8) Covariant tensor fields of type $\binom{0}{k}$

$$\alpha : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k \text{ times}} \xrightarrow[k\text{-linear}]{} \mathfrak{F}(M)$$

$\mathfrak{F}$-linear !!!

at $p \in M$

$$\alpha(X_1, \ldots X_k)(p) = \alpha_p(X_{1_p}, \ldots X_{k_p})$$

$$\alpha_p : T_p(M) \times \ldots \times T_p(M) \longrightarrow \mathbb{R}$$

k-$\mathbb{R}$ linear

- $\underline{k=1}$ fields of 1-forms (1-form fields).

$$\alpha \in \Lambda^1 M$$

<u>Example</u> differential of a function $f \in \mathfrak{F}(M)$

$$df(X) := X(f).$$

Local basis in $\Lambda^1 M$: $(\mathcal{U}, x) \Rightarrow \{dx^\mu\}_{\mu=1}^n$

gives n-linearly independent 1-forms at each point $p \in \mathcal{U}$.

$\{dx^\mu\}_{\mu=1}^n$ constitutes a basis in $\Lambda^1 \mathcal{U}$

$$\alpha = \alpha_\mu dx^\mu \quad ; \quad \alpha_\mu = \alpha\left(\frac{\partial}{\partial x^\mu}\right).$$

- $\underline{k=2}$ **METRIC:** $g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{F}(M)$

1° $g(X,Y) = g(Y,X)$

2° $\left(\forall X \quad g(X,Y) = 0\right) \Rightarrow (Y = 0)$

A metric g on M defines a map

$$\tilde{g} : \mathfrak{X}(M) \longrightarrow \Lambda^1 M$$

$$\underbrace{\tilde{g}(X)}_{\text{1-form}}(Y) := g(X,Y)$$

condition 2° means that $\tilde{g}$ is invertible.

(lowering and raising of an index)

9) Antisymmetric covariant tensor fields : $\Lambda^k M$

$$\Lambda^k M = \left[\begin{array}{l} \lambda - \text{tensor fields of type } \binom{0}{k} \text{ s.t.} \\ \boxed{\lambda(X_1, \dots X_i, \dots, X_j, \dots X_k) = -\lambda(X_1, \dots X_j, \dots, X_i, \dots X_k)} \\ \underline{\text{for all}} \text{ pairs } X_i, X_j \text{ from the set } X_1, \dots X_k. \end{array} \right.$$

- **Wedge product**

$\overset{k}{\lambda} \in \Lambda^k M$, $\overset{l}{\lambda} \in \Lambda^l M \Rightarrow \overset{k}{\lambda} \wedge \overset{l}{\lambda} \in \Lambda^{k+l} M$ and is defined by:

$$\boxed{(\overset{k}{\lambda} \wedge \overset{l}{\lambda})(X_1, \dots X_{k+l}) = \frac{1}{k!\, l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn} \sigma \, \overset{k}{\lambda}(X_{\sigma(1)}, \dots X_{\sigma(k)}) \overset{l}{\lambda}(X_{\sigma(k+1)}, \dots X_{\sigma(k+l)})}$$

Example

1) $\lambda, \mu \in \Lambda^1 \Rightarrow (\lambda \wedge \mu)(X_1, X_2) = \frac{1}{1!1!}[\lambda(X_1)\mu(X_2) - \lambda(X_2)\mu(X_1)] =$

$= (\lambda \otimes \mu)(X_1, X_2) - (\mu \otimes \lambda)(X_1, X_2) =$

$= (\lambda \otimes \mu - \mu \otimes \lambda)(X_1, X_2)$

$$\underline{\lambda \wedge \mu = \lambda \otimes \mu - \mu \otimes \lambda}$$

2) $\nu \in \Lambda^1$

$\Rightarrow [(\lambda \wedge \mu) \wedge \nu](X_1, X_2, X_3) = \frac{1}{2!\,1!}\Big[(\lambda(X_1)\mu(X_2) - \lambda(X_2)\mu(X_1))\nu(X_3) +$

