15) Local anholonomic frames and Frobenius theorem again

[A set of vector fields $\{X_\mu\}$ in $\mathcal{U}$ is a FRAME in $\mathcal{U}$ iff

$$\{X_\mu|_p\}_{\mu=1,\dots n} \text{ is a basis of } T_p(M) \ \forall p \in \mathcal{U}.]$$

1° example $\left(\frac{\partial}{\partial x^\mu}\right)$ is a frame in $(\mathcal{U}, x)$.

2° A frame $\{X_\mu\}$ is a holonomic frame if

$$[X_\mu, X_\nu] = 0 \quad \forall \mu, \nu$$

We know that locally holonomic frames are of the form $\{X_\mu = \frac{\partial}{\partial x^\mu}\}$.

3° But there are nonholonomic frames, i.e. such ~~for~~ that

$$[X_\mu, X_\nu] \neq 0.$$

In a coordinate system $\{x^\mu\}$ they are of the form

$$X_\mu = A_\mu{}^\nu(x) \frac{\partial}{\partial x^\nu}. \qquad A_\mu{}^\nu(x) \text{ invertible and GENERAL ENOUGH.}$$

4° Maurer-Cartan theorem

$\{X_\mu\}$ a frame $\Rightarrow$ $\boxed{[X_\mu, X_\nu] = C^\rho{}_{\mu\nu} X_\rho}$

where $C^\rho{}_{\mu\nu} = C^\rho{}_{\mu\nu}(x)$ functions in $\mathcal{U}$ s.t. ~~&~~

$$\boxed{C^\rho{}_{\mu\nu} = - C^\rho{}_{\nu\mu}}$$

$C^\rho{}_{\mu\nu} \leftarrow$ coefficients of anholonomy.

$$C^\rho{}_{\mu\nu} \equiv 0 \iff \exists x \text{ in } \mathcal{U} \text{ s.t. } X_\mu = \frac{\partial}{\partial x^\mu}.$$

Let $\{\omega^\mu\}$ be a basis in $\Lambda^1 U$ dual to $\{X_\mu\}$ i.e.

$$X_\mu \lrcorner\, \omega^\nu = \delta_\mu{}^\nu \qquad \left(\omega^\nu(X_\mu) = \delta_\mu{}^\nu\right)$$

$\{\omega^\mu\}$ is called a COFRAME, dual to $\{X_\mu\}$.

Thm Maurer-Cartan

$$\boxed{[X_\mu, X_\nu] = C^\rho{}_{\mu\nu} X_\rho \iff d\omega^\mu = -\tfrac{1}{2} C^\mu{}_{\nu\rho}\, \omega^\nu \wedge \omega^\rho}$$

Proof

$$X_\rho \lrcorner X_\nu \lrcorner d\omega^\mu = d\omega^\mu(X_\nu, X_\rho) = X_\nu\big(\cancel{\underbrace{\omega^\mu(X_\rho)}_{\delta^\mu_\rho}}\big) - X_\rho\big(\cancel{\underbrace{\omega^\mu(X_\nu)}_{\delta^\mu_\nu}}\big) - \omega^\mu([X_\nu, X_\rho]) =$$

$$= -C^\mu{}_{\nu\rho}$$

On the other hand if $d\omega^\mu = \frac{1}{2} b^\mu{}_{\nu\rho}\, \omega^\nu \wedge \omega^\rho$ then:

$$X_\rho \lrcorner X_\nu \lrcorner d\omega^\mu = X_\rho \lrcorner X_\nu \lrcorner \left(\tfrac{1}{2} b^\mu{}_{\alpha\beta}\, \omega^\alpha \wedge \omega^\beta\right) =$$

$$= X_\rho \lrcorner \left(\tfrac{1}{2} b^\mu{}_{\nu\beta}\, \omega^\beta - \tfrac{1}{2} b^\mu{}_{\alpha\nu}\, \omega^\alpha\right) =$$

$$= X_\rho \lrcorner \left(b^\mu{}_{\nu\beta}\, \omega^\beta\right) = b^\mu{}_{\nu\rho} \implies b^\mu{}_{\nu\rho} = -C^\mu{}_{\nu\rho} \quad \square.$$

6° Frobenius again!

