Lecture 7. 29.03.2011

$\boxed{\text{Summary of lecture 6}}$

k-forms $\alpha \in V \otimes \Lambda^k M$ s.t. $\alpha(ae) = \rho(a)\alpha$

connections $\omega \in End(V) \otimes \Lambda^1 M$ $\quad \omega(ae) = a\,\omega(e)a^{-1} - da\,a^{-1}$

covariant differential $\quad D\alpha = d\alpha + \rho'(\omega) \wedge \alpha$

$$\rho'(A) = \frac{d}{dt}\Big|_{t=0} \rho(\exp(tA))$$

$$D\alpha^A = d\alpha^A + \rho^A{}_B{}^\mu{}_\nu\, \omega^\nu{}_\mu \wedge \alpha^B$$

$$\rho^A{}_B{}^\mu{}_\nu = \frac{\partial \rho^A{}_B}{\partial a^\nu{}_\mu}\Big|_{a=1}$$

e.g.

$$\begin{cases} DK^\mu = dK^\mu + \omega^\mu{}_\nu \wedge K^\nu \\ DK_\mu = dK_\mu - \omega^\nu{}_\mu \wedge K_\nu \\ DK^\mu{}_\nu = dK^\mu{}_\nu + \omega^\mu{}_\rho \wedge K^\rho{}_\nu - \omega^\rho{}_\nu \wedge K^\mu{}_\rho \\ DK_{\mu\nu} = dK_{\mu\nu} - \omega^\rho{}_\mu \wedge K_{\rho\nu} - \omega^\rho{}_\nu \wedge K_{\mu\rho} \end{cases}$$

etc.

moving (co)frame: $\theta^\mu(e) = e^\mu$

torsion : $\Theta^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = D\theta^\mu$

curvature: $\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu$

Bianchi identities: $\begin{cases} D\Theta^\mu = \Omega^\mu{}_\nu \wedge \theta^\nu \\ D\Omega^\mu{}_\nu = 0 \end{cases}$

Ricci formula: $\quad D^2\alpha^A = \rho'^A{}_B(\Omega) \wedge \alpha^B$

Note that

$\Theta^\mu = \frac{1}{2} T^\mu{}_{\nu\rho}\, \theta^\nu \wedge \theta^\rho \qquad T^\mu{}_{\nu\rho}$ – torsion tensor

$\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\sigma}\, \theta^\rho \wedge \theta^\sigma \qquad R^\mu{}_{\nu\rho\sigma}$ – curvature tensor

7) Covariant differentiation (Koszul notation)

α^A - zero-form $\Rightarrow$

$$D\alpha^A \in V \otimes \Lambda^1 M \Rightarrow \boxed{D\alpha^A = \nabla_\mu \alpha^A \theta^\mu}$$

$\nabla_\mu \alpha^A$ - covariant derivative of α^A in the direction of the frame vector e_μ

$$\boxed{\nabla_\mu \alpha^A = e_\mu \lrcorner D\alpha^A}$$

$$\underbrace{X^\mu e_\mu \lrcorner D\alpha^A}_{= \nabla_X \alpha^A} = X \lrcorner D\alpha^A \quad \Rightarrow \quad \boxed{\nabla_X \alpha^A = X \lrcorner D\alpha^A}$$

∇_X is a map "from tensors to tensors of the same type"

More formally (in the language "without indices")

$$\mathfrak{X}(M) \times \mathfrak{X}^r_s(M) \ni (X, K) \longmapsto \nabla_X K \in \mathfrak{X}^r_s(M)$$

Properties:

1° ∇_X preserves type of a tensor field

2° $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$

3° ∇ is $\mathbb{R}$-linear in K

4° satisfies Leibnitz rule:

$$\nabla_X (K \otimes L) = \nabla_X K \otimes L + K \otimes \nabla_X L$$

5° ∇_X commutes with contractions

6° $\nabla_X f = X(f)$.

