Lecture 6; 22.03.2011

Cartan's formalism of vector-valued forms (cont'd)

19) Coordinate (holonomic) frames

Recall: V with basis $\{e_\mu\}_{\mu=1,\dots,n}$
V^* with basis $\{e^\mu\}_{\mu=1,\dots,n}$
$e^\mu(e_\nu)=\delta^\mu{}_\nu$
} in any vector space.

Change of bases:

Bases:	$e'^\mu = a^\mu{}_\nu e^\nu$;	$e'_\mu = e_\nu a^{-1\nu}{}_\mu$	$a=(a^\mu{}_\nu)\in$
Components:	$v'^\mu = a^\mu{}_\nu v^\nu$;	$v'_\mu = v_\nu a^{-1\nu}{}_\mu$	$GL(n,\mathbb{R})$

On manifold $M \supset \mathcal{U}, x$ - coord. system

$x=(x^\mu)$ defines a coordinate frame:

$$\boxed{e_\mu = \frac{\partial}{\partial x^\mu} \quad ; \quad e^\mu = dx^\mu}$$

which is holonomic; i.e.

$$[e_\mu, e_\nu]=0 \Leftrightarrow de^\mu = 0.$$

More generally we have frames

$$\begin{cases} e_\mu = A^\nu{}_\mu(x)\dfrac{\partial}{\partial x^\nu} \\ e^\mu = A^{-1\mu}{}_\nu(x)\,dx^\nu \end{cases}$$ nonholonomic

20) Representation-valued forms.

$V = \mathbb{R}^N$; M - manifold of dimension n.

$$\rho : GL(n,\mathbb{R}) \underset{\text{homomorphism}}{\longrightarrow} GL(N,\mathbb{R})$$

i.e. $\rho(a \cdot b) = \rho(a) \cdot \rho(b)$ and

$$\rho(a) \in GL(N,\mathbb{R}) \subset End(\mathbb{R}^N).$$

Def

A k-form of representation type ρ is a map α assigning a field of a k-form with values in V to each field of coframes e s.t.

$$\alpha(ae) = \rho(a)\alpha(e).$$

Here $\alpha(e) \in V \otimes \Lambda \underline{M}$

Locally:

V has basis $\{E_A\}$; $A = 1, \dots N$ and let $\mu = 1, \dots n = \dim M$.

$$\Rightarrow \quad \alpha \overset{\text{ory}}{=} \alpha^A \qquad \alpha^A = \frac{1}{k!} \alpha^A{}_{\mu_1 \dots \mu_k} e^{\mu_1} \wedge \dots \wedge e^{\mu_k}$$

to stress that α depends on e we write

$$\alpha^A = \alpha^A(e)$$

$$\Rightarrow \rho(a) = \left(\rho(a)^A{}_B\right)$$

$$\alpha^A(ae) = \rho^A{}_B(a)\alpha^B(e).$$

2.1) Examples

1° Canonical 1-form on M

$V = \mathbb{R}^n$, $\varrho = \mathrm{id}$ i.e. $\varrho(a) = a$; $k = 1$

$\theta^A = \theta^\mu$ $\quad A = \mu = 1, \dots n$

$\boxed{\theta^\mu(e) := e^\mu}$ $\qquad$ $\boxed{\theta^A \text{ – 1-form of type id}}$

ok. because: $\theta^\mu(ae) = (ae)^\mu = a^\mu{}_\nu e^\nu = \varrho^\mu{}_\nu(a) e^\nu$ ✓

2° A vector field X on M defines a 0-form of type id:

$$X^\mu(e) := X \lrcorner e^\mu \left(= e^\mu(X)\right) \in \mathbb{R}^n$$

$$X^\mu(ae) = X \lrcorner (ae)^\mu = a^\mu{}_\nu (X \lrcorner e^\nu) = a^\mu{}_\nu X^\nu(e).$$ ✓

3° Forms of type Ad

$V = \mathrm{End}\, \mathbb{R}^n \ni (A^\mu{}_\nu) = A$ $\quad$ here $A = (\alpha_m)$
index

$a \in GL(n, \mathbb{R})$, $\varrho = \mathrm{Ad}$ i.e.

$\varrho(a) A = a A a^{-1}$ or:

$$[\varrho(a) A]^\mu{}_\nu = [a A a^{-1}]^\mu{}_\nu = a^\mu{}_\rho A^\rho{}_\sigma a^{-1\sigma}{}_\nu$$

for example: components of energy momentum-tensor

$T^\mu{}_\nu$ $=$ 0-form of type Ad.