Let S - be a distribution of dim m on M

Define $S^* = \{\omega \in \Lambda^1 M \,;\ X \lrcorner\, \omega = 0 \quad \forall X \in S\}$.

Recall

a local frame for S is a set $\{X_i\}$ of vector fields in $U \subset M$ s.t. $\{X_i|_p\}$ is a basis for S_p for all $p \in U$.

Given $\{X_i\}$ consider 1-forms $\{\omega^\mu\}$ such that they constitute a basis of 1-forms spanning S^* in U.

~~We can always choose~~

By definition

$$X_i \lrcorner\, \omega^\mu = 0$$

and $i = 1, \dots m$, $\mu = m+1, \dots, n$ $\quad$ ($\dim S^* = n-m$, $n = \dim M$).

We can now extend $\{X_i\}$ to $\{X_A\} = \{X_i, X_\mu\}$ so that $\{X_A\}_{A=1,\dots n}$ is a local frame in $\mathcal{U}$,

We also extend $\{\omega^\mu\}$ to $\{\omega^A\} = \{\omega^i, \omega^\mu\}$ so that

$$X_A \lrcorner\, \omega^B = \delta_A{}^B$$

We just have:

$$\left.\begin{array}{lll} \{X_i\} & i = 1, \dots m & \text{spanning } S \\ \{X_\mu\} & \mu = m+1, \dots n & \end{array}\right\} \text{spaning } T(\mathcal{U})$$

$$\left.\begin{array}{lll} \{\omega^\mu\} & \mu = m+1, \dots n & \text{Spanning } S^* \\ \{\omega^i\} & i = 1, \dots m & \end{array}\right\} \text{spaning } \Lambda^1 M.$$

$\boxed{\text{Thm Frobenius}}$

the following conditions are equivalent:

1. S is involutive
2. through every point of M passes precisely one integral manifold of S
3. $[X_i, X_j] = c^k{}_{ij} X_k$
4. $\exists (x^A)$ - coordinate system in $\mathcal{U} \subset M$ s.t. $X_i = A_i{}^j(x^k, x^\mu) \frac{\partial}{\partial x^j}$
5. $d\omega^\mu \wedge \omega^{m+1} \wedge \dots \wedge \omega^n = 0 \quad \forall\, \mu = m+1, \dots n$
6. $\exists (x^A)$ s.t. $\omega^\mu = B^\mu{}_r(x^k, x^\alpha)\, dx^\mu$

Proof

(4) => (6) obvious since X_i must anihilate ω^μ.

(6) => (5) obvious

(5) => (3) $d\omega^\alpha = -\frac{1}{2} c^\alpha{}_{\beta\gamma}\omega^\beta \wedge \omega^\gamma - \frac{1}{2} c^\alpha{}_{ij}\omega^i \wedge \omega^j - \cancel{\tfrac{1}{2}}\, c^\alpha{}_{i\beta}\omega^i \wedge \omega^\beta$

(5) $\Rightarrow c^\alpha{}_{ij} \equiv 0 \overset{M.C.}{\Longrightarrow} [X_i, X_j] = c^k{}_{ij} X_k + \cancel{c^\alpha{}_{ij} X_\alpha}$ $\square$.

7° A related theorem

Let α be a 1-form and $x^1, x^2, \dots x^k$ be functions on M. Then we have

$$\begin{pmatrix} \alpha\wedge dx^1\wedge\dots\wedge dx^k\neq 0 \\ d\alpha\wedge dx^1\wedge\dots\wedge dx^k = 0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{there exist functions} \\ x, y_1, y_2, \dots y_k \text{ on } M \text{ s.t.} \\ \alpha = dx + y_1 dx^1 + y_2 dx^2 + \dots + y_k dx^k \\ dx\wedge dx^1\wedge\dots\wedge dx^k \neq 0 \end{pmatrix}$$

Proof

$\Leftarrow$ obvious

$\Rightarrow$ Since $\alpha\wedge dx^1\wedge\dots dx^k\neq 0$ then $dx^1\wedge\dots\wedge dx^k\neq 0$, and hence $x^1, \dots x^k$ are independent at each point of M. Supplement them by $x^A = (x^{k+1}, \dots x^n)$ to a coordinate system $(\underbrace{x^1, \dots x^k}_{x^\varrho}, \underbrace{x^{k+1}, \dots x^n}_{x^A})$.