Remark 1

if $K \in \mathcal{X}^r_s(M)$ and we have ∇ as above

$\Rightarrow$ we define ∇K as a tensor field of type $\binom{r}{s+1}$

by:

$$\nabla K(\lambda^1, \dots, \lambda^r; X_0, X_1, \dots X_s) = (\nabla_{X_0} K)(\lambda^1, \dots, \lambda^r; X_1, \dots X_s)$$

Exercise explain relation between ∇K and DK when K is a usual s-form here and 0-form with s antisymmetric indices here

Remark 2

If $\{e_\mu\}$ is a local frame in $U \subset M$ then

consider
$$\boxed{\nabla_{e_\mu} e_\nu = \Gamma^\rho{}_{\nu\mu} e_\rho}$$

Dual frame:

$$(\nabla_{e_\mu} e^\nu)(e_\rho) = C^1_1 (\nabla_{e_\mu} e^\nu \otimes e_\rho) =$$

$$= C^1_1 \nabla_{e_\mu}(e^\nu \otimes e_\rho) - C^1_1 e^\nu \otimes \nabla_{e_\mu} e_\rho =$$

$$= C^1_1 \cancel{\nabla_{e_\mu}}(\delta^\nu_\rho)$$

$$- C^1_1 (e^\nu \otimes \Gamma^\alpha{}_{\rho\mu} e_\alpha) =$$

$$= - \Gamma^\nu{}_{\rho\mu}$$

$$\Rightarrow \quad \boxed{\nabla_{e_\mu} e^\nu = - \Gamma^\nu{}_{\rho\mu} e^\rho}$$

Notation $\boxed{\nabla_{e_\mu} := \nabla_\mu}$

If

$$K = K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\, e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes e^{\nu_1} \otimes \dots \otimes e^{\nu_s}$$

$$\nabla_X K = X(K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s})\, e_{\mu_1} \dots \dots e^{\nu_s} +$$

$$+ K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\, \nabla_X e_{\mu_1} \otimes \dots \otimes e^{\nu_s} +$$

$$+ \dots \dots + K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\, e_{\mu_1} \otimes \dots \otimes \nabla_X e^{\nu_s} =$$

$$= X^\mu \Big[e_\mu(K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s})\, e_{\mu_1} \otimes \dots \otimes e^{\nu_s}$$

$$\Gamma^\rho{}_{\mu_1 \mu} K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\, e_\rho \otimes \dots \otimes e^{\nu_s} +$$

$$\vdots$$

$$- \Gamma^{\nu_s}{}_{\rho\mu} K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\, e_{\nu_1} \dots \dots \otimes e^\rho \Big] =$$

$$= X^\mu \Big[e_\mu(K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}) +$$

$$+ \Gamma^{\mu_1}{}_{\rho\mu} K^{\rho \dots \mu_r}{}_{\nu_1 \dots \nu_s} + \Gamma^{\mu_2}{}_{\rho\mu} K^{\mu_1 \rho \dots \mu_r}{}_{\nu_1 \dots \nu_s} + \dots +$$

$$- \Gamma^\rho{}_{\nu_1 \mu} K^{\mu_1 \dots \mu_r}{}_{\rho \dots \nu_s} - \dots \dots - \Gamma^\rho{}_{\nu_s \mu} K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \Big] \cdot$$

$$e_{\mu_1} \dots e^{\nu_s}$$

$$= X^\mu (\nabla_\mu K)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\, e_{\mu_1} \otimes \dots \otimes e^{\nu_s}$$

$$\boxed{\begin{aligned} (\nabla_\mu K)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = {} & e_\mu(K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}) + \\ & + \Gamma^{\mu_1}{}_{\rho\mu} K^{\rho \dots \mu_r}{}_{\nu_1 \dots \nu_s} + \dots + \Gamma^{\mu_r}{}_{\rho\mu} K^{\mu_1 \dots \rho}{}_{\nu_1 \dots \nu_s} + \\ & - \Gamma^\rho{}_{\nu_1 \mu} K^{\mu_1 \dots \mu_r}{}_{\rho \dots \nu_s} + \dots - \Gamma^\rho{}_{\nu_s \mu} K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \rho} \end{aligned}}$$

