Lecture 8 5.04.2011

Selfparallels: $\nabla_X X = \lambda X$

locally $\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}(x)\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = \lambda \frac{dx^\mu}{dt}$

reparametrization $t \to t' \Rightarrow \frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0$

in an affine parametrization.

Geodesics

extremals for the functional $s = \int_{t_0}^{t_1} \sqrt{|g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu|}\, dt$ i.e.

$\delta s = 0$ and $\delta x^\mu(t_0) = \delta x^\mu(t_1) = 0$ $\Longrightarrow$ $ds = \sqrt{|g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu|}\,dt$

Lagrangian: $\mathcal{L} = \mathcal{L}(x^\alpha, \dot{x}^\alpha) = \sqrt{|g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu|}$

Euler-Lagrange eqs: $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial}{\partial x^\mu}\mathcal{L} = 0$ $\left(\frac{\partial f}{\partial x^\alpha} = f_{,\alpha}\right)$

Calculations:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}^\mu}\sqrt{|g_{\mu\rho}\dot{x}^\mu\dot{x}^\rho|} - \frac{\partial}{\partial x^\mu}\sqrt{|g_{\nu\rho}\dot{x}^\nu\dot{x}^\rho|} = 0$$

$$\frac{d}{dt}\frac{g_{\nu\rho}\delta^\nu_\mu\dot{x}^\rho + g_{\nu\rho}\dot{x}^\nu\delta^\rho_\mu}{2\sqrt{|\quad|}} - \frac{g_{\nu\rho,\mu}\dot{x}^\nu\dot{x}^\rho}{2\sqrt{|\quad|}} = 0$$

$$\frac{d}{dt}\frac{g_{\mu\rho}\dot{x}^\rho}{\sqrt{|\quad|}} - \frac{1}{2}\frac{g_{\nu\rho,\mu}\dot{x}^\nu\dot{x}^\rho}{\sqrt{|\quad|}} = 0 \quad \Big| \quad \frac{1}{\sqrt{|\quad|}}, \quad \frac{d}{ds} = \frac{d}{\sqrt{\quad}\,dt}$$

$$\frac{d}{ds}\left(g_{\mu\rho}\frac{dx^\rho}{ds}\right) - \frac{1}{2}g_{\nu\rho,\mu}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

$$g_{\mu\rho,\nu}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} + g_{\mu\rho}\frac{d^2x^\rho}{ds^2} - \frac{1}{2}g_{\nu\rho,\mu}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

$$g_{\mu\rho}\frac{d^2x^\rho}{ds^2} + \frac{1}{2}\left(g_{\mu\rho,\nu} + g_{\mu\nu,\rho} - g_{\nu\rho,\mu}\right)\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

$$\boxed{\frac{d^2x^\sigma}{ds^2} + \frac{1}{2}g^{\sigma\mu}\left(g_{\mu\rho,\nu} + g_{\nu\mu,\rho} - g_{\rho\nu,\mu}\right)\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0}$$

Introducing:

$$\boxed{\left\{ {\sigma \atop \nu\rho} \right\} = \frac{1}{2} g^{\sigma\mu} \left(g_{\mu\rho,\nu} + g_{\nu\mu,\rho} - g_{\rho\nu,\mu} \right)}$$

we bring the equation for geodesics into the form:

$$\boxed{\frac{d^2x^\sigma}{ds^2} + \left\{ {\sigma \atop \nu\rho} \right\} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} = 0}$$

$\left\{ {\sigma \atop \nu\rho} \right\}$ – Christoffel symbols

2) Properties of Christoffel symbols

Obviously: $\left\{ {\sigma \atop \nu\rho} \right\} = \left\{ {\sigma \atop \rho\nu} \right\}$.

