Lecture 9, 12.04.2011

$\boxed{\text{Recall:}}$

$(M, g) \rightsquigarrow$ unique Levi-Civita connection $\nabla \leftrightarrow \omega$.

s.t. $\begin{cases} \Theta \equiv 0 & \text{no torsion} \\ Dg \equiv 0 & \text{metricity} \end{cases}$

$$\left\{ \begin{array}{l} \left(\begin{array}{c} \nabla_X X = 0\,, \quad X = \frac{dx}{dt} \\ \text{self parallels of L.C.} \end{array}\right) \Longleftrightarrow \left(\begin{array}{c} \delta \int \sqrt{|g(X,X)|}\, dt = 0 \\ \text{geodesics of L.C.} \end{array}\right) \\ \qquad\qquad \text{in affine parametrization} \\ \omega^\mu{}_\nu(e) = \Gamma^\mu{}_{\nu\rho} e^\rho \quad \text{and if } de^\rho = 0\,, \quad e_\rho = \frac{\partial}{\partial x^\rho} \\ \qquad \Gamma^\mu{}_{\nu\rho} = \left\{ {\mu \atop \nu\rho} \right\} = \frac{1}{2} g^{\mu\sigma} \left(g_{\sigma\rho,\nu} + g_{\nu\sigma,\rho} - g_{\rho\nu,\sigma} \right) \end{array} \right.$$

(Connection ~~is~~ metric) $\Leftrightarrow$ (parallel transport preserves scalar product defined by g)

Curvature $\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\sigma}\, \theta^\rho \wedge \theta^\sigma$, $\quad g_{\mu\alpha} R^\alpha{}_{\nu\rho\sigma} = R_{\mu\nu\rho\sigma}$

$$\Rightarrow \begin{cases} R_{\mu\nu\rho\sigma} = - R_{\mu\nu\sigma\rho} \\ R_{\mu\nu\rho\sigma} = - R_{\nu\mu\rho\sigma} \\ R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \\ R_{\mu[\nu\rho\sigma]} = 0 \end{cases} \qquad \begin{array}{c} \text{always} \\ Dg = 0 \\ \downarrow \\ \uparrow \\ \Theta = 0 \end{array} \qquad \# \text{comp} = \frac{1}{12} n^2 (n^2 - 1)$$

$+$ II$^{\text{nd}}$ Bianchi $\Rightarrow$ $R_{\mu[\nu\rho\sigma;\tau]} = 0$, $\qquad X_{\alpha;\tau} = \nabla_\tau X_\alpha$.

7) Vanishing of the Riemann tensor

$R_{\mu\nu\rho\sigma} \equiv 0 \iff$ locally there exists a coordinate system in which $g_{\mu\nu} = (1, 1, \ldots, 1, -1, \ldots -1)$.

Proof

$\Leftarrow$ obvious since $\left\{ {\alpha \atop \beta\gamma} \right\} \equiv 0$

$\Rightarrow$ $\Omega \equiv 0$ and $\Theta \equiv 0$ implies that there exists a coord system in which $\omega \equiv 0$ and $de = 0$

$\Rightarrow \left\{ {\mu \atop \nu\rho} \right\} = 0$ in this coord system

$\Rightarrow$ by permuting indices in

$$g_{\mu\nu,\rho} + g_{\rho\mu,\nu} - g_{\nu\rho,\mu} \equiv 0$$

we get that

$$g_{\mu\nu,\rho} \equiv 0 \Rightarrow g_{\mu\nu} = \text{const}$$

in this coord. system.

Since $g_{\mu\nu} = g_{\nu\mu}$ then there exists a linear transformation ~~setting it to diag(1, ..., 1, ...)~~. reducing it to

$$g_{\mu\nu} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \text{and } \lambda_i \neq 0$$

$$\Rightarrow g = \lambda_1 (dx^1)^2 + \ldots + \lambda_n (dx^n)^2 =$$

$$= \pm\, d\left(\sqrt{|\lambda_1|}\, x^1\right)^2 + \ldots \pm d\left(\sqrt{|\lambda_n|}\, x^n\right)^2$$

$\Rightarrow$ in new cordinates $y^i = \sqrt{|\lambda_i|}\, x^i$ we have

$$g = \pm\, dy^{1\,2} + \ldots \pm dy^{n\,2}$$

q.