Then $\boxed{\alpha = a_\varrho dx^\varrho + a_A dx^A}$ (*) where

$a_\varrho = a_\varrho(x^\alpha, x^A)$, $a_B = a_B(x^\varrho, x^A)$ are functions on M

But

$$0 = d\alpha\wedge dx^1\wedge\dots\wedge dx^k = \left(\cancel{da_\varrho\wedge dx^\varrho} + da_A\wedge dx^A\right)\wedge dx^1\wedge\dots dx^k$$

$$= \frac{\partial a_A}{\partial x^B} dx^B\wedge dx^A\wedge dx^1\wedge\dots dx^k =$$

$$= a_{A,B}\, dx^B\wedge dx^A\wedge dx^1\wedge\dots\wedge dx^k$$

$$\Rightarrow \quad a_{[A,B]} = 0 \quad \Leftrightarrow \quad \boxed{a_{A,B} = a_{B,A}} \Rightarrow$$

$\Rightarrow$ there exists a function $x = x(x^\varrho, x^A)$ s.t.

$$a_A = \frac{\partial x}{\partial x^A}$$

Thus we have:

$$dx = x_{,\varrho} dx^\varrho + x_{,A} dx^A = x_{,\varrho} dx^\varrho + a_A dx^A$$

$$\Rightarrow \quad \underline{a_A dx^A = dx - x_{,\varrho} dx^\varrho}$$

Inserting in (*)

$$\alpha = a_\varrho dx^\varrho + dx - x_{,\varrho} dx^\varrho =$$

$$= dx + \overbrace{(a_\varrho - x_{,\varrho})}^{y_\varrho} dx^\varrho =$$

$$= dx + y_\varrho dx^\varrho = \boxed{dx + y_1 dx^1 + \dots y_k dx^k.}$$

Since $\alpha \wedge dx^1 \wedge \dots dx^k \neq 0 \Rightarrow$

$$\Rightarrow dx \wedge dx^1 \wedge \dots dx^k \neq 0$$

$\square$.

8°

Theorems of Pfaff and Darboux

Let α - be a 1-form on $\underline{M}$.

It always defines two integers:

1) s such that $(d\alpha)^s \neq 0$ & $(d\alpha)^{s+1} = 0$

2) r such that $\alpha \wedge (d\alpha)^r \neq 0$ & $\alpha \wedge (d\alpha)^{r+1} = 0$.

Simple argument shows that only two cases may occur:

Either

(1) $s = r$

or

(2) $s = r+1$

<u>Thm</u>

If (1) then $\alpha = dy^1 + y^2 dy^3 + \dots + y^{2r} dy^{2r+1}$ & $dy^1 \wedge \dots \wedge dy^{2r+1} \neq 0$

If (2) then $\alpha = y^0 dy^1 + y^2 dy^3 + \dots + y^{2r} dy^{2r+1}$ & $dy^0 \wedge dy^1 \wedge \dots \wedge dy^{2r+1} \neq 0$

Dimension 4

$r=0 \quad \Rightarrow \qquad s=0 \quad \Rightarrow \quad \boxed{d\alpha=0} \quad \Rightarrow \quad \boxed{\alpha = dy^1}$

$\qquad\qquad\qquad\quad s=1 \quad \Rightarrow \quad \boxed{d\alpha\wedge d\alpha=0, \quad \alpha\wedge d\alpha=0 \quad d\alpha\neq 0} \quad \Rightarrow \quad \boxed{\alpha = y^0 dy^1}$

$r=1 \quad \Rightarrow \qquad s=1 \qquad d\alpha\neq 0, \ \boxed{d\alpha\wedge d\alpha=0, \ \alpha\wedge d\alpha\neq 0} \Rightarrow \boxed{\alpha = dy^1 + y^2 dy^3}$

$\qquad\qquad\qquad\quad s=2 \qquad \boxed{d\alpha\wedge d\alpha\neq 0, \ \alpha\wedge d\alpha\neq 0} \Rightarrow \boxed{\alpha = y^0 dy^1 + y^2 dy^3}$

16 An example of Exterior Differential System

Plebański !

