Stony Brook University
Differential Geometry -- Fall 2008

Instructor: Pawel Nurowski

Schedule and Syllabus for Fall 2008

Week of	Content
Aug 31	Lecture 1 and 2: Manifolds, differentiable maps, tangent vectors and tangent spaces, transport of vectors/ differentials of maps, immersions and embeddings, submanifolds, vector fields and their trajectories, commutator, (vector) distributions, Frobenius theorem, tensors, tensor fields
Sept 7	Lecture 3 and 4: Local frames, differential forms, Cartan algebra and its derivations, Maurer-Cartan Theorem, more on Frobenius theorem, theorems of Pfaff and Darboux
Sept 14	Lecture 5 and 6: Connections: Koszul axioms, parallelism; tensor-valued forms; connection 1-form, covariant exterior differential; curvature 2-form, torsion 2-form; Ricci formula; Ist and IInd Bianchi identities in terms of curvature 2-forms and torsion 2-forms; definition of curvature and torsion tensors in terms of Koszul notation
Sept 21	Lecture 7 and 8: torsion/curvature 2-forms vs torsion and curvature tensors; Cartan structure equations and Bianchi identities as a closed differential system; Bianchi identities in the Koszul notation; Riemannian manifolds, pseudo-Riemannian manifolds; isometry
Sept 28	Lecture 9: Examples of metrics; left, right and biinvariant metrics on Lie groups; Lobachevski metric on an upper half plane; product metrics; wrapped products
Oct 5	Lecture 10: Geodesics; how metric and torsion determines connection
Oct 12	Lecture 11 and 12: Levi-Civita connection; connection coefficients in orthonormal and holonomic frames; arc length; geodesics as curves locally

	minimalizing arc length; geodesics in pseudo-riemannian setting; energy functional
Oct 19	Lecture 13 and 14: Metric connections as connections which preserve scalar product under the parallel transport; Riemann tensor and its symmetries; symmetries of curvature tensor of general connection: the role of the metricity and vanishing torsion conditions; vanishing of the Riemann tensor as neccessary and sufficient condition for an existence of a local coordinate system in which the metric is flat; decomposition of the Riemann tensor onto SO(n)-irreducibles: Weyl, Ricci, Ricci sclar; conformal significance of the Weyl tensor; examples in low dimensions
Oct 26	Lecture 15 and 16: Cannonical metrics on quadrics in flat (pseudo)-Riemannian manifolds; their curvature; Eisntein manifolds; Einstein field equations; examples of DeSitter and antiDeSitter spaces; isometries; Killing equations; full solution to the system of Killing equations in terms of flat metrics; isometry groups of maximal dimension; spaces of constant curvature; construction of all local metrics that have constant curvature in n-diemnsions;
Nov 2	Lecture 17 and 18: Sectional curvature; spaces of constant sectional curvature; Spherical symmetry; 4-dimensional Lorentzian case: stationary vs static spaces; Schwarzschild metric; Homogeneity of geodesics; exponentila map; normal coordinates; normal ball; example of these concepts in case of Lobachevski metric;
Nov 9	Lecture 19 and 20: Jacobi fields (exactly as in the relevant chapter of Do Carmo)
Nov 16	Lecture 21: Local isometric embedding of hypersurfaces in R^(n+1); Gauss Codazzi equations
Nov 23	Lecture 22: Local isometric embedding in R^(n+k) of codimension k; Gauss-Codazzi-Ricci equations;
Nov 30	Lecture 23: Local isometric embedding of a Riemannian manifold (M^n,g) in a Riemannian manifold (M^(n+k), G); Gauss-Codazzi-Ricci equations
	Lecture 24 and 25: Gauss-Kronecker curvature; mean curvature; isoparametric

Dec 7	hypersurfaces in space forms, with particular emphasis on hypersurfaces in spheres; minimal surfaces; Enepper-Weierstrass formula for minimal surfaces in R^3
Thank you for the attention! Pawel Nurowski	

Paweł Nurowski

Differential Geometry

Fall 2008/2009

Stony Brook University
Math Department.