8) Riemann tensor and isometries

(M,g) , (M',g')

$$\varphi : M \xrightarrow{\text{diff.}} M'$$

$$g \xleftarrow{\varphi^*} g'$$

$$\varphi^* g' = ?$$

<u>Definition</u>

a diffeomorphism φ between two Riemannian manifolds (M,g) and (M',g') is called an isometry $\Leftrightarrow$ $\varphi^* g' = g$.

$\rightsquigarrow$ Object of study of Riemannian geometry

$$\{\text{set of all } (M,g)\} \big/ \text{isometries}.$$

<u>Thm</u>

$\varphi : (M,g) \longrightarrow (M',g')$ isometry

$$\Rightarrow \quad \tilde{\varphi} \text{ Riemann} = \text{Riemann}'$$

Riemann tensor is a Riemannian invariant!

9) Decomposition of Riemann into irreducibles.

$\overset{r}{\otimes} V^* = \{ A_{\alpha_1 \dots \alpha_r},\ \alpha_i \in 1, \dots n \}$ is a representation space for $G \subset GL(n, \mathbb{R})$

$$(\rho(a)A)_{\alpha_1 \dots \alpha_r} = A_{\beta_1 \dots \beta_r} a^{-1\beta_1}{}_{\alpha_1} \dots a^{-1\beta_r}{}_{\alpha_r}.$$

But this representation is <u>reducible</u>.

<u>Def</u>

$\rho : G \xrightarrow{\text{homo}} GL(N, \mathbb{R})$ is <u>irreducible</u> if the only invariant subspaces in $W = \mathbb{R}^N$ are $\{0\}$ and W.

A vector subspace $S \subset W = \mathbb{R}^N$ is invariant for ρ iff

$$\rho(a) S \subset S \qquad \forall a \in G.$$

<u>Example</u>

(1) <u>$G = GL(n, \mathbb{R})$ and $\overset{2}{\otimes} V^* \ni A_{\mu\nu}$</u>

$$A_{\mu\nu} = \tfrac{1}{2}(A_{\mu\nu} + A_{\nu\mu}) + \tfrac{1}{2}(A_{\mu\nu} - A_{\nu\mu}) =$$
$$= A_{(\mu\nu)} + A_{[\mu\nu]}$$
$$\overset{2}{\otimes} V^* = \underset{\substack{\Psi \\ A_{(\mu\nu)}}}{S^2 V^*} \oplus \underset{\substack{\Psi \\ A_{[\mu\nu]}}}{\Lambda^2 V^*}$$

This decomposition consists of invariant subspaces for ρ.

Indeed: i) $A_{\mu\nu} = A_{\nu\mu}$

$$A'_{\mu\nu} = [\rho(a)A]_{\mu\nu} = A_{\alpha\beta} a^{-1\alpha}{}_{\mu} a^{-1\beta}{}_{\nu} = A_{\beta\alpha} a^{-1\beta}{}_{\mu} a^{-1\alpha}{}_{\nu} =$$
$$= A_{\beta\alpha} a^{-1\alpha}{}_{\nu} a^{-1\beta}{}_{\mu} = A_{\alpha\beta} a^{-1\alpha}{}_{\nu} a^{-1\beta}{}_{\mu} =$$
$$= [\rho(a)A]_{\nu\mu} = A'_{\nu\mu}$$

so if $A_{\mu\nu} \in S^2V^* \Rightarrow [\rho(a)A]_{\mu\nu} \in S^2 V^*$

ii) similarly if $A_{\mu\nu} = -A_{\nu\mu}$

$$A'_{\mu\nu} = [\rho(a)A]_{\mu\nu} = A_{\alpha\beta}\, a^{-1\alpha}{}_{\mu}\, a^{-1\beta}{}_{\nu} = A_{\beta\alpha}\, a^{-1\beta}{}_{\mu}\, a^{-1\alpha}{}_{\nu} =$$
$$= -A_{\alpha\beta}\, a^{-1\alpha}{}_{\nu}\, a^{-1\beta}{}_{\mu} = -A'_{\nu\mu}.$$

$\Rightarrow$ $\otimes^2 V^*$ is reducible w.r.t. $GL(n,\mathbb{R})$

it decomposes onto invariant subspaces as:
$$\otimes^2 V^* = S^2 V^* \oplus \Lambda^2 V^*$$

<u>Fact</u> ~~...~~ This decomposition is a decomposition onto irreducibles!