Change of a frame:

$$e_\mu = a^\nu{}_\mu e'_\nu \qquad a \in GL(n,\mathbb{R})$$

$$\Gamma^\rho{}_{\nu\mu} e_\rho = \Gamma^\rho{}_{\nu\mu} a^\alpha{}_\rho e'_\alpha$$

$$\|$$

$$\nabla_\mu e_\nu = \nabla_\mu (a^\alpha{}_\nu e'_\alpha) = e_\mu(a^\alpha{}_\nu) e'_\alpha + a^\alpha{}_\nu (\nabla_\mu e'_\alpha)$$

$$= e_\mu(a^\alpha{}_\nu) e'_\alpha + a^\alpha{}_\nu a^\beta{}_\mu \nabla_{e'_\beta} e'_\alpha =$$

$$= e_\mu(a^\alpha{}_\nu) e'_\alpha + a^\alpha{}_\nu a^\beta{}_\mu \Gamma'^\rho{}_{\alpha\beta} e'_\rho =$$

$$= a^\rho{}_\nu a^\sigma{}_\mu \Gamma'^\alpha{}_{\rho\sigma} e'_\alpha + e_\mu(a^\alpha{}_\nu)$$

$$\boxed{a^\alpha{}_\rho \Gamma^\rho{}_{\nu\mu} = \Gamma'^\alpha{}_{\rho\sigma} a^\rho{}_\nu a^\sigma{}_\mu + e_\mu(a^\alpha{}_\nu)}$$

introduce: $\boxed{\Gamma^\rho{}_\nu := \Gamma^\rho{}_{\nu\mu} e^\mu}$

$$\Rightarrow \quad a^\alpha{}_\rho \Gamma^\rho{}_\nu = \Gamma'^\alpha{}_{\rho\sigma} a^\rho{}_\nu e'^\sigma + da^\alpha{}_\nu$$

$$\Rightarrow \quad a^\alpha{}_\rho \Gamma^\rho{}_\nu = \Gamma'^\alpha{}_\rho a^\rho{}_\nu + da^\alpha{}_\nu$$

$$\boxed{\Gamma'^\alpha{}_\rho = a^\alpha{}_\beta \Gamma^\beta{}_\delta a^{-1\delta}{}_\rho - da^\alpha{}_\beta a^{-1\beta}{}_\rho}$$

writing $\boxed{\Gamma^\rho{}_\nu = \omega^\rho{}_\nu(e)}$:

$$\boxed{\omega^\rho{}_\nu(ae) = a^\rho{}_\beta \omega^\beta{}_\delta(e) a^{-1\delta}{}_\rho - da^\alpha{}_\beta a^{-1\beta}{}_\rho}$$

$\Rightarrow$ we have a connection in the previous sense!

8) <u>Torsion tensor</u> $X, Y \in \mathfrak{X}(M)$, ∇ - covariant diff.

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

<u>Fact</u> T is a type $\binom{1}{2}$ tensor field!

<u>Proof</u>

$$T(X,Y) = -T(Y,X)$$

f linearity?

$$T(fX,Y) = f\nabla_X Y - \nabla_Y(fX) - [fX,Y] =$$

$$= f\nabla_X Y - \cancel{Y(f) X} - f\nabla_Y X - f[X,Y] + \cancel{Y(f) X}$$

$$= f\,T(X,Y).$$

$\square$.

In a local frame:

$$T(e_\mu, e_\nu) = \nabla_\mu e_\nu - \nabla_\nu e_\mu - [e_\mu, e_\nu] =$$

$$= \left(\Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu} - C^\rho_{\mu\nu}\right) e_\rho = T^\rho_{\mu\nu} e_\rho$$

$$\boxed{T^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu} - C^\rho_{\mu\nu}} \quad \underline{\text{NEW}} \qquad \underline{T^\rho_{\mu\nu} = -T^\rho_{\nu\mu}}$$

Note that in <u>a holonomic frame</u> $C^\rho_{\mu\nu} \equiv 0 \Rightarrow$

$$T^\rho_{\mu\nu} = 0 \text{ if } \Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$$

Historic name for a connection without torsion

$\equiv$ <u>symmetric</u> connection! $\underline{\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}}$

<u>Calculating:</u> (in the language of Lecture 6!)