$+ (\lambda(X_3)\mu(X_1) - \lambda(X_1)\mu(X_3))\nu(X_2) +$

$+ (\lambda(X_2)\mu(X_3) - \lambda(X_3)\mu(X_2))\nu(X_1)$

$- (\lambda(X_2)\mu(X_1) - \lambda(X_1)\mu(X_2))\nu(X_3)$

$- (\lambda(X_3)\mu(X_2) - \lambda(X_2)\mu(X_3))\nu(X_1)$

$- (\lambda(X_1)\mu(X_3) - \lambda(X_3)\mu(X_1))\nu(X_2)\Big] =$

+ 123
+ 312
+ 231
− 213
− 321
− 132

$= \lambda(X_1)\mu(X_2)\nu(X_3) + \lambda(X_3)\mu(X_1)\nu(X_2) + \lambda(X_2)\mu(X_3)\nu(X_1) +$

$- \lambda(X_2)\mu(X_1)\nu(X_3) - \lambda(X_1)\mu(X_3)\nu(X_2) - \lambda(X_3)\mu(X_2)\nu(X_1) =$

$= (\lambda \otimes \mu \otimes \nu + \mu \otimes \nu \otimes \lambda + \nu \otimes \lambda \otimes \mu - \mu \otimes \lambda \otimes \nu - \lambda \otimes \nu \otimes \mu - \nu \otimes \mu \otimes \lambda)(X_1, X_2, X_3)$

Define: $\Lambda^0 M = \mathcal{F}(M)$. and if $f \in \Lambda^0 M$ and $\lambda \in \Lambda^k M$

we have $f \wedge \lambda = f \cdot \lambda$.

$$\Rightarrow \quad \Lambda M = \underbrace{\bigoplus_{k=0}^{n} \Lambda^k M, \ }_{\text{Cartan algebra}} \wedge$$

product. (in particular ASSOCIATIVE).

Fact $\boxed{\overset{k}{\omega} \wedge \overset{l}{\omega} = (-1)^{kl} \, \overset{l}{\omega} \wedge \overset{k}{\omega}}$

10) Exterior differential

$$d : \Lambda M \longrightarrow \Lambda M \quad \text{s.t.}$$

1° $d \Lambda^k M \subset \Lambda^{k+1} M$

2° d - $\boxed{\mathbb{R}}$-linear

3° $f \in \Lambda^0 M \Rightarrow df(X) = X(f)$

4° $d(\overset{k}{\omega} \wedge \overset{l}{\omega}) = d\overset{k}{\omega} \wedge \overset{l}{\omega} + (-1)^k \overset{k}{\omega} \wedge d\overset{l}{\omega}$

5° $d^2 \equiv 0$.

The above properties define d uniquely!

11) Inner product (Hook operator)

X - a vector field on M; $X \in \mathfrak{X}(M)$.

$$X \lrcorner : \Lambda M \to \Lambda M \quad \text{s.t.}$$

0° $X \lrcorner f = 0 \quad \forall f \in \Lambda^0 M$

1° $X \lrcorner \Lambda^k M \subset \Lambda^{k-1} M \qquad \forall k \geq 1$

2° $X \lrcorner$ - $\boxed{\mathcal{F}}$-linear

3° $X \lrcorner df = df(X) = X(f)$

4° $X \lrcorner (\overset{k}{\omega} \wedge \overset{l}{\omega}) = (X \lrcorner \overset{k}{\omega}) \wedge \overset{l}{\omega} + (-1)^k \overset{k}{\omega} \wedge (X \lrcorner \overset{l}{\omega})$

One can check that $X\lrcorner : \Lambda^k M \longrightarrow \Lambda^{k-1}$ is given by:

$$(X_1 \lrcorner \overset{k}{\omega})(X_2, \dots X_k) = \overset{k}{\omega}(X_1, X_2, \dots X_k)$$

12) Derivations of $\mathbb{Z}$-gradded algebras

1° $A = \bigoplus_{k=-\infty}^{\infty} A_k$ A is an algebra with A_k vector spaces s.t.

$$A_k \cdot A_\ell \subset A_{k+\ell}$$

$$\boxed{\begin{array}{l}\text{A is called (anti)abelian } \mathbb{Z}\text{-gradded algebra iff:} \\ \qquad a_k \cdot a_\ell = (-)(-1)^{k\ell}\, a_\ell \cdot a_k \qquad a_k \in A_k\end{array}}$$

Example $\begin{pmatrix} \Lambda M \\ + \\ \wedge \end{pmatrix}$ - abelian $\mathbb{Z}$-gradded algebra with
$A_k = \Lambda^k M$ for $k = 0, \dots, n$
$A_k = \{0\}$ for $k \notin \{0, \dots, n\}$