Introduce $\Gamma^\sigma{}_{\nu\rho} = \left\{ {\sigma \atop \nu\rho} \right\}$ and $\Gamma^\sigma{}_\nu = \left\{ {\sigma \atop \nu\rho} \right\} dx^\rho$

This defines a connection $\omega^\sigma{}_\nu$ s.t.

$$\omega^\sigma{}_\nu (dx) = \left\{ {\sigma \atop \nu\rho} \right\} dx^\rho. \quad \Longleftrightarrow \quad \nabla$$

Properties of this connection?

① Geodesics of (M, g) are selfparallels for ∇ in an affine parametrization. And arc length is an affine parameter.

② $\Gamma^\sigma{}_{\nu\rho} = \Gamma^\sigma{}_{\rho\nu}$ and the connection is defined in coordinate frame.

Recall: that the torsion tensor (in any frame) is given by:

$$T^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\rho\nu} - \Gamma^\mu{}_{\nu\rho} - C^\rho{}_{\mu\nu} = \underset{\substack{\uparrow \\ \text{coord. frame} \\ C^\mu{}_{\nu\rho} \equiv 0}}{=} \Gamma^\mu{}_{\rho\nu} - \Gamma^\mu{}_{\nu\rho} = 0.$$

$\Rightarrow$ connection associated with $\left\{ {\sigma \atop \mu\nu} \right\}$ has NO torsion.

(3) Let us write down $\Gamma^\mu{}_\nu$ explicitly:

$$\Gamma^\sigma{}_\nu = \omega^\sigma{}_\nu(dx) = \left\{ {\sigma \atop \nu\rho} \right\} dx^\rho =$$
$$= \frac{1}{2} g^{\sigma\mu} \left(g_{\mu\nu,\rho} + g_{\rho\mu,\nu} - g_{\nu\rho,\mu} \right) dx^\rho$$
$$= \frac{1}{2} g^{\sigma\mu} dg_{\mu\nu} + \frac{1}{2} g^{\sigma\mu} (g_{\rho\mu,\nu} - g_{\nu\rho,\mu}) dx^\rho$$

$\rightsquigarrow$ we are on (M,g) $\Rightarrow$ $g = g_{\mu\nu} \underbrace{dx^\mu dx^\nu}$

$dx^\mu dx^\nu = \frac{1}{2}(dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu)$

$\Rightarrow$ metric tensor $g_{\mu\nu} = g_{\mu\nu}(x)$. $\leftarrow$ in a frame dx

+ connection $\Gamma^\mu{}_\nu$ $\leftarrow$ in a frame dx

$\Rightarrow$ exterior covariant differential in a frame dx:

$$(Dg)_{\mu\nu} = dg_{\mu\nu} - \Gamma^\rho{}_\mu g_{\rho\nu} - \Gamma^\rho{}_\nu g_{\mu\rho} =$$
$$= \cancel{dg_{\mu\nu}} - g_{\rho\nu}\left(\cancel{\tfrac{1}{2} g^{\rho\alpha} dg_{\alpha\mu}} + g^{\rho\alpha} \cancel{(g_{\beta\alpha,\mu} - g_{\mu\beta,\alpha})} dx^\beta \right)$$
$$- g_{\mu\rho}\left(\cancel{\tfrac{1}{2} g^{\rho\alpha} dg_{\alpha\nu}} + g^{\rho\alpha} \cancel{(g_{\beta\alpha,\nu} - g_{\nu\beta,\alpha})} dx^\beta \right) = 0,$$

$\Rightarrow$ $(Dg)_{\mu\nu} = 0$ in a coordinate frame $\Rightarrow$

in every frame.

<u>Proposition</u>

Connection defined in coordinate frame by christoffel symbols is

(1) torsion-free $T^\mu{}_{\nu\rho} \equiv 0$

(2) metric $(Dg)_{\mu\nu} \equiv 0$.

3) Levi-Civita connection. Determination of a connection in terms of Θ^μ and $(Dg)_{\mu\nu}$.