(2) <u>$G = O(n)$ and $\otimes^2 V^* \ni A_{\mu\nu}$</u>

of course $\otimes^2 V^* = S^2 V^* \oplus \Lambda^2 V^*$
is still a decomposition onto $SO(n)$ invariant subspaces. But it FURTHER DECOMPOSES!

Note that
$$G = O(n) = \{ a \in GL(n,\mathbb{R}) : g(a^{-1}X, a^{-1}Y) = g(X,Y) \}$$

$$g(a^{-1\alpha}{}_{\mu} e_\alpha, a^{-1\beta}{}_{\nu} e_\beta) = g(e_\mu, e_\nu) = g_{\mu\nu}$$
$$\parallel$$
$$g(e_\alpha, e_\beta)\, a^{-1\alpha}{}_{\mu}\, a^{-1\beta}{}_{\nu} = g_{\alpha\beta}\, a^{-1\alpha}{}_{\mu}\, a^{-1\beta}{}_{\nu} = [\rho(a) g]_{\mu\nu}$$

$\Rightarrow$ $g_{\mu\nu} \in S^2 V^*$ and $\rho(a) g = g$.

$\Rightarrow$ $\lambda \cdot g_{\mu\nu}$ is a 1-dim subspace in $S^2 V^*$ which is $O(n)$ invariant.

$\Rightarrow$ $$S^2 V^* \underset{O(n)}{=} S^2_0 V^* \oplus \lambda \cdot g$$

$$\cancel{A_{\mu\nu} = A_{(\mu\nu)} + A_{[\mu\nu]} = A_{(\mu\nu)} - \frac{1}{n} g_{\mu\nu} (g^{\alpha\beta} A_{\alpha\beta}) + \frac{1}{n} g_{\mu\nu} A + A_{[\mu\nu]}}$$

Projection from S^2V^* onto λg

$$S^2V^* \ni A_{(\mu\nu)} \xrightarrow{Tr} (g^{\alpha\beta}A_{\alpha\beta})\, g_{\mu\nu} \in \lambda g.$$

Denote: $g^{\alpha\beta}A_{\alpha\beta} = A$ and define $S^2_0V^*$ to be the kernel of the operator Tr:

$$Tr\,(S^2V^*_0) \equiv 0.$$

$$S^2V^*_0 = \{ \check{A}_{(\mu\nu)} \in S^2V : \quad \check{A}_{(\mu\nu)} = A_{(\mu\nu)} - \tfrac{1}{n} A g_{\mu\nu} \}$$

Fact: decomposition

$$A_{\mu\nu} = A_{[\mu\nu]} + \check{A}_{(\mu\nu)} + \tfrac{1}{n} A g_{\mu\nu}$$

$$\overset{2}{\otimes} V^* = \Lambda^2 V^* \oplus S^2_0 V^* \oplus \lambda g$$

is $O(n)$ invariant and if $n \neq 4$ is a decomposition of $\overset{2}{\otimes}V^*$ onto irreducibles w.r.t. $O(n)$.

(3) More generally: an $O(n)$ invariant decomposition of $\overset{n}{\otimes}V^*$ is obtained by symmetrizing, taking traces and antisymmetrizing the indices.

$\xrightarrow[\text{properly}]{\text{if done}}$ leads to decomposition onto irreducibles.

(4) Riemann tensor

$$R^{\mu}{}_{\nu\rho\sigma}.$$

Since we have metric, placement of the indices is not important!

But: for ANY curvature tensor $R^{\mu}{}_{\nu\rho\sigma}$ we always have

$$R^{\mu}{}_{\nu\rho\sigma} \longmapsto \boxed{R_{\nu\sigma} = R^{\mu}{}_{\nu\mu\sigma}}$$

↑ Ricci tensor.