2° Derivation of degree k

$$D : A \underset{\text{linear}}{\longrightarrow} A$$

- $D(A_j) \subset A_{j+k}$
- $D(a_i \cdot a_j) = Da_i \cdot a_j + (-1)^{ik} a_i \cdot Da_j$

Example In $(\Lambda M, \wedge)$ $\boxed{d}$ is a derivation of degree $\boxed{+1}$
$\boxed{X\lrcorner}$ is a derivation of degree $\boxed{-1}$

3° The set of all derivations of A

A - $\mathbb{Z}$-gradded

$\mathrm{Der}_k A = \{ D$ - derivations of degree k of $A \}$

$\mathrm{Der} A = \bigoplus_{k=-\infty}^{\infty} \mathrm{Der}_k A.$

Given $\boxed{D_1 \in \mathrm{Der}_{k_1} A}$ and $\boxed{D_2 \in \mathrm{Der}_{k_2} A}$ define

$$\boxed{[D_1, D_2] = D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1} \quad \in \underline{\mathrm{Der}_{k_1+k_2} A}$$

Then

$[D_1, D_2]$ is a derivation of degree k_1+k_2 of A.

Proof

$$[D_1, D_2](a_i a_j) = D_1 \left(D_2 a_i a_j + (-1)^{ik_2} a_i D_2 a_j \right) +$$

$$-(-1)^{k_1 k_2} D_2 \left(D_1 a_i a_j + (-1)^{ik_1} a_i D_1 a_j \right) =$$

$$= \underline{D_1 D_2 a_i \cdot a_j} + \cancel{(-1)^{(i+k_2)k_1} D_2 a_i D_1 a_j} +$$

$$\cancel{(-1)^{ik_2} D_1 a_i D_2 a_j} + (-1)^{ik_2} (-1)^{ik_1} a_i D_1 D_2 a_j$$

$$- (-1)^{k_1 k_2} \underline{D_2 D_1 a_i a_j} - \cancel{(-1)^{k_1 k_2} D_1 a_i D_2 a_j (-1)^{(i+k_1)k_2}} +$$

$$- \cancel{(-1)^{k_1 k_2} (-1)^{ik_1} D_2 a_i D_1 a_j} - (-1)^{k_1 k_2} (-1)^{ik_1} (-1)^{ik_2} a_i D_2 D_1 a_j$$

$$= \underline{[D_1, D_2](a_i) \cdot a_j} + (-1)^{i(k_1+k_2)} \left[a_i D_1 D_2 a_j - (-1)^{k_1 k_2} a_i D_2 D_1 a_j \right]$$

$$= \left([D_1, D_2] a_i \right) a_j + (-1)^{i(k_1+k_2)} a_i [D_1, D_2] a_j$$

$\uparrow$

ok!

Note that

a) $\left[D_1, D_2\right] = -(-1)^{k_1 k_2}\left[D_2, D_1\right]$

$\Rightarrow \left(\mathrm{Der}A, [\cdot,\cdot]\right)$ <u>anti</u> abelian $\mathbb{Z}$ gradded algebra.

b) one can check that:

$$[D_i,[D_j,D_k]] = [[D_i,D_j],D_k] + (-1)^{ij}[D_j,[D_i,D_k]]$$

and this differe from the Jacobi identity by $(-1)^{ij}$.

<u>Definiton</u>

B - an anti abelian $\mathbb{Z}$ gradded algebra is called gradded LIE algebra iff.

$$b_i \cdot (b_j \cdot b_k) = (b_i \cdot b_j) b_k + (-1)^{ij} b_j (b_i \cdot b_k)$$

As an example take $\underline{B = \mathrm{Der}A, \ \cdot = [\cdot,\cdot]}$.

4° <u>Returning to $(\Lambda M, \wedge)$</u>

$\mathrm{Der}_1(\Lambda M) \ni d$

$\mathrm{Der}_{-1}(\Lambda M) \ni X\lrcorner$

Note that $\mathrm{Der}_{-2}(\Lambda M) = \{0\}$.

because if $D \in \mathrm{Der}_{-2}(\Lambda M)$

$Df \equiv 0$ (no -2 forms)

$D\overset{1}{\lambda} \equiv 0$ (no -1 forms)

but functions and 1-forms generate entire ΛM.