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S.B. 2.09.2008

Notation

① M - n dimensional manifold (smooth)

M is a topological Hausdorff, paracompact space

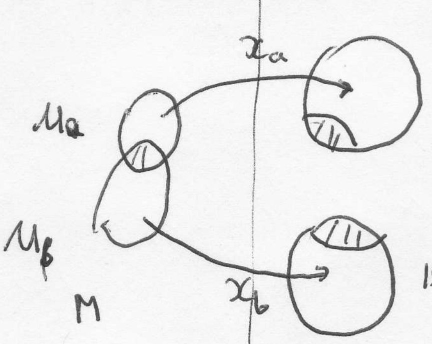
$p, q \in M$, $p \neq q$ $\Rightarrow \exists\, \mathcal{U}_a, \mathcal{U}_b$ s.t. $\mathcal{U}_a \cap \mathcal{U}_b = \phi$ and $p \in \mathcal{U}_a$, $q \in \mathcal{U}_b$

e.g. if it has a countable basis for its topology then it is paracompact

$M = \bigcup_{a \in I} \mathcal{U}_a$ $\qquad$ $\mathcal{U}_a$ - open sets

$\underbrace{(\mathcal{U}_a, x_a)}_{\text{chart}}$ $\qquad$ $x_a : \mathcal{U}_a \longrightarrow \mathbb{R}^n$ homeomorphism $\qquad$ $p \ni \mathcal{U}_a$ $\quad x_a(p) = (x^\mu)$ $\mu = 1, \dots n$

chart $\quad$ domain of a chart $\quad$ local coordinates



$x_b \circ x_a^{-1} \Big|_{x_a(\mathcal{U}_a \cap b)}$

are both smooth (have all partial derivatives)

$x_a \circ x_b^{-1} \Big|_{x_b(\mathcal{U}_a \cap \mathcal{U}_b)}$

$\mathcal{A} = \{ (\mathcal{U}_a, x_a), a \in I \}$ - atlas

② Differentiable map

M, N - manifolds $\qquad$ $\phi : M \to N$ is differentiable of class k

$\forall\, (\mathcal{U}, x) \in \mathcal{A}(M)$
$\quad (\mathcal{V}, y) \in \mathcal{A}(N)$ $\qquad$ $y \circ \phi \circ x^{-1}$ is of class k

- $M \subset \mathbb{R}^1 \Rightarrow \phi$ is called a curve

- $N \subset \mathbb{R}^1 \Rightarrow \phi$ is called a function

③ Tangent vector

$p \in M$, $\mathcal{F}(p)$ – algebra of ~~of~~ functions of class C^∞ defined in an neighbourhood of p.

$$\mathcal{F}(p) = \{ f : U_p \to \mathbb{R}^1, \ f \text{ of class } C^\infty \}$$

Two curves $\gamma, \tilde{\gamma}$ of class C^1 are tangent at $p = \gamma(0) = \tilde{\gamma}(0)$

iff $\quad \forall f \in \mathcal{F}(p) \quad \frac{d}{dt} f \circ \gamma(t) \Big|_{t=0} = \frac{d}{dt} f \circ \tilde{\gamma}(t) \Big|_{t=0}$

This is an equivalence relation on the set of curves passing through p.

Tangent vector ~~to γ~~ at p is an ~~an equivalence~~ equivalence class of curves tangent at p ~~with γ~~.

This defines a map $\quad X : \mathcal{F}(p) \longrightarrow \mathbb{R}$

$$X(f) = \frac{d}{dt}\left[f \circ \gamma(t) \right] \Big|_{t=0}$$

- Local representation: $\quad x$ – local coord. around $p \quad x \circ \gamma(t) = (x^\mu(t))$

$$X(f) = \frac{d}{dt} f \circ \gamma(t) \Big|_{t=0} = \frac{d}{dt} \left(f \circ x^{-1} \right) \circ (x \circ \gamma)(t) \Big|_{t=0} =$$

$$= \frac{\partial f}{\partial x^\mu}\Big|_p \frac{dx^\mu}{dt}\Big|_{t=0} = X^\mu \frac{\partial f}{\partial x_\mu}\Big|_p$$

$$X^\mu = \frac{dx^\mu}{dt}\Big|_{t=0}.$$

$$X = X^\mu \frac{\partial}{\partial x^\mu}\Big|_p \qquad (X^\mu) \in \mathbb{R}^n$$