For Riemann tensor, because of the symmetries, we have

$$\underline{R_{\nu\sigma} = R_{\sigma\nu}.} \quad !$$

Now we ~~add~~ decompose this as:

$$R_{\nu\sigma} = \check{R}_{\nu\sigma} + \frac{1}{n} R g_{\nu\sigma}$$

$$\boxed{R = g^{\mu\nu} R_{\mu\nu} \quad \text{— Ricci scalar}}$$

$$\boxed{\check{R}_{\nu\sigma} = R_{\nu\sigma} - \frac{1}{n} R g_{\nu\sigma} \quad \text{- traceless Ricci}}$$

We define

$$C^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} + a\, \delta^{[\mu}_{[\rho} \check{R}^{\nu]}{}_{\sigma]} + b R\, \delta^{[\mu}_{[\rho} \delta^{\nu]}_{\sigma]}$$

and find a and b by the requirement that $C^{\mu\nu}{}_{\rho\sigma}$ is traceless in every pair of indices.

$$C^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} + \frac{1}{4} a \left[\delta^{\mu}_{\rho} \check{R}^{\nu}_{\sigma} - \delta^{\nu}_{\rho} \check{R}^{\mu}_{\sigma} - \delta^{\mu}_{\sigma} \check{R}^{\nu}_{\rho} + \delta^{\nu}_{\sigma} \check{R}^{\mu}_{\rho}\right] +$$
$$+ \frac{R}{4} b \left[\delta^{\mu}_{\rho} \delta^{\nu}_{\sigma} - \delta^{\nu}_{\rho} \delta^{\mu}_{\sigma} - \delta^{\mu}_{\sigma} \delta^{\nu}_{\rho} + \delta^{\nu}_{\sigma} \delta^{\mu}_{\rho}\right]$$

$$\Longrightarrow 0 = R^{\nu}{}_{\sigma} + \frac{1}{4} a \left[n \check{R}^{\nu}_{\sigma} - \check{R}^{\nu}_{\sigma} - \check{R}^{\nu}_{\sigma}\right] + \frac{R}{4} b \left[n \delta^{\nu}_{\sigma} - \delta^{\nu}_{\sigma} - \delta^{\nu}_{\sigma} + n \delta^{\nu}_{\sigma}\right]$$

$$0 = \check{R}^{\nu}_{\sigma} \underbrace{\left[1 + \frac{n-2}{4} a\right]}_{=0} + R \underbrace{\left[\frac{2n-2}{4} b + \frac{1}{n}\right]}_{=0} \delta^{\nu}_{\sigma}$$

$$\Longrightarrow a = \frac{4}{2-n}, \quad b = -\frac{2}{n(n-1)}$$

$$\boxed{C^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} + \frac{4}{2-n}\,\delta^{[\mu}_{[\rho} R^{\nu]}{}_{\sigma]} + \frac{2R}{(1-n)(n)}\,\delta^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]}}$$

$C^{\mu\nu}{}_{\rho\sigma}$ or $C^{\mu}{}_{\nu\rho\sigma}$ or $C_{\mu\nu\rho\sigma}$ is called Weyl tensor. It has ALL THE ALG. SYMMETRIES OF RIEMANN + lack of trace

And

$$\boxed{R^{\mu\nu}{}_{\rho\sigma} = C^{\mu\nu}{}_{\rho\sigma} + \frac{4}{n-2}\,\delta^{[\mu}_{[\rho} R^{\nu]}{}_{\sigma]} + \frac{2}{(n-1)n}\,R\,\delta^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]}}$$

is $O(n)$ invariant decomposition of $R^{\mu\nu}{}_{\rho\sigma}$ with all the algebraic symmetries of Riemann.

<u>Low dimensions:</u>

	# comp. of Riemann	# Ricci	# Weyl
	$\underbrace{\frac{1}{12}n^2(n^2-1)}_{1}$	$\underbrace{\frac{n(n+1)}{2}}_{2}$	1 − 2
$n=1$	0	0	0
$n=2$	1	1	0
$n=3$	6	6	0
$n=4$	20	10	10 = 5+5

$n \geq 4$ decomposition is into irreducibles!