4° More generally: tensor calculus "with indices"

$\Rightarrow$ Calculus of 0-forms of representation type ϱ^r_s.

5° Usual k-forms = <u>scalar</u> k-forms i.e.

$k \geq 0$, $V = \mathbb{R}^1$, $\rho(a) = 1$

2) <u>Tensor products</u>

$\alpha_i, k_i, \rho_i, V_i \quad (i=1,2)$ k_i-forms of type ρ_i.

We define their tensor products by:

$$(\alpha_1 \wedge \alpha_2)^{A_1 A_2} = \alpha_1^{A_1} \wedge \alpha_2^{A_2}$$

This is a (k_1+k_2)-form of type $\rho_1 \otimes \rho_2$.

(4) <u>Covariant differentiation</u>

1) <u>How to differentiate $X^\mu(e)$?</u>

$\boxed{\text{Goal}}$: Start with a <u>k</u>-form of type ρ and obtain <u>$(k+1)$</u>-form of <u>the same type</u> ρ

$$d(X^\mu(ae)) = d(a^\mu{}_\nu X^\nu(e)) =$$

$$= \underbrace{a^\mu{}_\nu \, dX^\nu(e)}_{\text{OK}} + \underbrace{da^\mu{}_\nu \, X^\nu(e)}_{\text{BAD!}}$$

So: $X^\mu(e)$ was a 0-form of type id but because of the BAD term $dX^\mu(e)$ is NOT a 1-form of type id.

Remedy: correct $d(X^\mu(ae))$ by adding a term so that the composed object will be a 1-form of type id.

Instead of

$$d(X^\mu(ae))$$

consider

$$dX^\mu(ae) + \boxed{\omega^\mu{}_\nu(ae) \wedge X^\nu(ae)} =: (DX^\mu)(ae)$$

Here $\omega^\mu{}_\nu(e)$ has values in 1-forms.

Calculate:

$$(DX^\mu)(ae) = d(X^\mu(ae)) + \omega^\mu{}_\nu(ae) X^\nu(ae) =$$

$$= a^\mu{}_\nu dX^\nu(e) + da^\mu{}_\nu X^\nu(e) + \omega^\mu{}_\nu(ae) a^\nu{}_\rho X^\rho(e) =$$

$$= a^\mu{}_\nu \left[DX^\nu(e) - \omega^\nu{}_\rho(e) X^\rho(e)\right] + da^\mu{}_\nu X^\nu(e) + \omega^\mu{}_\nu(ae) a^\nu{}_\rho X^\rho(e) =$$

$$= \underbrace{a^\mu{}_\nu DX^\nu(e)}_{\text{good!}} + \underbrace{\left[\omega^\mu{}_\nu(ae) a^\nu{}_\rho - a^\mu{}_\nu \omega^\nu{}_\rho(e) + da^\mu{}_\rho\right] X^\rho(e)}_{\text{BAD!}}$$

Make BAD term ZERO:

$$\omega^\mu{}_\nu(ae) a^\nu{}_\rho - a^\mu{}_\nu \omega^\nu{}_\rho(e) + da^\mu{}_\rho = 0$$

$$\Longrightarrow \boxed{\omega^\mu{}_\nu(ae) = a^\mu{}_\alpha \omega^\alpha{}_\rho(a) a^{-1\rho}{}_\nu - da^\mu{}_\rho a^{-1\rho}{}_\nu}$$

or

$$\boxed{\omega(ae) = \underline{\underline{a\,\omega(e)a^{-1}}} - da\cdot a^{-1}}$$

or becaus $\quad 0 = d(a\cdot a^{-1}) = da\cdot a^{-1} + a\,da^{-1}$

$$\boxed{\omega(ae) = \underline{\underline{a\,\omega(e)a^{-1}}} + a\,da^{-1}}$$

If only doubly underlined terms were present, $\omega(ae)$ would be 1-form of type Ad.

But because of the remaining term $\omega(ae)$ is a NEW object:

- it is 1-form (because it is 1-form valued)
- but it has bad transformation properties.

We decide to live with such objects.