- Properties: $\quad X : \mathcal{F}(p) \longrightarrow \mathbb{R}$

1° X linear

2° $X(f \cdot g) = X(f) g(p) + f(p) X(g)$

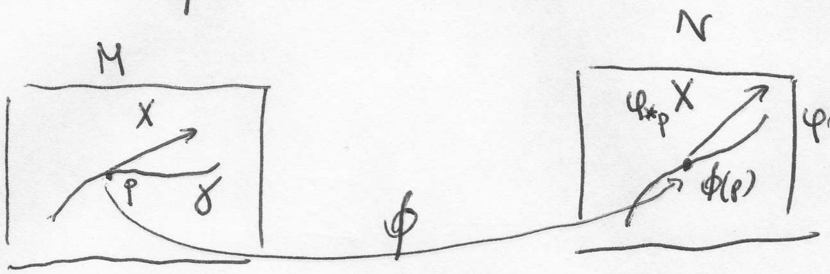
Leibniz rule.

④ Tangent space $T_p(M)$ at p

vector space of all X as above. Locally $\left(\frac{\partial}{\partial x^\mu}\Big|_p \right)$ $\mu = 1, \dots, n$, basis in $T_p(M)$.

(5) Transport of tangent vectors / differential of a map.

$$\phi : M \longrightarrow N \qquad \text{differentiable map.}$$



Example 1 e.g. $\phi : (x,y,z) \longrightarrow (y^2, x^3, z+x)$

$$X = A\partial_x + B\partial_y + C\partial_z$$

$$\gamma(t) = (At+x_0, Bt+y_0, Ct+z_0) \qquad \varphi(\gamma(t)) = \left((Bt+y_0)^2, (At+x_0)^3, (C+A)t+x_0+z_0\right)$$

$$\varphi_{*(x_0,y_0,z_0)} X = 2(Bt+y_0)B\big|_{t=0}\partial_x + 3(At+x_0)^2 A\big|_{t=0}\partial_y + (C+A)\big|_{t=0}\partial_z =$$

$$= 2By_0\,\partial_x + 3Ax_0^2\,\partial_y + (C+A)\,\partial_z$$

$$\varphi_{*(x_0,y_0,z_0)} : \begin{pmatrix} A \\ B \\ C \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 0 & 2y_0 & 0 \\ 3x_0^2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}}\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$\varphi_{*(x_0,y_0,z_0)} = \frac{\partial \phi^i}{\partial x^\mu}\bigg|_{(x_0,y_0,z_0)}$$

Properties

- ϕ_{*p} is linear
- locally $\left(\frac{\partial}{\partial x^\mu}\Big|_p\right)$ basis in $T_p(M)$ $\quad \mu = 1, \dots m$

  $\left(\frac{\partial}{\partial y^i}\Big|_{\varphi(p)}\right)$ basis in $T_{\varphi(p)}(N)$ $\quad i = 1, \dots n$

$$\varphi_{*p} \frac{\partial}{\partial x^\mu}\bigg|_p = \underbrace{\frac{\partial \varphi^i}{\partial x^\mu}\bigg|_p} \frac{\partial}{\partial y^i}\bigg|_{\varphi(p)}$$

$$\varphi_{*p} = d\phi_p$$

$$\varphi_{*p} \sim \frac{\partial \varphi^i}{\partial x^\mu}\bigg|_p \Rightarrow \text{hence the name differential.}$$

⑥ Immersions and embeddings.

$\phi: M \to N$ is an immersion if ϕ_{*p} is injective for all $p \in M$.

Example 1 continued

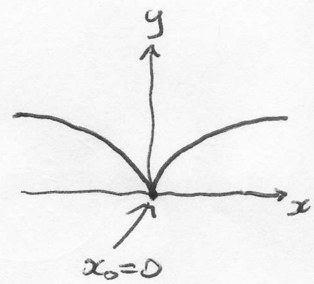
ϕ is not an immersion in $\mathbb{R}^3$ since it is not injective when either $x_0 = 0$ or $y_0 = 0$.

Ex 2

$\gamma: \mathbb{R} \to \mathbb{R}^2 \qquad x \longmapsto (x^3, x^2)$ has

$$\varphi_{*x_0} = (3x_0^2, 2x_0)$$

not an immersion.

