3) Extension of ΛM to include (-1)-forms

MINUS-1-FORMS

$p = 0, \dots n \quad : \quad \mathcal{L}^p M = \Lambda^p M \times \Lambda^{p+1} M$

$p = -1 \qquad \mathcal{L}^{-1} M = \underbrace{\{0\} \times \Lambda^0 M}_{\text{with a natural structure of a vector space over } \mathbb{R} \text{ (module over } \mathcal{F}(M))}$, $\quad \Lambda^0 M = \mathcal{F}(M)$

$\mathcal{L}^p M \ni \overset{p}{a} = (\overset{p}{\alpha}, \overset{p+1}{\alpha})$

$\mathcal{L}^{-1} M \ni \overset{-1}{a} = (0, f)$

$$\mathcal{L}M = \bigoplus_{p=-1}^{n} \mathcal{L}^p M$$

In $\mathcal{L}M$ we introduce a product $\wedge$

$$\overset{p}{a} \wedge \overset{q}{b} = \left(\overset{p}{\alpha} \wedge \overset{q}{\beta}, \ \overset{p}{\alpha} \wedge \overset{q+1}{\beta} + (-1)^q \overset{p+1}{\alpha} \wedge \overset{q}{\beta}\right)$$

and the differential

$$d\overset{p}{a} = \left(d\overset{p}{\alpha} + (-1)^{p+1} k \overset{p+1}{\alpha}, \ d\overset{p+1}{\alpha}\right)$$

where $\overset{p}{a} = (\overset{p}{\alpha}, \overset{p+1}{\alpha})$

$\overset{q}{b} = (\overset{q}{\beta}, \overset{q+1}{\beta})$

One checks that $\wedge$ is associative and distributive

$$\overset{p}{a} \wedge \overset{q}{b} = (-1)^{pq} \overset{q}{b} \wedge \overset{p}{a}$$

$$d(\overset{p}{a} \wedge \overset{q}{b}) = d\overset{p}{a} \wedge \overset{q}{b} + (-1)^p \overset{p}{a} \wedge d\overset{q}{b}$$

In particular $d^2 \equiv 0$.

~~In particular functions $\Lambda^0 M$ can be identified with $(0, \mathcal{F}(M))$~~

~~Moreover~~

~~$\Lambda M \longrightarrow (0, \Lambda M) \subset \mathcal{L}M$ and it is functorial~~ with respect to $\wedge$ and d.

MORE:

3) Extension of ΛM $\qquad$ $\dim M = n$

$p = 0, 1, \dots n$ $\qquad$ $\mathcal{L}^p M = \Lambda^p M \times \Lambda^{p+1} M$

$p = -1$ $\qquad$ $\mathcal{L}^{-1} M = \{0\} \times \underset{\underset{\mathcal{F}(M)}{\shortparallel}}{\Lambda^0 M}$

$$\mathcal{L} M = \bigoplus_{p=-1}^{n} \mathcal{L}^p M$$

$$\mathcal{L}^p M \ni \overset{p}{a} = (\overset{p}{\alpha}, \overset{p+1}{\alpha})$$

$$\mathcal{L}^{-1} M \ni \overset{-1}{a} = (0, f)$$

We introduce $\wedge$ and d in $\mathcal{L}M$

$\wedge$: $$\overset{p}{a} \wedge \overset{q}{b} = (\overset{p}{\alpha} \wedge \overset{q}{\beta}, \overset{p}{\alpha} \wedge \overset{q+1}{\beta} + (-1)^q \overset{p+1}{\alpha} \wedge \overset{q}{\beta})$$

d: $$d\overset{p}{a} = (d\overset{p}{\alpha} + (-1)^{p+1} k \overset{p+1}{\alpha}, d\overset{p+1}{\alpha})$$

$$\overset{p}{a} = (\overset{p}{\alpha}, \overset{p+1}{\alpha}) \qquad \overset{q}{b} = (\overset{q}{\beta}, \overset{q+1}{\beta})$$

$$\overset{p}{a} \wedge \overset{q}{b} = (\overset{p}{\alpha} \wedge \overset{q}{\beta}, \overset{p}{\alpha} \wedge \overset{q+1}{\beta} + (-1)^q \overset{p+1}{\alpha} \wedge \overset{q}{\beta}) =$$

$$= \left((-1)^{p \cdot q} \overset{q}{\beta} \wedge \overset{p}{\alpha}, (-1)^{p(q+1)} \overset{q+1}{\beta} \wedge \overset{p}{\alpha} + (-1)^q (-1)^{q(p+1)} \overset{q}{\beta} \wedge \overset{p+1}{\alpha}\right)$$