So $D\overset{k}{\omega} \equiv 0$. $\Rightarrow$ $\boxed{\mathrm{Der}(\Lambda M) = \bigoplus_{k=-1}^{n} \mathrm{Der}_k(\Lambda M)}$

~~in particular~~ $\cancel{X\lrcorner \circ Y\lrcorner + Y\lrcorner \circ X\lrcorner = [X\lrcorner, Y\lrcorner] = 0}$.

In particular:

$$\mathrm{Der}_{-2}\Lambda M \ni [X\lrcorner, Y\lrcorner] = X\lrcorner\circ Y\lrcorner - (-1)^{(-1)(-1)} Y\lrcorner\circ X\lrcorner = \boxed{X\lrcorner\circ Y\lrcorner + Y\lrcorner\circ X\lrcorner = 0}$$

What about

$$[d, X\lrcorner] = d\circ X\lrcorner - (-1)^{(-1)\cdot 1} X\lrcorner\circ d\lrcorner = d\circ X\lrcorner + X\lrcorner\circ d \; ?$$

this is a derivation of degree $-1+1=0$.

. We denote it by

$$\boxed{\mathcal{L}_X = d\circ X\lrcorner + X\lrcorner\circ d = [d, X\lrcorner]}$$

We have in addition

$$\mathrm{Der}_{+1}\Lambda M \ni [\mathcal{L}_X, d] = \mathcal{L}_X\circ d - (-1)^{0\cdot(-1)} d\circ\mathcal{L}_X = \mathcal{L}_X\circ d - d\circ\mathcal{L}_X$$

$$= (d\circ X\lrcorner + \cancel{X\lrcorner\circ d})\circ d - d\circ(\cancel{d\circ X\lrcorner} + X\lrcorner\circ d) = 0.$$

$$\boxed{d\circ\mathcal{L}_X = \mathcal{L}_X\circ d}$$

$$\mathrm{Der}_{-1}\Lambda M \ni [\mathcal{L}_X, Y\lrcorner] = \mathcal{L}_X\circ Y\lrcorner - (-1)^{0(-1)} Y\lrcorner\circ\mathcal{L}_X =$$

$$= \mathcal{L}_X\circ Y\lrcorner - Y\lrcorner\circ\mathcal{L}_X \underset{\uparrow}{=} [X,Y]\lrcorner$$

almost obvious!

$$\boxed{\mathcal{L}_X\circ Y\lrcorner - Y\lrcorner\circ\mathcal{L}_X = [X,Y]\lrcorner}$$

$$[\mathcal{L}_X, \mathcal{L}_Y] = ?$$

13) Lie derivative of tensor fields

$\mathfrak{X}^r_s(M) \ni$ T. - tensor field of type $\binom{r}{s}$

$$T : \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{s} \times \underbrace{\Lambda^1 M \times \dots \times \Lambda^1 M}_{r} \longrightarrow \mathcal{F}(M)$$

$(r+s)$-$\mathcal{F}$ linear.

TRANSPORT

Let M and M' two manifolds, $\boxed{\varphi : M \xrightarrow[\text{map}]{\text{smooth}} M'}$

$p \in M$.

1°) Pull-back of a function

$$\varphi^* : \mathcal{F}(M') \longrightarrow \mathcal{F}(M)$$

$$(\varphi^* f')(p) := f'(\varphi(p)) \qquad \text{i.e.} \quad \varphi^* f' = f' \circ \varphi$$

2°) Push forward of tangent vectors

$$\varphi_{*p} : T_p M \longrightarrow T_{\varphi(p)} M'$$

X_p - a vector tangent to M at p

$\gamma(t)$ - a curve passing through p, ~~tang~~
with a tangent vector X_p

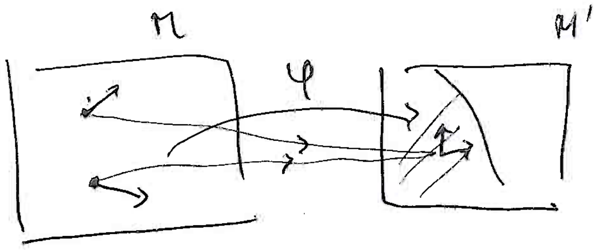
$\tilde{\gamma}(t) = \varphi(\gamma(t))$ - a curve in M' passing through $\varphi(p)$: $\tilde{\gamma}(0) = \varphi(p)$

$\varphi_{*p} X_p$ = tangent vector to $\tilde{\gamma}(t)$ at $t=0$ i.e. in $\varphi(p)$.