Integrate:

$$\begin{cases} d\Gamma_{42} + \Gamma_{42}\wedge(\Gamma_{12}+\Gamma_{34}) = 0 \\ d(\Gamma_{12}+\Gamma_{34}) + 2\Gamma_{42}\wedge\Gamma_{31} = 0 \\ d\Gamma_{31} + (\Gamma_{12}+\Gamma_{34})\wedge\Gamma_{31} = 0 \end{cases} \qquad (*).$$

i.e. find 1-forms $\Gamma_{12}+\Gamma_{34}$, Γ_{42}, Γ_{31} on M such that $(*)$ holds.

① $\Gamma_{42}\wedge\Gamma_{31}\neq 0$

$(*) \Rightarrow \qquad d\Gamma_{42}\wedge\Gamma_{42}=0 \quad \xRightarrow{FR} \quad \Gamma_{42} = e^{\phi+\psi}dx$

$\qquad\qquad\;\; d\Gamma_{31}\wedge\Gamma_{31}=0 \quad \Rightarrow \quad \Gamma_{31} = e^{\phi-\psi}dy$

and the assumption ① says that $\quad dx\wedge dy\neq 0$

Inserting back:

$$e^{\phi+\psi}\left[d\phi + d\psi - \Gamma_{12} - \Gamma_{34}\right]\wedge dx = 0$$
$$e^{\phi-\psi}\left[d\phi - d\psi + \Gamma_{12} + \Gamma_{34}\right]\wedge dy = 0$$

$\Rightarrow$

$$d\phi + d\psi - \Gamma_{12} - \Gamma_{34} = -\rho\, dx$$
$$d\phi - d\psi + \Gamma_{12} + \Gamma_{34} = \sigma\, dy$$

$+$

$$\Rightarrow \quad 2d\phi = \sigma\, dy - \rho\, dx \qquad |\ \wedge dx \wedge dy$$

$$\Rightarrow \quad d\phi \wedge dx \wedge dy = 0$$

$$\Rightarrow \quad \phi = \phi(x,y) \,. \quad \text{only !!!}$$

moreover:

$$2\, d\phi = 2\phi_x\, dx + 2\phi_y\, dy = \sigma\, dy - \rho\, dx$$

$$\Rightarrow \quad \rho = -2\phi_x$$
$$\sigma = 2\phi_y$$

$$\Rightarrow \quad \underline{\underline{\Gamma_{12} + \Gamma_{34}}} = d\phi + d\psi + \rho\, dx =$$
$$= d\psi + \phi_x\, dx + \phi_y\, dy - 2\phi_x\, dx =$$
$$= \underline{\underline{d\psi - \phi_x\, dx + \phi_y\, dy}} \,.$$

Inserting this into the middle equation (*):

$$d\left[\cancel{d\psi} - \phi_x\, dx + \phi_y\, dy\right] + 2e^{2\phi}\, dx \wedge dy = 0$$

$$2\left(\phi_{xy} + e^{2\phi}\right) dx \wedge dy = 0$$

$$\boxed{\phi_{xy} + e^{2\phi} = 0}$$

ϕ must satisfy Liouville equation!

17) Integration of the Liouville equation!

One of the most famous nonlinear equations.
Because — it may be integrated!

$$\phi_{xy} + e^{2\phi} = 0 \Rightarrow \phi_{xxy} + 2\phi_x e^{2\phi} = 0$$

$$\Rightarrow \phi_{xxy} - 2\phi_x \phi_{xy} = 0$$

$$\Rightarrow \left(\phi_{xx} - \phi_x^2\right)_y = 0.$$

Symmetry

$$\Rightarrow \left(\phi_{yy} - \phi_y^2\right)_x = 0$$

But:

$$\phi_{xx} - \phi_x^2 = -e^{\phi}\left(e^{-\phi}\right)_{xx} = -F(x)$$

$$\phi_{yy} - \phi_y^2 = -e^{\phi}\left(e^{-\phi}\right)_{yy} = -G(y)$$

$$\Rightarrow \left(e^{-\phi}\right)_{xx} - F(x)e^{-\phi} = 0$$

$$\Rightarrow \left[\partial_x^2 - F(x)\right]e^{-\phi} = 0 \qquad (1)$$

$$\left[\partial_y^2 - G(y)\right]e^{-\phi} = 0 \qquad (2)$$

Think about (1) and (2) as ODEs for $e^{-\phi}$.