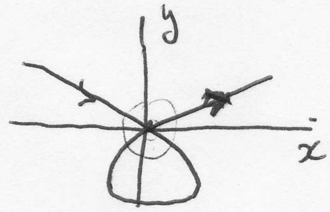


An immersion $\phi: M \to N$ is an embedding if ϕ is a homeomorphism onto $\phi(M) \subset N$ (topology on $\phi(M)$ induced from N)

Ex 3

$\gamma: \mathbb{R} \to \mathbb{R}^2 \qquad x \longmapsto (x^3 - 4x, x^2 - 4)$

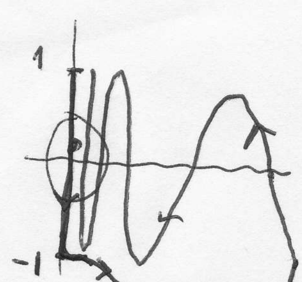
$$\varphi_{*x_0} = (3x_0^2 - 4, 2x_0) \quad \text{is an immersion}$$



Not embedding because of selfintersection

Ex 4

$$\gamma(t) = \begin{cases} (0, -(x+2)) & x \in (-3, -1) \\ \text{regular curve} & x \in (-1, -\frac{1}{\pi}) \\ (-x, -\sin\frac{1}{x}) & x \in (-\frac{1}{\pi}, 0) \end{cases}$$



immersion, but not embedding since the neighb. of a point on vertical line consists of disjoint interv.

(7) Submanifold.

If $M \subset N$ and the inclusion is an embedding then M is called submanifold of N.

(8) Codimension

If M is a submanifold of N and $\dim M = m$, $\dim N = n$ then $n-m$ is called a codimension of M in N.

Hypersurface a submanifold of codimension 1.

1 Vector field

$$M \ni p \xmapsto{\ X\ } X_p \in T_p(M)$$

i.e. a vector field is an assignment to every point $p \in M$ of a vector X_p from $T_p(M)$.

$\mathcal{F}(M)$ – algebra of C^∞ functions on M

one can think about X as a map

$$X: \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$$
$$f \longmapsto X(f)$$

$$X(f)(p) = X_p(f)$$

X is <u>of class C^k</u> if $\forall f \in \mathcal{F}(M)$ $X(f)$ is of class C^k

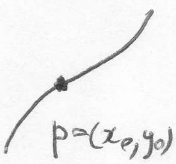
locally $(\mathcal{U}, x)$: $\quad X = X^\mu \dfrac{\partial}{\partial x^\mu}$ where $X^\mu: M \longrightarrow \mathbb{R}^n$

if X is of class C^k then $X^\mu = X^\mu(p)$ are of class C^k and vice versa.

$\mathfrak{X}(M)$ – vector space of vector fields of class C^∞ on M (infinite dimensional!)

2 <u>Trajectory of a vector field</u> passing through p

or <u>integral curve of X</u> passing through p



<u>Ex</u> $\quad X = y\partial_x - x\partial_y \quad$ vector field on $\mathbb{R}^2$

$$X^\mu = \begin{pmatrix} y \\ -x \end{pmatrix} \qquad \gamma^\mu(t) = \begin{pmatrix} x \\ y \end{pmatrix}(t) \quad \text{s.t.} \quad \frac{d\gamma^\mu}{dt} = X^\mu$$

$$\left.\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \frac{d(x+iy)}{dt} &= -i(x+iy) \\ x+iy &= (x_0+iy_0)e^{-it} \end{aligned}\right\} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\gamma^\mu(t) = \varphi_t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\varphi_t = \begin{pmatrix} \cos t & +\sin t \\ -\sin t & \cos t \end{pmatrix} \qquad \varphi_0 = id$$
$$\varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$$