$$\Theta^\rho(e) = de^\rho + \Gamma^\rho{}_\mu \wedge e^\mu = -\tfrac{1}{2} C^\rho_{\mu\nu} + \Gamma^\rho_{\mu\nu} e^\nu \wedge e^\mu$$

$$\shortparallel$$

$$\tfrac{1}{2} T^\rho_{\mu\nu} e^\mu \wedge e^\nu \Rightarrow \boxed{T^\rho_{\mu\nu} = -\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\nu\mu} - C^\rho_{\mu\nu}} \quad \underline{\text{OLD}}$$

$$\boxed{\begin{aligned} &T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \\ &\qquad \Longleftrightarrow \quad \Theta^\rho = d\theta^\rho + \omega^\rho{}_\mu \wedge \theta^\mu = \tfrac{1}{2} T^\rho{}_{\mu\nu}\, \theta^\mu \wedge \theta^\nu \end{aligned}}$$

9) Curvature tensor

$X, Y \ni \mathfrak{X}(M)$, ∇

$$R(X,Y) : \mathcal{T}(M) \longrightarrow \mathcal{T}(M)$$

$$R(X,Y)K = [\nabla_X, \nabla_Y]K - \nabla_{[X,Y]}K$$

Fact the second term makes R to be f-linear in K, X, Y

Properties of $R(X,Y)$:

1. $\mathcal{F}$-linear in K, X, Y
2. Preserves type
3. commutes with contractions
4. satisfies Leibnitz rule.

e.g.

$$R(X,Y)f = [\nabla_X, \nabla_Y]f - [X,Y](f) =$$

$$= \nabla_X Y(f) - \nabla_Y X(f) - [X,Y]f = [X,Y]f - [X,Y]f = 0.$$

$\Rightarrow$ f-linear in K

$\models 1$.

in particular:

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X,Y,Z) \longmapsto R(X,Y)Z \in \mathfrak{X}(M)$$

is f-linear in X, Y, Z.

Proof

$$R(fX,Y)Z = [f\nabla_X, \nabla_Y]Z - \nabla_{[fX,Y]}Z =$$

$$= f[\nabla_X, \nabla_Y]Z - \cancel{\nabla_Y(f)\nabla_X Z} - f\nabla_{[X,Y]}Z + \cancel{Y(f)\nabla_X Z} =$$

$$= f R(X,Y)Z$$

In second argument: $R(X,Y) = -R(Y,X)$ □.

$R(X,Y)$ defines a tensor of type $\binom{1}{3}$:

$$\boxed{R(e_\mu, e_\nu)e_\rho = R^\sigma{}_{\rho\mu\nu}\, e_\sigma}$$

$$R^\sigma{}_{\rho\mu\nu} = -R^\sigma{}_{\rho\nu\mu}$$

$$\Rightarrow \quad \boxed{\Omega^\sigma{}_\rho(e) = \tfrac{1}{2} R^\sigma{}_{\rho\mu\nu}\, e^\mu \wedge e^\nu} \qquad (*)$$

Fact

$$\Omega^\sigma{}_\rho = d\omega^\sigma{}_\rho + \omega^\sigma{}_\mu \wedge \omega^\mu{}_\nu$$

Indeed:

$$R^\sigma{}_{\rho\mu\nu} e_\sigma = R(e_\mu, e_\nu)e_\rho = [\nabla_\mu, \nabla_\nu]e_\rho - \nabla_{[e_\mu, e_\nu]} e_\rho =$$

$$= \ldots\ldots$$

$$R^\sigma{}_{\rho\mu\nu} = e_\mu(\Gamma^\sigma{}_{\rho\nu}) - e_\nu(\Gamma^\sigma{}_{\rho\mu}) - c^\tau{}_{\mu\nu}\Gamma^\sigma{}_{\rho\tau} + \Gamma^\tau{}_{\rho\nu}\Gamma^\sigma{}_{\tau\mu}$$
$$- \Gamma^\tau{}_{\rho\mu}\Gamma^\sigma{}_{\tau\nu}$$

and this easily proves $(*)$

q.

$$\boxed{\begin{aligned} &R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \\ &\iff \Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu = \tfrac{1}{2} R^\mu{}_{\nu\rho\sigma}\, \theta^\rho \wedge \theta^\sigma \end{aligned}}$$

10) Parallel transport

γ - a curve of class C^1, $\quad X = \frac{dx}{dt}$

$\nabla_X K$ is called in such a case an absolute derivative of K along a curve γ.