A formula:

$$(\varphi_{*p} X_p)(f') = X(\varphi^* f')$$

3°) Trouble with vector fields



For transporting vector fields one needs <u>diffeomorphisms</u>

$$\varphi : M \xrightarrow{\text{diffeo}} M'$$

Then one transforms X by transporting point by point tangent vectors representing X. This defines $\varphi_* : \mathfrak{X}(M) \to \mathfrak{X}(M')$

4°) <u>Transport of tensorfields of type $\binom{0}{s}$</u> : $\mathfrak{X}^0_s(M)$

given $\varphi : M \to M'$ there exist a unique map $\varphi^* : \mathfrak{X}^0_s(M') \longrightarrow \mathfrak{X}^0_s(M)$ s.t.

a) φ^* is $\mathbb{R}$-linear

b) $\varphi^*(\omega' \otimes \tau') = \varphi^*(\omega') \otimes \varphi^*(\tau')$

c) $\varphi^*(\omega' \wedge \tau') = \varphi^*(\omega') \wedge \varphi^*(\tau')$ if $\omega', \tau' \in \Lambda^k M', \Lambda^\ell M'$

d) $\varphi^*(\Lambda^k M') \subset \Lambda^k M$

e) $\varphi^* \circ d = d \circ \varphi^*$

f) $\varphi^* f' = f' \circ \varphi \quad \forall f' \in \mathcal{F}(M')$

5°) <u>Tensor fields $\binom{r}{s}$</u>

$\varphi : M \to M'$ diffeomorphism.

$$\tilde{\varphi} : \mathfrak{X}^r_s(M) \longrightarrow \mathfrak{X}^r_s(M')$$

We have $\varphi^* : \mathfrak{X}^0_s(M') \longrightarrow \mathfrak{X}^0_s(M)$ and

$$(\varphi^{-1})^* : \mathfrak{X}^0_s(M) \longrightarrow \mathfrak{X}^0_s(M')$$

We define $\varphi_* : \mathfrak{X}^r_0(M) \longrightarrow \mathfrak{X}^r_0(M')$ by $\varphi_*(X \otimes Y \otimes \dots \otimes Z) =$
$= (\varphi_* X) \otimes \varphi_*(Y) \dots \otimes \varphi_* Z$

and finally

$$\tilde{\varphi}(X_1 \otimes \dots \otimes X_r \otimes \lambda^1 \otimes \dots \otimes \lambda^s) =$$

$$= (\varphi_* X_1) \otimes \dots \otimes \varphi_* X_r \otimes (\varphi^{-1})^* \lambda^1 \otimes \dots \otimes (\varphi^{-1})^* \lambda^s$$

6°) Transport map $\tilde{\varphi} : \mathfrak{X}^r_s(M) \longrightarrow \mathfrak{X}^r_s(M')$

<u>commutes with contractions</u>,

where a contraction C_i^j is a map:

$$C_i^j : \mathfrak{X}^r_s(M) \xrightarrow[\mathcal{F}\text{-linear}]{} \mathfrak{X}^{r-1}_{s-1}(M)$$

$$C_i^j (X_1 \otimes \dots \otimes X_r \otimes \lambda^1 \otimes \dots \otimes \lambda^s) =$$

$$= X_1 \otimes \underset{\substack{\uparrow \\ i\text{th}}}{\dots} \otimes X_r \otimes \lambda^1 \otimes \underset{\substack{\uparrow \\ j\text{th}}}{\dots} \otimes \lambda^s \cdot \langle X_i , \lambda^j \rangle$$

removed

7°) <u>LIE DERIVATIVE</u>

X - vector field on M.

φ_t - its local 1-parameter group of transformations in U around p.