Ex $\quad X = \partial_y - x^{-2}\partial_y \quad$ on $\quad \mathbb{R}_+ \setminus \{0\}$

$$\frac{d\gamma^\mu}{dt} = X^\mu \qquad \left.\begin{aligned}\frac{dx}{dt} &= 1\\ \frac{dy}{dt} &= -x^{-2}\end{aligned}\right\} \Rightarrow \begin{cases} x = t + x_0 \\ y = \dfrac{1}{t+x_0} + y_0 - \dfrac{1}{x_0}\end{cases}$$

$$\gamma^\mu(t) = \varphi_t\begin{pmatrix}x_0\\ y_0\end{pmatrix} = \begin{pmatrix} t + x_0 \\ \dfrac{1}{t+x_0} + y_0 - \dfrac{1}{x_0}\end{pmatrix}$$

$$\varphi_0\begin{pmatrix}x_0\\ y_0\end{pmatrix} = \begin{pmatrix}x_0\\ y_0\end{pmatrix} \qquad \varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$$

← this only holds locally. If t and t' are to far from 0 the second component blows up.

1-parameter transformation groups of M	Local 1 par. t. groups of M
$\mathbb{R}\times M \ni (t,p) \xrightarrow{\text{smooth}} \varphi_t(p) \in M$	$\forall_p \ \exists\, U_p, \varepsilon > 0$
1° $\varphi_0 = id$	$]-\varepsilon,\varepsilon[\times U_p \ni (t,p) \to \varphi_t(p) \in M$
2° $\varphi_t \circ \varphi_{t'} = \varphi_{t+t'}$	1°
	2° as above for $\lvert t\rvert, \lvert t'\rvert, \lvert t+t'\rvert < \varepsilon$

Def

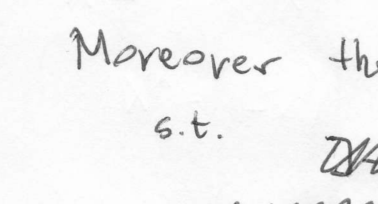
$t \longrightarrow \gamma(t)$ is called an integral curve of X passing through p if $\gamma(0) = p$,

↑ differentiable

$$\frac{d\gamma}{dt} = X_{\gamma(t)}$$

$\rightsquigarrow$ locally $\quad \dfrac{dx^\mu}{dt} = X^\mu(x^\nu(t))$ where $X = X^\mu(x^\alpha)\dfrac{\partial}{\partial x^\mu}$

~~a uniqu~~ theory of autonomous systems of ODEs:



There exist $\varepsilon > 0$ and a unique curve

$$]\varepsilon,\varepsilon[\ni t \longmapsto \gamma(t) = \varphi(t,p) = \varphi_t(p)$$

s.t. $\gamma(0) = p$ and $\dfrac{d\gamma}{dt} = X_{\gamma(t)}$.

Moreover there exists a neighbourhood $U_p \subset M$ s.t.

if $t \in]-\varepsilon,\varepsilon[$ then $\varphi_t : p \longrightarrow \varphi_t(p)$ is diffeo with the properties

1) $\varphi_0 = id$

2) $\varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$ when $t, t' \in]-\varepsilon,\varepsilon[$.

$\boxed{\varphi_t \text{ is called a } \underline{\text{flow}} \text{ of } X}$

Of course $X \in \mathfrak{X}(M)$ then the map

$$X : \mathcal{F}(M) \longrightarrow \mathcal{F}(M) \text{ is}$$

1) linear
2) satisfies $X(f \cdot g) = X(f) \cdot g + f \cdot X(g)$

③ Commutator

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X,Y) \longmapsto [X,Y] \in \mathfrak{X}(M)$$

$$[X,Y](f) = X(Y(f)) - Y(X(f))$$

obviously $[X,Y] : \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$ and is linear

but also $[X,Y](f \cdot g) = [X,Y](f) \cdot g + f [X,Y](g)$ check!

- locally $X = X^\mu \partial_\mu \quad , \quad Y = Y^\mu \partial_\nu$

$$[X,Y] = \left(X^\mu Y^\nu{}_{,\mu} - Y^\mu X^\nu{}_{,\mu} \right) \partial_\nu$$

- properties
    1° $[\ ,\ ]$ - bilinear
    2° $[X,Y] = -[Y,X]$ antisymmetric
    3° $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$, Jacobi.

$\left(\mathfrak{X}(M), [\ ,\ ] \right)$ Lie algebra of smooth vector fields over M (infinite dim).