Definition

Linear connection in M is an assignment ~~of~~ a field of an endomorphism valued 1-form $\omega^\mu{}_\nu(e)$ to each coframe (e) such that

$$\mathrm{End}(\mathbb{R}^n) \ni \omega(ae) = a\,\omega(e)a^{-1} - da\cdot a^{-1}$$

Then

$$(DX^\mu)(e) = dX^\mu(e) + \omega^\mu{}_\nu(e)\wedge X^\nu(e)$$

is a 1-form of type id as $X^\mu(e)$.

2) Linear connections and their curvatures

$$\begin{cases} e \longmapsto \omega(e) \quad \text{s.t.} \\ \omega(ae) = a\,\omega(e)\,a^{-1} - da\,a^{-1}. \end{cases} \qquad \omega(e) \in \mathrm{End}\,\mathbb{R}^n \otimes \Lambda^1$$

Note that:

At every point $p \in M$ there exists e s.t. $\omega(e)_p = 0$

Indeed:

if $\omega(e)_p \neq 0 \Rightarrow$

$$0 \underset{(*)}{=} \omega(ae)_p = a(p)\,\omega(e)_p\,a^{-1}(p) - (da)_p\,a(p)^{-1}$$

$$\Rightarrow \quad (da)_p = a(p)\,\omega(e)_p$$

So taking $a(p) = \mathbb{1}$, $(da)_p = \omega(e)_p$

I will satisfy $(*)$.

$\square$

QUESTION

When I can achieve $\omega(ae) = 0$ in a neighbourhood?

$$0 = \omega(ae) = a\,\omega(e)\,a^{-1} - da\,a^{-1}$$

$$\Rightarrow \quad da = a\,\omega(e) \qquad | \, d$$

$$\Rightarrow \quad 0 = d^2 a = da \wedge \omega(e) + a\,d\omega(e) =$$

$$= a\,\omega(e) \wedge \omega(e) + a\,d\omega(e) =$$

$$= a\left[d\omega(e) + \omega(e) \wedge \omega(e)\right]$$

$$\Rightarrow \quad \boxed{\Omega(e) = d\omega(e) + \omega(e) \wedge \omega(e) \equiv 0}$$

So vanishing of $\Omega(e)$ is a neccessary cond. for an existence of a s.t. $\omega(ae) \equiv 0$.

It is actually sufficient:

If $d\omega(e) + \omega(e) \wedge \omega(e) \equiv 0$ then

$$(*)\quad da = a \cdot \omega(e) \quad \text{is a linear equation.}$$

Write

$$a^{\mu}{}_{\nu} = a^{\mu}{}_{\nu}(0) + x^{\rho} b^{\mu}{}_{\nu\rho} + \dots$$

and see that all the further coeficients are determined by $(*)$. So, in analytic category, we have a solution. Also in smooth — Cauchy–Kowalewske

Thm

There exists a frame e' s.t. $\omega(e') \equiv 0$ in a neighbourhood if and only if

$$\Omega(e) = d\omega(e) + \omega(e) \wedge \omega(e) \equiv 0$$

Exercise

Check that $\Omega(ae) = a\,\Omega(e)\,a^{-1}$, i.e. that $\Omega(ae)$ is a 2-form of type Ad.

~~Def~~ **FACT**

Given a connection $\omega(e)$ there is a natural 2-form of type Ad associated with it which is called <u>curvature 2-form</u> of ω.

3) Covariant exterior differential

example

$$(DX^{\mu})(e) = dX^{\mu}(e) + \omega^{\mu}{}_{\nu}(e) X^{\nu}(e).$$

A bit of calculations:

$$\rho : GL(n,\mathbb{R}) \longrightarrow GL(N,\mathbb{R})$$

homomorphism of groups.

This induces a homomorphism of the corresponding Lie algebras $\mathrm{End}(\mathbb{R}^n)$ (Lie algebra for $GL(n,\mathbb{R})$) and $\mathrm{End}(\mathbb{R}^N)$ (Lie alg. for $GL(N,\mathbb{R})$).