Then (1) should be solved as

$$e^{-\phi} = A_2(y)A_1(x) + B_2(y)B_1(x)$$

where $A_1(x)$, $B_1(x)$ are two indep. solutions of (1) and $A_2(y)$, $B_2(y)$ are arbitrary.

On the other hand (2) solves as:

$$e^{-\phi} = A_1(x)\,A_2(y) + B_1(x)\,B_2(y)$$

where now A_2, B_2 are two indep. solutions $\Rightarrow$

$$\Rightarrow \quad e^{-\phi} = A_1(x)A_2(y) + B_1(x)B_2(y)$$

where A_i, B_i are two sets of indep. solutions for the ODE (1).

We have:

$$0 = e^{-2\phi}(\phi_{xy} + e^{2\phi}) = 1 + e^{-2\phi}\phi_{xy} =$$

$$\ldots = 1 - (A_{1x}B_1 - B_{1x}A_1)(A_{2y}B_2 - B_{2y}A_2)$$

$$\frac{1}{A_1B_1A_2B_2} = \left[(\log A_1)_x - (\log B_1)_x\right]\left[(\log A_2)_y - (\log B_2)_y\right]$$

$$\frac{1}{A_1B_1A_2B_2} = \frac{\left(\frac{A_1}{B_1}\right)_x\left(\frac{A_2}{B_2}\right)_y}{\frac{A_1}{B_1}\,\frac{A_2}{B_2}} \iff \left(\frac{A_1}{B_1}\right)_x\left(\frac{A_2}{B_2}\right)_y = \frac{1}{(B_1B_2)^2}$$

Now

$$e^{2\phi} = \frac{1}{(A_1A_2+B_1B_2)^2} = \frac{1}{(B_1B_2)^2}\,\frac{1}{\left(1+\frac{A_1}{B_1}\frac{A_2}{B_2}\right)^2} =$$

$$= \frac{\left(\frac{A_1}{B_1}\right)_x\left(\frac{A_2}{B_2}\right)_y}{\left(1+\left(\frac{A_1}{B_1}\right)\left(\frac{A_2}{B_2}\right)\right)^2} = - \frac{\left(\frac{B_1}{A_1}\right)_x\left(\frac{A_2}{B_2}\right)_y}{\left(\frac{B_1}{A_1}+\frac{A_2}{B_2}\right)^2}$$

Introduce

$$p = p(x) = \frac{B_1}{A_1}, \quad q = q(y) = -\left(\frac{A_2}{B_2}\right)$$

$$\Rightarrow \quad e^{2\phi} = \frac{p'(x)q'(y)}{(p(x)-q(y))^2}$$

Check that

$$\boxed{e^{\phi} = \frac{\sqrt{p'(x)q'(y)}}{p(x)-q(y)}} \qquad (*)$$

with arbitrary smooth functions $p=p(x)$, $q=q(y)$ satisfy

$$\phi_{xy} + e^{2\phi} = 0,$$

Referring to the EDS from point 16) ①

we now have ϕ given by $(*)$.

We redefine ψ to $\psi \to \psi + \frac{1}{2}\log\frac{p'}{q'}$

This leads to

$$\Gamma_{42} = e^{\phi+\psi}dx = \frac{\sqrt{p'q'}}{p-q}\, e^{\psi}\sqrt{\frac{p'}{q'}}\,dx = \frac{e^{\psi}p'dx}{p-q} = \frac{e^{\psi}}{p-q}dp$$

$$\Gamma_{31} = \text{----} \qquad = \frac{e^{-\psi}}{p-q}dq$$

$$\Gamma_{12} + \Gamma_{34} = d\psi + \frac{dp+dq}{p-q}$$

$$\boxed{\begin{aligned} &\Gamma_{12}+\Gamma_{34} = d\psi + \frac{dp+dq}{p-q} \\ &\Gamma_{42} = \frac{e^{\psi}}{p-q}dp \\ &\Gamma_{31} = \frac{e^{-\psi}}{p-q}dq \end{aligned} \qquad \begin{gathered} \psi, p, q \\ \text{arbitrary} \\ \text{functions} \\ \text{such that} \\ dp\wedge dq \neq 0 \end{gathered}} \qquad (**)$$

We solved assuming ① i.e. that $\Gamma_{42}\wedge\Gamma_{31}\neq 0$. $\Leftrightarrow dp\wedge dq \neq 0$.

