega). \quad \checkmark$$

- We are interested in the set of orbits, i.e. in the set of equivalence classes of the relation:

$$(e,w) \sim (e',w') \iff \exists a \in GL(n,\mathbb{R}) \text{ s.t. } (e',w') = \Psi_a(e,\omega).$$

The set of orbits is denoted by

$$W_\rho = S/_\sim = P(V)\times W/_\sim$$

- <u>Facts</u>

1° W_ρ is a vector space. The linear structure in W_ρ is defined as follows:

$(e,w) \longmapsto$ equivalence class $[(e,\omega)]$

$(e',\omega') \longmapsto$ . . . $[(e',w')] \sim [(e,\tilde{w})]$

$$\alpha \cdot [(e,w)] + \alpha' [(e',w')] := [(e, \alpha w + \alpha' \tilde{w})]$$

Check that this def does NOT depend on the choice of representative (e,ω) of $[(e,\omega)]$ and (e',ω') of $[(e',\omega')]$.

2° $\dim(P(V)\times W) = \dim GL(n,\mathbb{R}) + \dim W$

$\dim W_\rho = \dim W$

Important examples

(I) $\boxed{\text{TENSORS}}$

$V = \mathbb{R}^n \quad , \quad W = \mathbb{R}^{n(r+s)}$

$W \ni K = \left(K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\right)$ $\qquad \mu_i = 1, \dots n$, $\nu_i = 1, \dots n$

$$(\alpha K + \beta L)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = \alpha K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} + \beta L^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}$$

$$\rho^r_s : GL(n, \mathbb{R}) \longrightarrow GL(W)$$

$$\left[\rho^r_s(a) K\right]^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = a^{\mu_1}{}_{\rho_1} a^{\mu_2}{}_{\rho_2} \dots a^{\mu_r}{}_{\rho_r} K^{\rho_1 \dots \rho_r}{}_{\tau_1 \dots \tau_s} a^{-1\tau_1}{}_{\nu_1} \dots a^{-1\tau_s}{}_{\nu_s}$$

↑ this is a good representation

$$S = P(\mathbb{R}^n) \times \mathbb{R}^{n(r+s)},$$

$$\psi_a(e, K) = \left(ea^{-1}, \rho^r_s(K)\right)$$

$W_{\rho^r_s}$ – vector space of tensors of type $\binom{s}{r}$ – s-times covariant and r-times contravariant.

(II) $\boxed{\text{Tensor densities of weight } \omega.}$

$W = \mathbb{R}^{n(r+s)}$ as before (but now take

$$^{\omega}\rho^r_s(a) = (\det a)^{\omega} \rho^r_s(a).$$

e.g.

a) Levi-Civita symbol

$\boxed{\text{definition:}}$

① $\begin{cases} \varepsilon_{\mu_1 \dots \mu_n} \quad \text{s.t.} \quad \varepsilon_{\mu_1 \dots \mu_n} = \operatorname{sgn} \pi \quad \text{if } \pi = \begin{pmatrix} 1 & \dots & n \\ \mu_1 & \dots & \mu_n \end{pmatrix} \\ \varepsilon_{\mu_1 \dots \mu_n} = 0 \quad \text{if at least one of } \mu_i\text{'s is repeated.} \end{cases}$

$\Rightarrow \quad \epsilon_{\mu_1 \dots \mu_n} = \epsilon_{[\mu_1 \dots \mu_n]} \leftarrow$ totally antisymmetric

② $\epsilon'_{\mu_1 \dots \mu_n} = \epsilon_{\mu_1 \dots \mu_n}$ i.e. ϵ looks the same in all bases.

Consider:

$$a^{-1\,\nu_1}{}_{\mu_1} \dots a^{-1\,\nu_n}{}_{\mu_n} \, \epsilon_{\nu_1 \dots \nu_n} = \det(a^{-1}) \, \epsilon_{\mu_1 \dots \mu_n}$$

Example

$n=2, \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$(a^{-1})^\alpha{}_\beta = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$$a^{-1\,\alpha}{}_\gamma \, a^{-1\,\beta}{}_\delta \, \epsilon_{\alpha\beta} =$$
$$= a^{-1\,\alpha}{}_\gamma \, \epsilon_{\alpha\beta} \, a^{-1\,\beta}{}_\delta$$
$$(a^{-1})^T \epsilon \cdot a^{-1} =$$
$$= \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} =$$
$$= \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ -\alpha & -\beta \end{pmatrix} = \begin{pmatrix} 0 & \alpha\delta - \beta\gamma \\ \beta\gamma - \alpha\delta & 0 \end{pmatrix} =$$
$$= \det(a^{-1}) \cdot \varepsilon$$

$$\Rightarrow \quad \det a \cdot a^{-1\,\nu_1}{}_{\mu_1} \dots a^{-1\,\nu_n}{}_{\mu_n} \, \epsilon_{\nu_1 \dots \nu_n} = \epsilon_{\mu_1 \dots \mu_n} = \epsilon'_{\mu_1 \dots \mu_n}$$

$$\epsilon'_{\mu_1 \dots \mu_n} = \left[(\det a)^{\boxed{+1}} \rho_n(a) \, \varepsilon \right]_{\mu_1 \dots \mu_n} = \left[\rho(a) \, \varepsilon \right]_{\mu_1 \dots \mu_n}$$

$\Rightarrow$ ε tensor of type $\binom{0}{n}$ density of weight $\boxed{+1}$

b) $\det(g_{\mu\nu})$ $\quad g_{\mu\nu} = g_{\nu\mu} \quad \det g_{\mu\nu} \neq 0,$

$$\det(g'_{\mu\nu}) = \det(a^{-1\alpha}{}_{\mu}\, a^{-1\beta}{}_{\nu}\, g_{\alpha\beta}) = (\det a)^{-1}(\det a)^{-1}\det(g_{\mu\nu})$$

$$\det(g'_{\mu\nu}) = (\det a)^{-2}\det(g_{\mu\nu})$$

$\Rightarrow$ <u>scalar density of weight -2</u>

3) Pseudotensors

$W = \mathbb{R}^{n(r+s)}$ $\qquad \rho(a) = \mathrm{sgn}(\det a)\, \rho^r_s(a).$

e.g.

$$\eta_{\mu_1 \dots \mu_n} = \underbrace{\underbrace{\sqrt{|\det g|}}_{\text{as } |\det a|^{-1}} \; \underbrace{\varepsilon_{\mu_1 \dots \mu_n}}_{\text{as } \det a}}_{\text{everything as } \mathrm{sgn}(\det a),}$$