We denote it by ρ' and define by:

$$\rho' : \mathrm{End}(\mathbb{R}^n) \longrightarrow \mathrm{End}(\mathbb{R}^N)$$

$$A \in \mathrm{End}(\mathbb{R}^n) \rightsquigarrow a(t) = \exp(tA) \in GL(n,\mathbb{R})$$

$$\rho(\exp(tA)) \in GL(N,\mathbb{R})$$

$$\frac{d}{dt}\rho(\exp(tA))\Big|_{t=0} \in \mathrm{End}\,\mathbb{R}^n$$

Define

$$\boxed{\rho'(A) = \frac{d}{dt}\rho(\exp(tA))\Big|_{t=0}}$$

Exercise

Check that

$$\rho'([A,B]) = [\rho'(A), \rho'(B)] \quad \text{i.e. that}$$

ρ' is homomorphism of algebras $\mathrm{End}\,\mathbb{R}^n$ and $\mathrm{End}\,\mathbb{R}^N$

With indices

$$a = (a^\mu{}_\nu) = \left(\exp(tA)\right)^\mu{}_\nu$$

$$\rho(a) = \left(\rho^A{}_B(a)\right) = \rho^A{}_B(\exp tA)$$

$$\frac{d}{dt}\rho^A{}_B(\exp tA)\Big|_{t=0} = \frac{\partial \rho^A{}_B}{\partial a}\Big|_{a=1} \cdot A = \frac{\partial \rho^A{}_B}{\partial a^\mu{}_\nu}\Big|_{a=1} A^\mu{}_\nu$$

$$\Rightarrow \quad \boxed{\rho'^A{}_B(A) = \frac{\partial \rho^A{}_B}{\partial a^\mu{}_\nu}\Big|_{a=1} A^\mu{}_\nu}$$

or

$$\boxed{\boxed{\rho'^A{}_B{}^\nu{}_\mu = \frac{\partial \rho^A{}_B}{\partial a^\mu{}_\nu}\Big|_{a=1}}}$$

Def

If α is a k-form of type ρ then

$$\boxed{D\alpha = d\alpha + \rho'(\omega) \wedge \alpha}$$

is a covariant exterior differential of α

With indices:

$$\boxed{D\alpha^A = d\alpha^A + \rho'^A{}_B{}^\nu{}_\mu \, \omega^\mu{}_\nu \wedge \alpha^B}$$

or in full beauty:

$$(D\alpha^A)(e) = d\alpha^A(e) + \rho'^A{}_B{}^\nu{}_\mu \, \omega^\mu{}_\nu(e) \wedge \alpha^B(e)$$

Proposition

If α is a k-form of type ρ then $D\alpha$ is $(k+1)$-form of type ρ.

Examples

1º) α is a scalar form (form of type $\rho = 1$, $V = \mathbb{R}^1$)

$\Rightarrow \quad D\alpha = d\alpha$

e.g. $F_{\mu\nu} \to F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$

$\Rightarrow \quad DF = dF.$

2º) $D\theta^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu := \Theta^\mu$

θ^μ is a natural - GOD-GIVEN - 1-form of type id on M

Θ^μ is a natural 2-form of type id on M
characterizing connection $\omega^\mu{}_\nu$

Def
Given a connection ω on M the 2-form of type id

$$\Theta^\mu = D\theta^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu$$

is called torsion 2-form

Recall ω a connection

$$\Rightarrow \quad \left.\begin{aligned} \Omega^\mu{}_\nu &= d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu \\ \Theta^\mu &= d\theta^\mu + \omega^\mu{}_\rho \wedge \theta^\rho \end{aligned}\right] \text{ two 2-forms characterizing connection}$$

3) Characterization of torsion

Note that

$$\Theta^\mu(e) = d\theta^\mu(e) + \omega^\mu{}_\nu(e)\wedge\theta^\nu(e) =$$

$$= de^\mu + \omega^\mu{}_\nu(e)\wedge e^\nu$$

and if $(de^\mu)_p = 0$ and $\omega^\mu{}_\nu(e)_p = 0 \Rightarrow \Theta^\mu(e)_p = 0$

The other way around:

if $\Theta^\mu(e)_p = 0$ we choose e' s.t. $\omega^\mu{}_\nu(e')_p = 0$

$$\Rightarrow (de'^\mu)_p = 0$$

Thus

$$\boxed{\Big((de^\mu)_p = 0 \text{ and } \omega(e)_p = 0\Big) \Leftrightarrow \Big(\Theta(e)_p = 0\Big)}$$

4) Important very simple fact.

Let β be an l-form of type σ depending on connection (e.g. $\beta = D\alpha$).

Proposition

$$\Big(\omega(e)_p = 0 \;;\; \beta(e)_p = 0\Big) \Rightarrow \Big(\beta(\cdot)_p = 0\Big)$$

because $\beta(ae)_p = \sigma(a)_p\, \beta(e)_p = 0$.