Definition

the connection associated with Christoffel symbols on a Riemannian manifold (M,g) is called Levi-Civita connection.

$$(M,g) \longrightarrow \nabla = \overset{L.C.}{\nabla}$$

$$\overset{L.C.}{\nabla} \to \Theta \equiv 0, \quad Dg \equiv 0.$$

Let us now fix (M,g)

$g_{\mu\nu}(e) = g(e_\mu, e_\nu)$ $\qquad$ $g_{\mu\nu}$ - 0-form of type $\binom{0}{2}$

Suppose now that we have

1) $A^\mu(e) = \frac{1}{2} A^\mu{}_{\nu\rho}\, e^\nu \wedge e^\rho$ $\qquad$ A^μ - 2-form of type $\binom{1}{0}$

and

2) $B_{\mu\nu}(e) = B_{\mu\nu\rho} e^\rho$ $\qquad$ $B_{\mu\nu}$ - 1-form of type $\binom{0}{2}$ s.t.

$B_{\mu\nu} = B_{\nu\mu}$

Does there exist a connection $\boxed{\omega}$ s.t.

$$\begin{cases} (A) & \Theta^\mu = A^\mu \\ (B) & (Dg)_{\mu\nu} = B_{\mu\nu} \end{cases} \quad ?$$

Preparations:

- $g_{\mu\rho}(e)\, \omega^\rho{}_\nu(e) := \omega_{\mu\nu}(e) = \Gamma_{\mu\nu}$; $\omega^\rho{}_\nu(e) = \Gamma^\rho{}_\nu$
- we fix a frame e.

We have:

$$B_{\mu\nu} \overset{(B)}{=} (Dg)_{\mu\nu} = dg_{\mu\nu} - \Gamma_{\mu\nu} - \Gamma_{\nu\mu}$$

$$\Gamma_{\mu\nu} = \Gamma_{\mu\nu\rho}\, e^\rho$$

$$(\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho})\, e^\rho = dg_{\mu\nu} - B_{\mu\nu} \qquad \Big|\ e_\rho \lrcorner$$

$$\boxed{\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho} = e_\rho \lrcorner (dg_{\mu\nu} - B_{\mu\nu})} \qquad (1)$$

$$A^\mu \overset{(A)}{=} \Theta^\mu = de^\mu + \Gamma^\mu{}_\nu \wedge e^\nu$$

$$\Rightarrow \quad \Gamma^\mu{}_\nu \wedge e^\nu = A^\mu - de^\mu \qquad \Big|\ g_{\rho\mu}$$

$$\Gamma_{\mu\nu} \wedge e^\nu = A_\mu - g_{\mu\nu}\, de^\nu$$

$$\Gamma_{\mu\nu\rho}\, e^\rho \wedge e^\nu = A_\mu - g_{\mu\nu}\, de^\nu \qquad \Big|\ e_\rho \lrcorner e_\nu \lrcorner$$

$$\boxed{\Gamma_{\mu\rho\nu} - \Gamma_{\mu\nu\rho} = e_\rho \lrcorner e_\nu \lrcorner (A_\mu - g_{\mu\sigma}\, de^\sigma)} \qquad (2)$$

(1) + (2)

$$\Gamma_{\nu\mu\rho} + \Gamma_{\mu\rho\nu} = \underbrace{e_\rho \lrcorner \left[(dg_{\mu\nu} - B_{\mu\nu}) + e_\nu \lrcorner (A_\mu - g_{\mu\sigma}\, de^\sigma) \right]}_{H_{\nu\mu\rho}}$$

Thus introducing

$$\boxed{H_{\nu\mu\rho} = \underbrace{e_\rho \lrcorner (dg_{\mu\nu} - B_{\mu\nu}) + e_\rho \lrcorner e_\nu \lrcorner (A_\mu - g_{\mu\sigma}\, de^\sigma)}_{\text{everything here is given}}}$$

we have

$$\boxed{\Gamma_{\nu\mu\rho} + \Gamma_{\mu\rho\nu} = H_{\nu\mu\rho}} \qquad (3)$$