$\mathfrak{X}^r_s(M) \ni T$

$$\boxed{(\mathcal{L}_X T)_p = \lim_{t \to 0} \frac{T_p - (\tilde{\varphi}_t T)_p}{t} = -\frac{d}{dt}\Big|_{t=0} (\tilde{\varphi}_t T)_p}$$

8°) Properties:

a) $\mathcal{L}_X$ preserves the type of a tensor $\binom{r}{s} \to \binom{r}{s}$

b) $\mathcal{L}_X$ is $\mathbb{R}$-linear

c) $\mathcal{L}_X (K \otimes L) = (\mathcal{L}_X K) \otimes L + K \otimes \mathcal{L}_X L$

d) $\mathcal{L}_X$ commutes with contractions

e) $\mathcal{L}_X (\Lambda M) \subset \Lambda M$

f) $\mathcal{L}_X (\omega \wedge \tau) = \mathcal{L}_X \omega \wedge \tau + \omega \wedge \mathcal{L}_X \tau$

g) $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$ on $T \in \Lambda M$

h) $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$.

9°) <u>Lie derivative of a function</u>

$$(\mathcal{L}_X f)_p = -\frac{d}{dt}\Big|_{t=0} \left((\varphi_{-t})^* f \right)(p) = -\frac{d}{dt}\Big|_{t=0} (f \circ \varphi_{-t})(p) =$$

$$= -\frac{d}{dt}\Big|_{t=0} f(\varphi_{-t}(p)) = \frac{d}{dt} f(\varphi_t(p)) = X_p(f)$$

$$\Rightarrow \quad \boxed{\mathcal{L}_X f = X(f)}$$

10°) <u>Lie derivative of a k-form</u>

$$\boxed{\mathcal{L}_X \omega = X \lrcorner (d\omega) + d(X \lrcorner \omega)}$$ i.e. ~~[illegible]~~

on forms, the operator $\mathcal{L}_X$ is equal to this what we had before.

Proof

on functions:

$$\underset{X}{\mathcal{L}} f = X(f)$$

but:

$$(X \lrcorner d + d X \lrcorner)(f) = X \lrcorner df + d(\cancel{X \lrcorner f}) = X(f)$$

ok

on 1-forms (it is enough on differentials)

$$\underset{X}{\mathcal{L}} df = d(\underset{X}{\mathcal{L}} f) = d\, X(f)$$

but:

$$(X \lrcorner d + d \lrcorner X)(df) = \cancel{X \lrcorner d^2 f} + d(X \lrcorner df) = d(X(f))$$

ok

Functions and 1-forms generate ΛM. $\square$.

11° Lie derivative of a vector field

$$\boxed{\underset{X}{\mathcal{L}} Y = [X, Y]} \qquad (*)$$

To prove it we need

14) Usefull formula for the exterior derivative

$$\boxed{(d \overset{k}{\omega})(X_0, X_1, \dots X_k) = \sum_{i=0}^{k} (-1)^i X_i (\overset{k}{\omega}(X_0, \overset{\vee}{\dots} X_k)) + \sum_{0 \le i < j \le k} (-1)^{i+j} \overset{k}{\omega}([X_i, X_j], X_0, \overset{\vee\ \vee}{\dots}, X_k)}$$

without X_i

without X_i and X_j.

In particular for 1-forms:

$$\boxed{(d\omega)(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])}$$

Proof of (*) at point 11°) $\quad$ Y - vector field, ω - 1-form:

Note that $Y \lrcorner \omega = C^1_1 (Y \otimes \omega)$

$$\mathcal{L}_X (Y \lrcorner \omega) = \mathcal{L}_X C^1_1(Y \otimes \omega) = C^1_1 \, \mathcal{L}_X (Y \otimes \omega) =$$

$$= C^1_1 \left(\mathcal{L}_X Y \otimes \omega + Y \otimes \mathcal{L}_X \omega \right) =$$

$$= \omega(\mathcal{L}_X Y) + Y \lrcorner (\mathcal{L}_X \omega)$$

$$\Rightarrow \quad \omega(\mathcal{L}_X Y) = \mathcal{L}_X (Y \lrcorner \omega) - Y \lrcorner \left(d(X \lrcorner \omega) + X \lrcorner d\omega \right) =$$

$$= X(\omega(Y)) - Y \lrcorner d(\omega(X)) - Y \lrcorner X \lrcorner d\omega =$$

$$= X(\omega(Y)) - Y(\omega(X)) - d\omega(X, Y) =$$

$$= \omega([X, Y]) \qquad \forall \omega \Rightarrow$$

$$\Rightarrow \boxed{\mathcal{L}_X Y = [X, Y]}$$

15) Local anholonomic frames an Frobenius thm revisited

$\{X_\mu\}$ in $\mathcal{U} \subset M$ (open) is a frame in $\mathcal{U}$ iff ~~[illegible]~~ $\{X_\mu|_p\}_{\mu=1}^n$ is a basis of $T_p(M)$ $\forall p \in \mathcal{U}$