To define this one needs only **K** along γ !!!

Indeed: $\gamma(t) = (x^\mu(t)) \qquad X^\mu = \frac{dx^\mu}{dt}$

$$(\nabla_X K)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} =$$

$$= X^\rho (\nabla_\rho K)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} =$$

$$= \underbrace{\frac{dx^\rho}{dt} \frac{\partial}{\partial x^\rho}}_{\frac{d}{dt}} \left(K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \right) + \underbrace{X^\rho \Gamma^{\mu_1}_{\alpha\rho} K^{\alpha \dots \mu_r}{}_{\nu_1 \dots \nu_s} + \dots}_{\text{no problem here}}$$

$$= \frac{d}{dt} \left(K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}(t) \right) + \dots$$

$$= \left(\frac{D}{dt} K \right)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}$$

$$\boxed{\left(\frac{D}{dt} K \right)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = \frac{d}{dt} K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} + \frac{dx^\rho}{dt} \Gamma^{\mu_1}_{\alpha\rho} K^{\alpha \dots \mu_r}{}_{\nu_1 \dots \nu_s} + \dots - \frac{dx^\rho}{dt} \Gamma^{\alpha}_{\nu_s \rho} K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \alpha}}$$

Definition

Tensor field K is parallely transported along γ

iff $\nabla_X K = 0$ $\left(\frac{D}{dt} K \equiv 0 \right)$

Note that if $\Gamma \equiv 0 \Rightarrow \left(\frac{D}{dt} K = 0 \Leftrightarrow \frac{d}{dt} K \equiv 0 \right)$,

i.e. K is parallely transported in an affine space with $\Gamma \equiv 0$ iff K has constant components!

frame dependendent!

So in such a case this is the usual parallel transport.

We have

$\gamma : [0,1] \longmapsto M$ and let's assume that γ is piecewise continuous

$$P_\gamma : \begin{pmatrix} \text{Tensor algebra} \\ \text{at } \gamma(0) \end{pmatrix} \longmapsto \begin{pmatrix} \text{Tensor algebra} \\ \text{at } \gamma(1) \end{pmatrix}$$

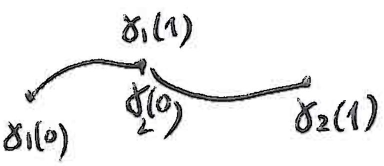
$$K_{\gamma(0)} \longmapsto K_{\gamma(1)}$$

propagate the tensor $K_{\gamma(0)}$ along γ in such a way that $\frac{DK}{dt} = 0$ all the time.

Properties of P_γ

1) does not depend on parametrization

2) if two curves γ_1, γ_2 s.t. $\gamma_2(0) = \gamma_1(1)$



$$\Rightarrow (\gamma_2 * \gamma_1)(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \frac{1}{2} < t \le 1 \end{cases}$$

$$\boxed{P_{\gamma_2 * \gamma_1} = P_{\gamma_2} \circ P_{\gamma_1}}$$

3°) P_γ is an isomorphism of tensor algebras that preserves the type and commutes with contractions.

11) Selfparallels

· Def

γ - a curve of class C^2 is selfparallel iff

$$\nabla_X X = \lambda X, \qquad X = \frac{d\gamma}{dt}.$$

Locally:

$$\frac{d}{dt} X^\mu + X^\beta \Gamma^\mu_{\alpha\beta} X^\alpha = \lambda X^\mu, \qquad X^\mu = \frac{dx^\mu}{dt}$$

$$\Rightarrow \boxed{\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = \lambda \frac{dx^\mu}{dt}}$$

Reparametrization:

$$t \to t' = f(t) \qquad \dot{f} \neq 0.$$

$$\frac{d}{dt} = \frac{df}{dt}\frac{d}{dt'} = \dot{f}\frac{d}{dt'}$$

$$\frac{d^2x^\mu}{dt^2} = \frac{d}{dt}\left(\dot{f}\frac{dx^\mu}{dt'}\right) = \ddot{f}\frac{dx^\mu}{dt'} + \dot{f}^2 \frac{d^2x^\mu}{dt'^2}$$

$$\Rightarrow \dot{f}^2\left(\frac{d^2x^\mu}{dt'^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{dt'}\frac{dx^\beta}{dt'}\right) = (\lambda\dot{f} - \ddot{f})\frac{dx^\mu}{dt'}$$

$$\Rightarrow \lambda \longrightarrow \lambda' = \frac{\lambda\dot{f} - \ddot{f}}{\dot{f}^2}$$