Now we cyclicly permute indices in (3):

$$
\begin{aligned}
+\quad & \underline{\Gamma_{\nu\mu\rho}} + \cancel{\Gamma_{\mu\rho\nu}} = H_{\nu\mu\rho} \\
& \cancel{\Gamma_{\rho\nu\mu}} + \underline{\Gamma_{\nu\mu\rho}} = H_{\rho\nu\mu} \\
-\quad & \cancel{\Gamma_{\mu\rho\nu}} + \cancel{\Gamma_{\rho\nu\mu}} = H_{\mu\rho\nu}
\end{aligned}
$$

$$2\Gamma_{\nu\mu\rho} = H_{\nu\mu\rho} + H_{\rho\nu\mu} - H_{\mu\rho\nu}$$

$$\Gamma_{\nu\mu\rho} = \frac{1}{2}\left(H_{\nu\mu\rho} + H_{\rho\nu\mu} - H_{\mu\rho\nu}\right)$$

better:

$$\boxed{\Gamma_{\mu\nu\rho} = \frac{1}{2}\left(H_{\mu\nu\rho} + H_{\rho\mu\nu} - H_{\nu\rho\mu}\right)}$$

Theorem On a (pseudo)Riemannian manifold (M, g)

(1) Connection is totally determined by
(A) its torsion Θ
and (B) its (non)metricity Dg

(2) The Levi-Civita connection is a UNIQUE connection for which $\Theta \equiv 0$ and $Dg \equiv 0$.

Terminology:

1) $\Gamma^\mu{}_\nu$ — connection 1-forms

$\Gamma_{\mu\nu\rho}$, $\Gamma^\mu{}_{\nu\rho}$ — connection coefficients

2) ~~If $de^\mu \equiv 0$ i.e. when the frame is holonomic~~

Note that if $\boxed{A^\mu \equiv 0 \text{ and } B_{\mu\nu} \equiv 0}$ i.e. in the case of Levi-Civita connection we have

$$\boxed{H_{\nu\mu\rho} = e_\rho \lrcorner\, dg_{\mu\nu} - e_\rho \lrcorner\, e_\nu \lrcorner\, (g_{\mu\sigma}\, de^\sigma)}$$

Now assume that: (i) $de \equiv 0$

(ii) $dg_{\mu\nu} \equiv 0$

In case (i) the frame is holonomic

$$H_{\nu\mu\rho} = e_\rho \lrcorner\, dg_{\mu\nu} = e_\rho(g_{\mu\nu}) \overset{\text{notation}}{=:} g_{\mu\nu|\rho} = g_{\nu\mu|\rho}$$

$$\Rightarrow \quad \Gamma_{\mu\nu\rho} = \frac{1}{2}\left(H_{\mu\nu\rho} + H_{\rho\mu\nu} - H_{\nu\rho\mu}\right) =$$

$$= \frac{1}{2}\left(g_{\nu\mu|\rho} + g_{\rho\mu|\nu} - g_{\nu\rho|\mu}\right)$$

$$\Rightarrow \quad \boxed{\Gamma^{\sigma}{}_{\nu\rho} = \frac{1}{2} g^{\sigma\mu}\left(g_{\nu\mu|\rho} + g_{\rho\mu|\nu} - g_{\nu\rho|\mu}\right) = \left\{ {\sigma \atop \nu\rho} \right\}}$$

Christoffel symbols.