(4) (Vector) distribution

Def An m-dimensional distribution S on M is a map

$$M \ni p \xrightarrow{S} S_p \subset T_p(M)$$

S_p: m-dimensional (vector) subspace of $T_p(M)$

if $\forall p \in M \; \exists \, \mathcal{U}_p \; \exists \, (X_i)_{i=1,\dots m}$ of class C^∞ s.t.

$\forall q \in \mathcal{U}_p \; \left(X_i \big|_q \right)$ is a basis for $S_q \Rightarrow S_q$ is smooth

Only SMOOTH distributions from now on.

- $X \in S \iff \forall_p \; X_p \in S_p$

- S is involutive $\iff X, Y \in S \Rightarrow [X, Y] \in S$

- M_S is an integral manifold of S iff
  - M_S is a submanifold of M s.t.

$$\forall p \in M_S \quad T_p(M_S) = S_p$$

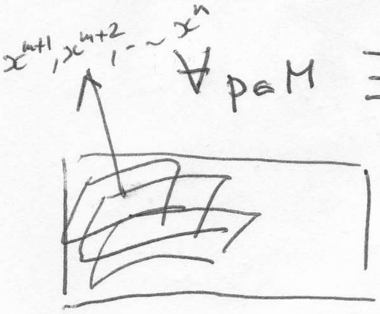
Note: Vector field is a distribution of dim 1.
Its integral manifolds $\equiv$ integral curves.

What about existence of integral manifolds for $m > 1$ dimensional distributions?

⑤ Frobenius theorem

$$\left(S \text{ is involutive} \right) \Longleftrightarrow \left(\begin{array}{l} \text{through every point } p \in M \\ \text{posses precisely one} \\ \text{(maximal) integral manifold} \\ M_S \text{ of } S \end{array} \right)$$

in such a case



$\forall p \in M \ \exists (\mathcal{U}, x)$ s.t. $p \in \mathcal{U}$ and ~~intersections of~~ all $E_S \cap \mathcal{U}$ are given by $x^{m+1}, x^{m+2}, \dots x^n = \text{const.}$

then $X_i = A^j{}_i(x^\mu) \frac{\partial}{\partial x^j}$ $j = 1, \dots m$ is a local basis in S

Fact

X_i on $\mathcal{U}$ linearly independent vector fields.

$[X_i, X_j] = 0 \quad \forall i,j = 1, \dots m \Longleftrightarrow$ there exists a coordinate system x^μ in $V \subset \mathcal{U}$ s.t. $X_i = \frac{\partial}{\partial x^i}$ $i = 1, \dots m$.

(6) Tensors.

$n < \infty$

V – n-dimensional vector space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$

$$V^* = \{\omega : V \xrightarrow[\text{linear}]{} \mathbb{K}\}$$

$$V^r_s = \underbrace{V \otimes \dots \otimes V}_{r} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{s} =$$

$$= L(\underbrace{V^*, V^*, \dots, V^*}_{r}, \underbrace{V, \dots V}_{s}; \mathbb{R})$$

$\uparrow$

multilinear maps fra $V^* \times V^* \times \dots V^* \times V \times \dots V \longrightarrow \mathbb{R}$.

$\{e_\mu\}$ – basis in V
$\mu = 1, \dots n$

$\{e^\mu\}$ – dual basis in V^* defined by
$\mu = 1, \dots n$

$$e^\mu(e_\nu) = \delta^\mu{}_\nu$$

$$V \ni v = v^\mu e_\mu \, ; \quad \omega = \omega_\mu e^\mu \in V^*$$

Basis in V^r_s:

$$e_{\mu_1} \otimes \dots e_{\mu_r} \otimes e^{\nu_1} \otimes \dots e^{\nu_s} \text{ defined by}$$

$$\left(e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes e^{\nu_1} \otimes \dots \otimes e^{\nu_s}\right)(e^{\alpha_1}, \dots e^{\alpha_r}, e_{\beta_1}, \dots e_{\beta_s}) =$$

$$= \delta_{\mu_1}^{\alpha_1} \cancel{\otimes} \dots \delta_{\mu_r}^{\alpha_r} \delta_{\beta_1}^{\nu_1} \dots \delta_{\beta_s}^{\nu_s}$$

$$V^r_s \ni K = K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes e^{\nu_1} \otimes \dots \otimes e^{\nu_s}$$