$$= (-1)^{pq} (\overset{q}{\beta} \wedge \overset{p}{\alpha}, \overset{q}{\beta} \wedge \overset{p+1}{\alpha} + (-)^p \overset{q+1}{\beta} \wedge \overset{p}{\alpha}) =$$

$$= (-1)^{pq} \overset{q}{b} \wedge \overset{p}{a} \quad \checkmark \qquad \Rightarrow \quad \overset{p}{a} \wedge \overset{q}{b} = (-1)^{pq} \overset{q}{b} \wedge \overset{p}{a}$$

$$d^2 \overset{p}{a} = ((-1)^{p+1} dk \wedge \overset{p+1}{\alpha} + \cancel{(-1)^{p+1} k \, d\overset{p+1}{\alpha}} + \cancel{(-1)^{p+2} k \, d\overset{p+1}{\alpha}}, 0)$$

$$\boxed{dk = 0} \qquad k = \text{const}$$

$\Rightarrow$ $d^2 \equiv 0.$

$$d(\overset{p}{\alpha} \wedge \overset{q}{b}) =$$

$$= d\left(\overset{p}{\alpha}\wedge\overset{q}{\beta},\ \overset{p}{\alpha}\wedge\overset{q+1}{\beta} + (-1)^q \overset{p+1}{\alpha}\wedge\overset{q}{\beta}\right) =$$

$$= \Big(d(\overset{p}{\alpha}\wedge\overset{q}{\beta}) + (-1)^{p+q+1} k\,(\overset{p}{\alpha}\wedge\overset{q+1}{\beta} + (-1)^q \overset{p+1}{\alpha}\wedge\overset{q}{\beta}),$$
$$d(\overset{p}{\alpha}\wedge\overset{q+1}{\beta} + (-1)^q \overset{p+1}{\alpha}\wedge\overset{q}{\beta})\Big)$$

$$= \Big(d\overset{p}{\alpha}\wedge\overset{q}{\beta} + (-)^p \overset{p}{\alpha}\wedge d\overset{q}{\beta} + (-1)^{p+q+1} k(\overset{p}{\alpha}\wedge\overset{q+1}{\beta} + (-1)^q \overset{p+1}{\alpha}\wedge\overset{q}{\beta}),$$
$$d\overset{p}{\alpha}\wedge\overset{q+1}{\beta} + (-1)^p \overset{p}{\alpha}\wedge d\overset{q+1}{\beta} + (-1)^q d\overset{p+1}{\alpha}\wedge\overset{q}{\beta} +$$
$$(-1)^q \overset{p+1}{\alpha}\wedge d\overset{q}{\beta}\Big)$$

$$d\overset{p}{a}\wedge\overset{q}{b} + (-1)^p \overset{p}{a}\wedge d\overset{q}{b} = \ldots = \text{:)}$$

$(\mathcal{L}M, \wedge, d)$ ~~algebra Cartan~~
generalized Cartan algebra.

Extension of $(\Lambda M, \wedge, d)$

Indeed:

$$\Lambda M \hookrightarrow (\Lambda M, 0) \subset \mathcal{L}M$$

$\wedge|_{(\Lambda M, 0)}$, $d|_{(\Lambda M, 0)}$ usual $\wedge$ and d.

In particular functions on M can be identified with

$$\mathcal{F}(M) = \Lambda^0 M \hookrightarrow (\mathcal{F}(M), 0) \subset \mathcal{L}M.$$

And usual forms ΛM with

$$\Lambda M \hookrightarrow (\Lambda M, 0)$$

Take $\xi = (0,1) \in \mathcal{L}^{-1}M$

$d\xi = (k, 0) \in (\mathcal{F}M, 0) \simeq \mathcal{F}M$

So ξ is a form whose differential is a constant!

ξ - standard MINUS 1-form

Moreover

$$\overset{p}{a} = (\overset{p}{\alpha}, \overset{p+1}{\alpha}) = \underbrace{(\overset{p}{\alpha}, 0)}_{\overset{p}{\alpha}} + \underbrace{(\overset{p+1}{\alpha}, 0)}_{\overset{p+1}{\alpha}} \wedge (0,1)$$

$$\overset{p}{a} \rightsquigarrow \overset{p}{\alpha} + \overset{p+1}{\alpha} \wedge \xi$$

$$\boxed{\overset{p}{a} = \overset{p}{\alpha} + \overset{p+1}{\alpha} \wedge \xi}$$

$d\xi = k$ ξ - (-1)-form

and all the rules follow.