In case (ii) the metric ~~frame~~ has constant ~~metric~~ coefficients, (e.g. orthonormal frame $g_{\mu\nu} = \mathrm{diag}(1, .., 1, -1, .., -1)$)

$$\Rightarrow \quad H_{\nu\mu\rho} = + g_{\mu\sigma} C^{\sigma}{}_{\nu\rho} =: C_{\mu\nu\rho}$$

$$\Rightarrow \quad \boxed{\Gamma_{\mu\nu\rho} = \frac{1}{2}\left(H_{\mu\nu\rho} + H_{\rho\mu\nu} - H_{\nu\rho\mu}\right) = \frac{1}{2}\left(C_{\nu\mu\rho} + C_{\mu\rho\nu} - C_{\rho\nu\mu}\right)}$$

connection is totally determined by the anholonomy coefficients.

in such a case $\Gamma^{\mu}{}_{\nu\rho}$ are called Ricci rotation coefficients

note that in this case

$$\Gamma_{\mu\nu} + \Gamma_{\nu\mu} = \left(\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho}\right) e^{\rho} = 0$$

since $\Gamma_{\mu\nu\rho} = \frac{1}{2}\left(\underbrace{C_{\nu\mu\rho} - C_{\mu\nu\rho}}_{\text{antisym in } \mu\nu} - \underbrace{C_{\rho\nu\mu}}_{\text{antisym in } \mu\nu}\right)$

Prop
In a frame in which $dg_{\mu\nu} = 0$

$$\boxed{\omega_{\mu\nu} + \omega_{\nu\mu} = 0}$$

4) Riemann tensor and its symmetries

In (pseudo) Riemannian geometry (M, g) we have a DISTINGUISHED connection characterized by $\Theta \equiv 0$ and $Dg \equiv 0$. This is the Levi-Civita connection and it is this connection that is used to characterize invariant properties of Riemannian manifolds.

$\nabla = \overset{L.C.}{\nabla}, \omega$ has no torsion, but it may have CURVATURE:

$$\underline{\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu} = \frac{1}{2} \underline{R^\mu{}_{\nu\rho\sigma}} \, \theta^\rho \wedge \theta^\sigma$$

On a Riemannian manifold the curvature tensor $R^\mu{}_{\nu\rho\sigma}(\cdot)$ of the Levi-Civita connection is called

<u>RIEMANN TENSOR</u> $\quad R^\mu{}_{\nu\rho\sigma}(e)$ –

– 0-form of type $Ad \otimes \check{id} \otimes \check{id}$

$\uparrow$ contragredient representation

(Asside:

$$\rho : GL(n, \mathbb{R}) \xrightarrow{\text{homo}} GL(V) \quad \Rightarrow$$

$$\check{\rho} : GL(n, \mathbb{R}) \longrightarrow GL(V^*)$$

$$\check{\rho}(a) = \rho(a^{-1})^* \quad)$$

$$R^\mu{}_{\nu\rho\sigma} \xrightarrow{g} R_{\mu\nu\rho\sigma} := g_{\mu\alpha} R^\alpha{}_{\nu\rho\sigma}$$

$\uparrow$ also called the Riemann tensor

– 0-form of type $\check{id} \otimes \check{id} \otimes \check{id} \otimes \check{id}$

Symmetries of Riemann

1) $R_{\mu\nu\rho\sigma} = - R_{\mu\nu\sigma\rho}$ - - - - - - - .
2) $R_{\mu\nu\rho\sigma} = - R_{\nu\mu\rho\sigma}$ - - - - - -
3) $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$ - - - - - - -
4) $R_{\mu[\nu\rho\sigma]} = 0$ - - - - - - -

if Dg and Θ g not necessarily 0

always

$Dg = 0$
$\Downarrow$
$Dg = \Theta = 0$
$\Uparrow$
$\Theta = 0$

Proof

1) obvious from definition

4) First Bianchi: $D\Theta^\mu = \Omega^\mu{}_\nu \wedge \theta^\nu$

$$\Rightarrow \Omega^\mu{}_\nu \wedge \theta^\nu \equiv 0 \Rightarrow \frac{1}{2} R^\mu{}_{\nu\rho\sigma}\, \theta^\rho \wedge \theta^\sigma \wedge \theta^\nu = 0$$

$$\Rightarrow \frac{1}{2} R^\mu{}_{[\nu\rho\sigma]}\, \theta^\rho \wedge \theta^\sigma \wedge \theta^\nu \equiv 0 \Rightarrow \boxed{R^\mu{}_{[\nu\rho\sigma]} = 0}$$

$$\Rightarrow \boxed{R_{\mu[\nu\rho\sigma]} = 0} .$$

Note that this equality means:

$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} - R_{\mu\rho\nu\sigma} - R_{\mu\sigma\rho\nu} - R_{\mu\nu\sigma\rho} = 0$$

so

$$\boxed{R_{\mu[\nu\rho\sigma]} = 0 \iff R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0}$$

2) ~~Second Bianchi~~ Metricity!: $(Dg)_{\mu\nu} = 0$

$$\Rightarrow 0 = (D^2 g)_{\mu\nu} \underset{\substack{\uparrow \\ \text{Ricci formula}}}{=} -\Omega^\rho{}_\mu g_{\rho\nu} - \Omega^\rho{}_\nu g_{\mu\rho} = - \Omega_{\nu\mu} - \Omega_{\mu\nu}$$

$$\Rightarrow \Omega_{\mu\nu} = -\Omega_{\nu\mu} \Rightarrow R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} \quad \square$$

3) Exercise!

5) Riemann tensor and IInd Bianchi identity

What about Second Bianchi?

$$D\Omega^\mu{}_\nu = 0.$$

Note that

$$D\Omega_{\mu\nu} = D(g_{\mu\alpha}\Omega^\alpha{}_\nu) = g_{\mu\alpha} D\Omega^\alpha{}_\nu = 0$$

$$0 = D\Omega_{\mu\nu} = \tfrac{1}{2} D(R_{\mu\nu\rho\sigma}\theta^\rho\wedge\theta^\sigma) =$$

$$= \tfrac{1}{2} DR_{\mu\nu\rho\sigma}\wedge\theta^\rho\wedge\theta^\sigma + \tfrac{1}{2}R_{\mu\nu\rho\sigma}\cancel{(D\theta^\rho\wedge\theta^\sigma + \ldots)}$$

no torsion

$$= \tfrac{1}{2} DR_{\mu\nu\rho\sigma}\wedge\theta^\rho\wedge\theta^\sigma =$$

$$= \tfrac{1}{2} \nabla_\tau R_{\mu\nu\rho\sigma}\,\theta^\tau\wedge\theta^\rho\wedge\theta^\sigma$$

Notation $\nabla_\tau f_\alpha := f_{\alpha;\tau}$

$$\Rightarrow \quad 0 = D\Omega_{\mu\nu} = \tfrac{1}{2} R_{\mu\nu\rho\sigma;\tau}\,\theta^\tau\wedge\theta^\rho\wedge\theta^\sigma =$$

$$= \tfrac{1}{2} R_{\mu[\nu\rho\sigma;\tau]}\,\theta^\tau\wedge\theta^\rho\wedge\theta^\sigma$$

$\Rightarrow$ Riemann tensor satisfies

$$\boxed{R_{\mu[\nu\rho\sigma;\tau]} \equiv 0}$$

IInd Bianchi identity for the Riemann tensor.

~~IInd geodesic~~

6) Metricity of the connection and parallel transport.

ω - a connection on (M, g). Not necessarily Levi-Civita, but such that $Dg \equiv 0$.

Such connection is called metric connection

$(Dg_{\mu\nu} \equiv 0 \quad \Longleftrightarrow \quad \nabla_\rho g_{\mu\nu} \equiv 0$

Proposition

$$\boxed{\nabla_\mu g_{\nu\rho} \equiv 0 \quad \Longleftrightarrow \quad \left(\begin{array}{l}\text{Parallel transport } \cancel{\text{does not}} \\ \text{preserves scalar product of vectors}\end{array}\right)}$$

Proof

$\Leftarrow$ Y^μ, Z^ν - parallelly transported along γ.