<u>Remark</u>

Some people define still another tensor $P_{\mu\nu}$ by:

$$R^{\mu\nu}{}_{\rho\sigma} = C^{\mu\nu}{}_{\rho\sigma} + \delta^{[\mu}_{[\rho} P^{\nu]}{}_{\sigma]}$$

Schouten tensor.

$P_{\mu\nu}$ is a trace-corrected Ricci. <u>Exercise</u> find relation between $P_{\mu\nu}$ and $R_{\mu\nu}$.

10) Weyl tensor and conformal invariance

$C^{\rho}{}_{\mu\nu\sigma}$ – Weyl tensor.

Def

$\varphi : (M, g) \xrightarrow{\text{diff}} (M', g')$ is called a conformal transformation iff

$$\varphi^* g' = e^{2f} g. \quad \text{where } f \in \mathcal{F}(M).$$

$$\left(\text{(pseudo) Riemannian manifolds} \right) \Big/ \text{conformal transformations} = \text{conformal structures}$$

Fact

$\varphi : (M, g) \to (M', g')$ conformal

$$\Rightarrow \quad \varphi^* C'^{\mu}{}_{\nu\rho\sigma} = C^{\mu}{}_{\nu\rho\sigma}$$

Weyl tensor is a conformal invariant!

Thm (Weyl) $\quad n \geq 4$

$C^{\mu}{}_{\nu\rho\sigma} \equiv 0 \iff (M, g)$ is conformally equivalent to a flat manifold.

i.e. $\exists\ g' = e^{2f} g$ s.t.

$$R'^{\mu}{}_{\nu\rho\sigma} \equiv 0.$$

For $n = 3$ Different characterization; for the conformal flatness vanishing of a higher order tensor is needed. $\Rightarrow$ Cotton tensor.

11) <u>Contracted second Bianchi identity:</u>

$$R_{\mu\nu[\rho\sigma;\tau]} = 0$$

$$\Rightarrow \quad R_{\mu\nu\rho\sigma;\tau} + R_{\mu\nu\tau\rho;\sigma} + R_{\mu\nu\sigma\tau;\rho} = 0$$

Contracting in $\mu \to \rho$

$$R_{\nu\sigma;\tau} - R_{\nu\tau;\sigma} + \nabla^\mu R_{\mu\nu\sigma\tau} = 0$$

$$\boxed{\nabla^\mu R_{\mu\nu\sigma\tau} = R_{\nu\tau;\sigma} - R_{\nu\sigma;\tau}}$$

Contracting in $\nu \to \tau$

$$\nabla^\mu R_{\mu\sigma} = R_{;\sigma} - \nabla^\tau R_{\tau\sigma}$$

$$\cancel{2\nabla^\mu R_{\mu\sigma} = (R g_{\mu\sigma})}$$

$$2\nabla^\mu R_{\mu\sigma} = \nabla_\sigma R$$

$$2\nabla^\mu R_{\mu\sigma} = \nabla^\mu (g_{\mu\sigma} R)$$

$$\Rightarrow \quad \nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = 0$$

$$\boxed{G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}$$ Einstein tensor

$$\boxed{\nabla^\mu G_{\mu\nu} = 0}$$ Contracted B.I.

(6) Formulation of General Relativity

1) What elements every relativistic theory of gravitations should contain?

- it contains Newton theory and special relativity theory as special limiting cases
- it respects an equivalence principle i.e. a principle static that "the acceleration imparted to a body by a gravitational field is independent of the nature of the body." For Newton's equation of motion in gravitational field, it is:

$$(\text{inertial mass}) \cdot (\text{acceleration}) = \left(\begin{matrix}\text{gravitational}\\ \text{mass}\end{matrix}\right) \cdot \left(\begin{matrix}\text{intensity}\\ \text{of gravitational}\\ \text{field}\end{matrix}\right)$$

So the equivalence principle is based on an experimental fact that body's

$$\text{inertial mass} \equiv \text{gravitational mass}$$

for every body!

This principle ~~step~~ is also equivalent that locally one can not distinguish between a gravitational field and inertial forces.