Corollaries:

1) definition is parametrization independent $\lambda \to \lambda' = \frac{\lambda \dot{f} - \ddot{f}}{\dot{f}^2}$

2) there exist a parameter t' s.t. $\lambda' = 0$.
   Such an parameter is called an affine parameter

3) Affine parameter is given up to $t' \to at' + b$
   $0 \neq a, b =$ const.

Equation of selfparallels in an affine parametrization:

$$\boxed{\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} = 0.}$$

(5) Riemannian manifolds

(M, g) where g is symmetric, nondegenerate, tensor of type $\binom{0}{2}$.

$$g : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{F}(M)$$

1) $g(X,Y) = g(Y,X)$

2) $g(X,Y) = 0 \ \forall X \Rightarrow Y = 0$.

Signature of g:

at every point g can be brought to $g = g_{\mu\nu} e^\mu \otimes e^\nu$

$$g_{\mu\nu} = \mathrm{diag}(\underbrace{1, \dots 1}_{p}, \underbrace{-1, \dots -1}_{q}) \quad \text{by } GL(n, \mathbb{R}) \text{ transf.}$$

if a signature is
$(1, \dots 1, -1, \dots -1)$ at a point $\Rightarrow$
$\Rightarrow$ by continuity is the same in a neighbourhood.

1) <u>Geodesics</u> $\gamma = \gamma(t)$ curve of class C^2 in M

<u>Arc length:</u> γ , $X = \frac{d\gamma}{dt}$

$$s := \int_{t_0}^{t_1} \sqrt{|g(X,X)|}\, dt \quad , \quad \gamma = (x^\mu)$$

$$s = \int_{t_0}^{t} \sqrt{\left|g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}\right|}\, dt$$

<u>note:</u> s is parametrization independent!

<u>Def</u>

$\gamma = \gamma(t)$ is a geodesic iff γ satisfies the necessary condition for extremalizing s.

I.e.

$$\left(\gamma \text{ is geodesic} \right) \Leftrightarrow \left(\begin{array}{l} \delta \int_{t_0}^{t} \sqrt{\left|g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}\right|}\, dt = 0 \\ \delta x^\mu(t_0) = \delta x^\mu(t_1) = 0 \end{array} \right)$$

<u>Euler-Lagrange equations:</u>

$$\mathcal{L} = \mathcal{L}(x^\mu, \dot{x}^\mu) = \sqrt{|g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu|}$$

$$\Rightarrow \frac{d}{dt} \frac{\partial}{\partial \dot{x}^\mu} \mathcal{L} - \frac{\partial}{\partial x^\mu} \mathcal{L} = 0 \Leftrightarrow$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{x}^\mu} \sqrt{|g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu|} - \frac{\partial}{\partial x^\mu} \sqrt{|g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu|} = 0$$

$\frac{\partial}{\partial x^\mu} = \partial_\mu$:

$$\frac{d}{dt} \frac{g_{\mu\nu} \dot{x}^\nu}{\sqrt{|\quad|}} - \frac{1}{2} \frac{g_{\alpha\nu,\mu} \dot{x}^\alpha \dot{x}^\nu}{\sqrt{|\quad|}} = 0 \qquad (*)$$

formulation is parametrization independent; we can choose any parameter. We take <u>the arc length</u>: $s = \int \sqrt{\ }\, dt$

$$\Rightarrow \frac{d}{ds} = \frac{d}{\sqrt{\ }\, dt}$$

$$\Rightarrow (*) \Big| \frac{1}{\sqrt{\ }}$$

$$\frac{d}{ds}\left(g_{\mu\nu} \frac{dx^\nu}{ds}\right) - \frac{1}{2} g_{\alpha\nu,\mu} \frac{dx^\alpha}{ds} \frac{dx^\nu}{ds} = 0$$