But one can see that $(**)$ is also a solution when $dp\wedge dq = 0$.

Actually $(**)$ is a general solution to <u>16)</u> ⑧

18) Flat $\underline{sl}(2,4)$ connections on M^4

Define

$$\Gamma = \begin{pmatrix} \frac{1}{2}(\Gamma_{12}+\Gamma_{34}) & \Gamma_{31} \\ -\Gamma_{42} & -\frac{1}{2}(\Gamma_{12}+\Gamma_{34}) \end{pmatrix}$$

Note that

$$\Gamma = \Gamma_{42} E_- + \frac{1}{2}(\Gamma_{12}+\Gamma_{34}) E_0 + \Gamma_{31} E_+$$

where

$$E_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad E_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$[E_0, E_\pm] = \pm 2 E_\pm, \quad [E_-, E_+] = E_0$$

Matrices (E_-, E_0, E_+) are closed w.r.t. taking of the commutator.

$$\underline{sl}(2,\mathbb{C}) = \{ \alpha_+ E_+ + \alpha_0 E_0 + \alpha_- E_-, \ \alpha_0, \alpha_+, \alpha_- \in \mathbb{C} \}$$

Γ is an $\underline{sl}(2,\mathbb{C})$-valued ~~matrix~~ form,

i.e. coefficients in the decomposition
$\Gamma = \Gamma_+ E_+ + \Gamma_0 E_0 + \Gamma_- E_-$ are 1-forms on M.

Define

$$\Omega = d\Gamma + \Gamma \wedge \Gamma$$

We calculate:

$$\Omega = \begin{pmatrix} \frac{1}{2} d(\Gamma_{12}+\Gamma_{34}) & d\Gamma_{31} \\ -d\Gamma_{42} & -\frac{1}{2} d(\Gamma_{12}+\Gamma_{34}) \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(\Gamma_{12}+\Gamma_{34}) & \Gamma_{31} \\ -\Gamma_{42} & -\frac{1}{2}(\Gamma_{12}+\Gamma_{34}) \end{pmatrix} \wedge \begin{pmatrix} \frac{1}{2}(\Gamma_{12}+\Gamma_{34}) & \Gamma_{31} \\ -\Gamma_{42} & -\frac{1}{2}(\Gamma_{12}+\Gamma_{34}) \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2} d(\Gamma_{12}+\Gamma_{34}) - \Gamma_{31}\wedge\Gamma_{42} & d\Gamma_{31} + (\Gamma_{12}+\Gamma_{34})\Gamma_{31} \\ -d\Gamma_{42} - \Gamma_{42}(\Gamma_{12}+\Gamma_{34}) & -\frac{1}{2} d(\Gamma_{12}+\Gamma_{34}) - \Gamma_{42}\Gamma_{31} \end{pmatrix} = >$$

$$\Omega = \left(d\Gamma_{42} + \Gamma_{42} \wedge \left(\Gamma_{12} + \Gamma_{34} \right) \right) E_-$$

$$+ \frac{1}{2} \left(d\left(\Gamma_{12} + \Gamma_{34} \right) + 2 \Gamma_{42} \wedge \Gamma_{31} \right) E_0$$

$$+ \left(d\Gamma_{31} + \left(\Gamma_{12} + \Gamma_{34} \right) \Gamma_{31} \right) E_+$$

Thus

$$\Omega = 0 \iff \begin{cases} d\Gamma_{42} + \Gamma_{42} \wedge \left(\Gamma_{12} + \Gamma_{34} \right) = 0 \\ d\left(\Gamma_{12} + \Gamma_{34} \right) + 2 \Gamma_{42} \wedge \Gamma_{31} = 0 \\ d\Gamma_{31} + \left(\Gamma_{12} + \Gamma_{34} \right) \wedge \Gamma_{31} = 0 \end{cases}$$

Hence $\binom{*}{*}$ of 17) represents the most general flat $\mathfrak{sl}(2,\mathbb{C})$ connection on M^4.