Let X^μ tangent vector to γ. $X^\mu = \frac{dx^\mu}{dt}$ $\gamma = (x^\mu(t))$

If $g_{\mu\nu} Y^\mu Z^\nu = \text{const}$ ~~and~~ along γ we have

$$0 = \nabla_X (\text{const}) = \nabla_X (g_{\mu\nu} Y^\mu Z^\nu) = X^\rho (\nabla_\rho g_{\mu\nu}) Y^\mu Z^\nu$$

for all $Y, Z \Rightarrow \nabla_\rho g_{\mu\nu} = 0$.

$$\Rightarrow \quad \nabla_X (g_{\mu\nu} Y^\mu Z^\nu) = X^\alpha (\nabla_\alpha g_{\mu\nu}) Y^\mu Z^\nu + g_{\mu\nu} \frac{DY^\mu}{dt} Z^\nu + g_{\mu\nu} Y^\mu \frac{DZ^\nu}{dt} = 0.$$

~~If $\nabla_\mu g_{\rho\sigma} \neq 0$~~

~~This ... in particular implies that~~ $\square$.

If $\nabla_\mu g_{\rho\nu} = 0$ the in particular we have:

$$\nabla_\mu X_\nu = \nabla_\mu (g_{\nu\rho} X^\rho) = g_{\nu\rho} \nabla_\mu X^\rho$$

"the metric can be commuted with ∇".

7) Vanishing of the Riemann tensor

$R_{\mu\nu\rho\sigma} \equiv 0 \iff$ locally there exists a coordinate system in which
$$g_{\mu\nu} = (1, 1, \dots, 1, -1, \dots -1).$$

Proof

$\Leftarrow$ obvious since $\left\{ {\alpha \atop \beta\gamma} \right\} \equiv 0$

$\Rightarrow$ $\Omega \equiv 0$ and $\Theta \equiv 0$ implies that there exists a coord system in which $\omega \equiv 0$ and $de = 0$

$\Rightarrow \left\{ {\mu \atop \nu\rho} \right\} = 0$ in this coord system

$\Rightarrow$ by permuting indices in
$$g_{\mu\nu,\rho} + g_{\rho\mu,\nu} - g_{\nu\rho,\mu} \equiv 0$$

we get that
$$g_{\mu\nu,\rho} \equiv 0 \Rightarrow g_{\mu\nu} = \text{const}$$
in this coord. system.

Since $g_{\mu\nu} = g_{\nu\mu}$ then there exists a linear transformation ~~setting it to diag(1, ..., 1, ...)~~. reducing it to

$$g_{\mu\nu} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \text{and} \quad \lambda_i \neq 0$$

$$\Rightarrow \quad g = \lambda_1 (dx^1)^2 + \dots + \lambda_n (dx^n)^2 =$$

$$= \pm\, d\left(\sqrt{|\lambda_1|}\, x^1\right)^2 + \dots \pm d\left(\sqrt{|\lambda_n|}\, x^n\right)^2$$

$\Rightarrow$ in new cordinates $y^i = \sqrt{|\lambda_i|}\, x^i$ we have

$$g = \pm\, dy^{1\,2} + \dots \pm dy^{n\,2} \quad \square.$$

8) Riemann tensor and isometries

(M,g), (M',g')

$$\varphi: M \xrightarrow{\text{diff.}} M'$$
$$g \xleftarrow{\varphi^*} g'$$

$$\varphi^* g' = ?$$

Definition

a diffeomorphism φ between two Riemannian manifolds (M,g) and (M',g') is called an isometry $\Longleftrightarrow$ $\varphi^* g' = g$.

$\rightsquigarrow$ Object of study of Riemannian geometry

$$\{\text{set of all } (M,g)\} \big/ \text{isometries}.$$

Thm

$\varphi: (M,g) \to (M',g')$ isometry

$\Rightarrow$ $\tilde{\varphi}$ Riemann $=$ Riemann$'$

Riemann tensor is a Riemannian invariant!