$$g_{\mu\nu} \frac{d^2x^\nu}{ds^2} + g_{\mu\nu,\alpha} \frac{dx^\alpha}{ds} \frac{dx^\nu}{ds} - \frac{1}{2} g_{\alpha\nu,\mu} \frac{dx^\alpha}{ds} \frac{dx^\nu}{ds} = 0$$

$$g_{\mu\nu} \frac{d^2x^\nu}{ds^2} + \frac{1}{2}\left(g_{\mu\nu,\alpha} + g_{\mu\alpha,\nu} - g_{\alpha\nu,\mu}\right) \frac{dx^\alpha}{ds} \frac{dx^\nu}{ds} = 0$$

$$\frac{d^2x^\nu}{ds^2} + \underbrace{\frac{1}{2} g^{\nu\varrho}\left(g_{\varrho\mu,\alpha} + g_{\alpha\varrho,\mu} - g_{\mu\alpha,\varrho}\right)}_{\left\{ {\nu \atop \alpha\mu} \right\}} \frac{dx^\alpha}{ds} \frac{dx^\mu}{ds} = 0$$

$$\Rightarrow \boxed{\frac{d^2x^\nu}{ds^2} + \left\{ {\nu \atop \alpha\mu} \right\} \frac{dx^\alpha}{ds} \frac{dx^\mu}{ds} = 0}$$

$$\boxed{\left\{ {\nu \atop \alpha\mu} \right\} = \frac{1}{2} g^{\nu\varrho}\left(g_{\varrho\mu,\alpha} + g_{\alpha\varrho,\mu} - g_{\mu\alpha,\varrho}\right)}$$

This defines a connection $\omega^\mu{}_\nu$, which in a coordinate frame dx^μ is

$$\omega^\mu{}_\nu(dx^\alpha) = \left\{ {\mu \atop \alpha\nu} \right\} dx^\alpha .$$

$\Longrightarrow$ <u>geodesics are selfparallels in this connection.</u>

Note that

① $\Gamma^\mu{}_{\alpha\nu} = \left\{ {\mu \atop \alpha\nu} \right\} = \left\{ {\mu \atop \nu\alpha} \right\} = \Gamma^\mu{}_{\nu\alpha}$

and dx^μ is a coordinate frame. Then $C^\mu{}_{\nu\rho} = 0$

and $T^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu} - \Gamma^\mu{}_{\nu\rho} = 0$

$\Rightarrow$ this connection has <u>NO</u> torsion

② $$\underset{\Gamma^\mu{}_\nu}{\overset{\parallel}{\omega^\mu{}_\nu}}(dx^\alpha) = \frac{1}{2} g^{\mu\rho}(g_{\rho\nu,\alpha} + g_{\alpha\rho,\nu} - g_{\nu\alpha,\rho})dx^\alpha =$$

$$= \frac{1}{2} g^{\mu\rho} dg_{\rho\nu} + \frac{1}{2} g^{\mu\rho}(g_{\alpha\rho,\nu} - g_{\nu\alpha,\rho})dx^\alpha$$

Thus:

$$(Dg)_{\alpha\nu} = dg_{\alpha\nu} - \Gamma^\rho{}_\alpha g_{\rho\nu} - \Gamma^\rho{}_\nu g_{\alpha\rho} =$$

$$= \cancel{dg_{\alpha\nu}} - g_{\rho\nu}\left[\cancel{\frac{1}{2} g^{\rho\mu} dg_{\mu\alpha}} + \frac{1}{2} g^{\rho\mu}(\cancel{g_{\gamma\mu,\alpha}} - g_{\alpha\gamma,\mu})dx^\gamma\right]$$

$$- g_{\alpha\rho}\left[\cancel{\frac{1}{2} g^{\rho\mu} dg_{\mu\nu}} + \frac{1}{2} g^{\rho\mu}(g_{\gamma\mu,\nu} - \cancel{g_{\nu\gamma,\mu}})dx^\gamma\right] = 0$$

$\Rightarrow$ $\omega^\mu{}_\nu(dx^\alpha) = \left\{ {\mu \atop \alpha\nu} \right\} dx^\alpha$

is a connection which is

① torsionless $\Theta = 0$ ✓

② such that $Dg \equiv 0$ ✓

such connection is called <u>metric</u>

this is called Levi-Civita connection!