Possible applications:

α_1, k_1, ϱ_1 and α_2, k_2, ϱ_2 two forms.

We have

$$\boxed{D(\alpha_1\wedge\alpha_2) = D\alpha_1\wedge\alpha_2 + (-1)^{k_1}\alpha_1\wedge D\alpha_2}$$

Proof on both sides we have (k_1+k_2)-forms of the same type

But at each point we enter to a frame in which $\omega(e)_p = 0 \Rightarrow D = d$.

ok.

3°) Ricci formula

$$\boxed{D^2\alpha = \rho'(\Omega)\wedge\alpha}$$

Proof

$D(D\alpha) = D(d\alpha + \rho'(\omega)\wedge\alpha) = \rho'(d\omega)\wedge\alpha +$ terms homogeneous in ω WITHOUT DIFFERENTIALS of ω.

$\underset{\substack{\uparrow \\ \text{frame at } p \\ \text{in which } \omega_p = 0}}{=} \rho'(d\omega)\wedge\alpha = \rho'(d\omega + \omega\wedge\omega)\wedge\alpha = \underbrace{\rho'(\Omega)\wedge\alpha}_{\text{this is a } k+2 \text{ form of the same type as } \alpha}$

$\underset{\substack{\uparrow \\ \text{in any frame;} \\ \text{at any point!}}}{=} \rho'(\Omega)\wedge\alpha$ $\square$.

4) Bianchi identities

$\omega \to \Omega = d\omega + \omega\wedge\omega$

$(D\Omega)^\mu{}_\nu = d\Omega^\mu{}_\nu + \omega^\mu{}_\rho\wedge\Omega^\rho{}_\nu - \omega^\rho{}_\nu\wedge\Omega^\mu{}_\rho =$

$\underset{\substack{\uparrow \\ \omega_p = 0}}{=} d(d\omega^\mu{}_\nu + \omega^\mu{}_\rho\wedge\omega^\rho{}_\nu) \underset{\substack{\uparrow \\ \omega_p = 0}}{=} dd\omega^\mu{}_\nu \underset{\substack{\uparrow \\ \text{every frame} \\ \text{every point}}}{=} dd\omega^\mu{}_\nu = 0.$

$\boxed{D\Omega^\mu{}_\nu = 0}$ second Bianchi identity

$(D\Theta)^\mu \underset{\substack{\uparrow \\ \text{Ricci formula}}}{=} \Omega^\mu{}_\nu\wedge\theta^\nu$ $\Rightarrow$ $\boxed{(D\Theta)^\mu = \Omega^\mu{}_\nu\wedge\theta^\nu}$ first Bianchi identity.

5) Full system of Cartan's structure equations

$\omega^\mu{}_\nu(e)$ – linear connection on M

$\theta^\mu(e) := e^\mu$ – canonical 1-form of type id.

$$\Rightarrow \quad D\theta^\mu = \begin{cases} d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = \Theta^\mu & \text{Ist structure equation} \\ d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu = \Omega^\mu{}_\nu & \text{IInd structure equation} \end{cases}$$

$$\begin{cases} D\Theta^\mu = \Omega^\mu{}_\nu \wedge \theta^\nu & \text{Ist Bianchi identity} \\ D\Omega^\mu{}_\nu = 0 & \text{IInd Bianchi identity} \end{cases} \Bigg\} \text{ Compatibility conditions for the structure equations}$$

6) A comment on notation:

α^A – 0-form of type ρ

$$(D\alpha)^A(e) = \underset{\substack{\uparrow \\ \text{1-form of} \\ \text{type } \rho}}{\square^A{}_\mu} e^\mu := \nabla_\mu \alpha^A e^\mu$$

$\nabla_\mu \alpha^A$ – covariant derivative of α in the direction of a frame vector e_μ.

$$\boxed{(D\alpha)^A(e) = \nabla_\mu \alpha^A e^\mu} \quad \text{or}$$

$$\boxed{\nabla_\mu \alpha^A = e_\mu \lrcorner (D\alpha)^A(e)}$$

$$\boxed{\nabla_X \alpha^A = X \lrcorner (D\alpha)^A(e)}$$

Exercise Calculate

$$\nabla_{[\alpha} \nabla_{\beta]} \alpha^A = ?$$