1° example $\left(\frac{\partial}{\partial x^\mu} \right)$ is a frame in $(\mathcal{U}, x)$.

by definition it is called holonomic frame

But there are anholonomic i.e. such that $[X_\mu, X_\nu] \neq 0$.

e.g.

$$X_\mu = A_\mu{}^\nu(x) \frac{\partial}{\partial x^\nu}$$

$A_\mu{}^\nu(x)$ invertible at every point p.

2° Maurer – Cartan formula

$\{X_\mu\}$ local frame :

$$[X_\mu, X_\nu] = C^\rho{}_{\mu\nu} X_\rho$$

coefficients of anholonomy

$C^\rho{}_{\mu\nu} = C^\rho{}_{\mu\nu}(x)$. functions in $\mathcal{U}$ s.t. $C^\rho{}_{\mu\nu} = -C^\rho{}_{\nu\mu}$.

$$\boxed{C^\rho{}_{\mu\nu} \equiv 0 \iff \underset{x \text{ in } \mathcal{U}}{\exists} \; X_\mu = \frac{\partial}{\partial x^\mu}}$$

Let $\{\omega^\mu\}$ be a basis in $\Lambda^1\mathcal{U}$ dual to $\{X_\mu\}$ i.e.

$$X_\mu \lrcorner\, \omega^\nu = \delta_\mu{}^\nu. \qquad (\text{i.e. } \omega^\nu(X_\mu) = \delta_\mu{}^\nu).$$

Thm M–C.

$$\boxed{[X_\mu, X_\nu] = C^\rho{}_{\mu\nu} X_\rho \iff d\omega^\mu = -\tfrac{1}{2} C^\mu{}_{\nu\rho}\,\omega^\nu \wedge \omega^\rho}$$

Proof

$$d\omega^\rho(X_\mu, X_\nu) = X_\mu\big(\cancel{\underbrace{\omega^\rho(X_\nu)}_{\delta^\rho_\nu}}\big) - X_\nu\big(\cancel{\underbrace{\omega^\rho(X_\mu)}_{\delta^\rho_\mu}}\big) - \omega^\rho([X_\mu, X_\nu]) =$$

$$= -\omega^\rho(C^\alpha{}_{\mu\nu} X_\alpha) = -C^\rho{}_{\mu\nu}$$

q.

3° Fröbenius thm once again

S – distribution on M

$S^* = \{ \omega \in \Lambda^1 M : \omega(X) = 0 \quad \forall X \in S \}$

$\{X_i\}$ – vector fields spanning S in $\mathcal{U}$ $\{\theta^\alpha\}$ – 1 forms spanning S^* in $\mathcal{U}$.

$X_i : i = 1, \dots, m$ $\theta^\alpha : \alpha = m+1, \dots n.$

$\boxed{\text{Thm Fröbenius}}$

the following conditions are equivalent:

1) S is involutive
2) through every point passes precisely one integral manifold of S
3) $[X_i, X_j] = c^k{}_{ij} X_k$
4) $X_i = a^j{}_i(x^k, x^\gamma) \frac{\partial}{\partial x^j}$
5) $d\theta^\alpha \wedge \theta^{m+1} \wedge \dots \wedge \theta^n = 0 \quad \forall \alpha = m+1, \dots n$
6) $\theta^\alpha = b^\alpha{}_\beta (x^k, x^\gamma) dx^\beta$

Proof

4) $\Rightarrow$ 6) obvious $(\theta^\alpha(X_i) = 0.)$

6) $\Rightarrow$ 5) obvious

5) $\Rightarrow$ 3) $d\theta^\alpha = -\frac{1}{2} c^\alpha{}_{\beta\gamma} \theta^\beta \wedge \theta^\gamma - \frac{1}{2} c^\alpha{}_{ij} \theta^i \wedge \theta^j - \frac{1}{2} c^\alpha{}_{i\beta} \theta^i \wedge \theta^\beta$

5) $\Rightarrow$ $c^\alpha{}_{ij} = 0$