(Einstein's lift)

2) Model

$$\left\{ \begin{array}{l} \text{Space-time} \\ (M,g) \end{array} \right. \equiv \begin{array}{l} \text{4-dimensional manifold } M \\ \text{equipped with a (pseudo) Riemannian} \\ \text{metric } g, \text{ of signature } +---. \end{array}$$

In the absence of gravitational field space-time is (M,g) with a flat (Minkowski) metric η. And this is a model of special relativity. (SR)

Gravity caused by g with nonvanishing Riemann

In SR there is a distinguished class of frames - called ~~...~~ inertial frames. ~~A~~ A frame is inertial if a free-body moves on a straight worldline without acceleration. In a Cartesian coordinates associated with any inertial frame the body's equation of motion is

$$\boxed{\frac{d^2x^\mu}{ds^2}=0} \qquad x^\mu = x^\mu(s)$$

s - arc length (proper time)

or introducing $v^\mu = \frac{dx^\mu}{dt}$

$$0 = \frac{dv^\mu}{ds} = \frac{dx^\mu}{ds}\cdot \partial_\mu v^\mu = v^\nu \partial_\nu v^\mu$$

$$\boxed{v^\nu \partial_\nu v^\mu = 0}$$

$$\boxed{\partial_v v = 0}$$

In the presence of gravitational field we cannot distinguish ~~ef~~ locally if we are in an accelerated lift or in an unaccelerated box inserted in a constant gravitational field.

We will eliminated a vague concept of gravitational FORCE by considering all frames.

We want that equations of SR will be still valid in the absence of gravitational field.

We adapt a principle ~~to~~ which translates equations from SR to GR by replacing

$$\partial_\mu \quad \text{by} \quad \nabla_\mu$$

$$\begin{pmatrix} \text{SR} \\ \partial_\mu \end{pmatrix} \longrightarrow \begin{pmatrix} \text{GR} \\ \nabla_\mu \end{pmatrix}$$

this is needed to have SR equations valid in any frame!

minimal coupling principle

$$\underset{\text{SR}}{\begin{pmatrix} M, \eta \\ \text{straight} \\ \text{worldlines} \end{pmatrix}} \longrightarrow \begin{pmatrix} \text{GR} \\ M, g \\ \nabla \end{pmatrix}$$

Levi-Civita connection for g.

Einstein ~~[illegible]~~ (1912)

In 1922 Cartan choose ∇ to be a connection with torsion. ; i.e. he had $\nabla g \equiv 0$ but $\Theta \neq 0$.
It is a Good theory of GR. An alternative to Einstein's one.

Example for minimal coupling.

$$\begin{cases} \partial_\mu F^{\mu\nu} = -4\pi j^\nu \\ \partial_\mu {*F}^{\mu\nu} = 0 \end{cases} \overset{SR}{\Longleftrightarrow} \begin{cases} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \\ \partial_\mu A^\mu = 0 \\ \partial_\mu \partial^\mu A^\nu = -4\pi j^\nu \end{cases}$$

$${*F}^{\mu\nu} = \frac{1}{2}\eta^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

$$\downarrow$$

$$\begin{cases} \nabla_\mu F^{\mu\nu} = -4\pi j^\nu \\ \nabla_\mu {*F}^{\mu\nu} = 0 \end{cases} \not\Longleftrightarrow \begin{cases} F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \\ \nabla_\mu A^\mu = 0 \\ \nabla_\mu \nabla^\mu A^\nu = -4\pi j^\nu \end{cases}$$

because $\nabla_{[\mu}\nabla_{\nu]} \neq 0$ and is proportional to Riemann.

these are the correct eqs. for GR.

3) Free particles (including particles in free fall).

$$v^\mu \partial_\mu v^\nu = 0 \Rightarrow \boxed{v^\mu \nabla_\mu v^\nu = 0}$$

$$v^\mu \nabla_\mu v^\nu = 0 \Leftrightarrow \boxed{\frac{dv^\mu}{ds} + \left\{ {\mu \atop \nu\rho} \right\} v^\nu v^\rho = 0}$$

$$\Leftrightarrow \boxed{\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0} \quad \boxed{\Gamma^\mu{}_{\nu\rho} = \left\{ {\mu \atop \nu\rho} \right\}}$$

Free particles move along geodesics parametrized by a proper time $s = \int_{t_0}^{t} \sqrt{g(v(t), v(t))}\, dt$

Note that $\Gamma^\mu{}_{\nu\rho}$ incorporates both the inertial forces and gravitational ones.

4) Newtonian limit for the equation of free particles

1° $\frac{\vec{v}}{c} \ll 1$

2° $|g_{\mu\nu} - \eta_{\mu\nu}| \ll 1 \iff$ gravitational field is weak

3° $|\partial_0 g_{\mu\nu}| \ll |\partial_a g_{\alpha\beta}| \iff$ gravitational field is slowly changing in time

$\uparrow$ time $\qquad \uparrow$ space

In Minkowski metric

$$\eta = d(ct)^2 - dx^{1\,2} - dx^{2\,2} - dx^{3\,2} = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

$$x^\mu = (ct, x^1, x^2, x^3) \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

proper time

$$ds = \sqrt{dx^{0\,2} - d\vec{x}^{\,2}} = \sqrt{1 - \left(\frac{d\vec{x}}{c\,dt}\right)^2}\, c\,dt = \sqrt{1 - \frac{\vec{v}^{\,2}}{c^2}}\, c\,dt$$

my assumptions 1° - 2° mean that

$$v^\mu = \left(1, \frac{v^a}{c}\right)$$

$ds \hat{=} c\,dt$ Then the geodesic equation is

$$\frac{dv^\mu}{ds} + \left\{ {\mu \atop \nu\rho} \right\} v^\nu v^\rho = 0,$$

and when applied to v^a reads:

$$\frac{dv^a}{c^2 dt} + \left\{ {a \atop 00} \right\} 1 \cdot 1 + O\left(\frac{1}{c\,n}\right) = 0$$

$$\left\{ {a \atop 00} \right\} = \frac{1}{2} g^{ab} \left(\cancel{g_{b0,0}} + \cancel{g_{0b,0}} - g_{00,b} \right) =$$

$$\simeq \frac{-1}{2} \eta^{ab}\, \partial_b g_{00} = \frac{1}{2} \partial_a g_{00}$$

$$\Rightarrow \qquad \frac{dv^a}{dt} + \frac{c^2}{2}\partial_a g_{00} = 0$$

In vectorial notation:

$$\frac{d\vec{v}}{dt} = -\vec{\nabla}\left(\frac{c^2}{2} g_{00}\right)$$

In Newtonian theory:

$$\frac{d\vec{v}}{dt} = \vec{F} = -\vec{\nabla}\phi$$

$$\Rightarrow \qquad \frac{c^2}{2} g_{00} = \phi + \text{const}$$

$$\Rightarrow \qquad \boxed{g_{00} = 1 + \frac{2\phi}{c^2}}$$

gravitational potential is related to g_{00} component of the metric.

5) Field equations

Newtonian theory:

$$\Delta\varphi = 4\pi G\rho$$

— Poisson equation for the gravitational potential

density of ~~mass~~ matter

potential ≡ component of the metric enters into the field equations with second derivatives.

The searched equation should be of the form

$$E_{\mu\nu}(g, \partial g, \partial^2 g) = T_{\mu\nu}$$

energy momentum tensor is a counterpart of density of mass in SR.

In SR matter always satisfies

$$\nabla^\mu T_{\mu\nu} = 0$$

(No creation and anihilation of matter!)

$$\Longrightarrow \quad \underset{①}{\nabla^\mu E_{\mu\nu} = 0}, \quad \underset{②}{E_{\mu\nu} = E_{\mu\nu}(g, \partial g, \partial^2 g)}$$

$E_{\mu\nu} = ?$

<u>Thm</u>

The most general tensor having properties is

$$E_{\mu\nu} = \frac{1}{\varkappa}\left(G_{\mu\nu} + \Lambda g_{\mu\nu}\right)$$

$\varkappa, \Lambda = \text{const.}$

$$\Longrightarrow \quad \boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = \varkappa T_{\mu\nu}}$$

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu}$$

Interpretation of $\varkappa$? Λ ?