Contraction $\quad C^i_j : V^r_s \xrightarrow{\text{linear}} V^{r-1}_{s-1}$

$1 \le i \le r$
$1 \le j \le s$

$$C^i_j(v_1 \otimes \dots \otimes v_r \otimes \omega^1 \otimes \dots \otimes \omega^s) = $$

$$= \langle \omega^j, v_i \rangle \, v_1 \otimes \dots \overset{\overset{i}{\vee}}{} \dots \otimes v_r \otimes \omega^1 \otimes \dots \overset{\overset{j}{\vee}}{} \dots \otimes \omega^s$$

(7) Change of the basis

$$\boxed{e'^{\mu} = a^{\mu}{}_{\nu}\, e^{\nu}} \qquad e' = a e$$

$$a = (a^{\mu}{}_{\nu}) \in GL(n, \mathbb{K})$$

$$e'_{\mu} = e_{\nu}\, b^{\nu}{}_{\mu}$$

$$\underset{\delta^{\mu}_{\nu}}{\underset{\|}{e'^{\mu}(e'_{\nu})}} = a^{\mu}{}_{\rho}\, b^{\sigma}{}_{\nu}\, e^{\rho}(e_{\sigma}) = a^{\mu}{}_{\rho}\, b^{\rho}{}_{\nu}$$

$$\Rightarrow \quad a \cdot b = 1 \Rightarrow b = a^{-1}$$

$$\boxed{e'_{\mu} = e_{\nu}\, a^{-1\,\nu}{}_{\mu}}$$

$$v = v^{\mu} e_{\mu} = v'^{\mu} e'_{\mu} = v'^{\mu} e_{\nu}\, a^{-1\,\nu}{}_{\mu}$$

$$\Rightarrow v^{\nu} = v'^{\mu} a^{-1\,\nu}{}_{\mu} \Rightarrow \boxed{v'^{\mu} = a^{\mu}{}_{\nu} v^{\nu}}$$

$$\omega = \omega_{\mu} e^{\mu} = \omega'_{\mu} a^{\mu}{}_{\nu} e^{\nu}$$

$$\Rightarrow \omega_{\nu} = \omega'_{\mu} a^{\mu}{}_{\nu} \Rightarrow \boxed{\omega'_{\mu} = \omega_{\nu}\, a^{-1\,\nu}{}_{\mu}}$$

$$K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} \longmapsto a^{\mu_1}{}_{\alpha_1} \dots a^{\mu_r}{}_{\alpha_r} K^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s} a^{-1\,\beta_1}{}_{\nu_1} \dots a^{-1\,\beta_s}{}_{\nu_s}$$

$$= K'^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}.$$

~~old style definition of tensor.~~

~~$T = P(V) \times W$~~ ~~vector space~~

Old style view on tensors

$$K \sim (e, K^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s})$$

$$e \mapsto e' = e a^{-1}$$

$$K^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s} \longmapsto K'^{\mu_1 \dots \mu_s}{}_{\nu_1 \dots \nu_s}$$

How to introduce a vector space structure on such pairs?

(8) Objects of type ρ.

$$\rho : G \xrightarrow{\text{hom}} GL(W)$$ i.e. ρ is a representation of G in W.

vector space of $\dim W = N < \infty$

$a \in G$; $\quad W \ni w \xrightarrow{\rho(a)} \rho(a)w \in W$ is a left action of G on W.

- $\rho(1) = id$
- $\rho(a \cdot b) = \rho(a)\rho(b)$

V another vector space of $\dim V = n < \infty$; $P(V)$ set of all basis in V

$G = GL(V)$ naturally acts on $P(V)$:

$$P(V) \ni e = (e_\mu) \xrightarrow{a \in GL(V)} e' = e \cdot a^{-1}$$
$$e'_\mu = e_\nu a^{-1\,\nu}{}_\mu$$

now in

$$P(V) \times W \ni (e, w) \xrightarrow{\varphi_a} \varphi_a(e, w) = (e a^{-1}, \rho(a) w)$$

this is a left action of $GL(V)$ on $P(V) \times W$

$(e, w) \sim (e', w') \iff \exists a \in GL(n, \mathbb{R})$ s.t.

$$(e', w') = \varphi_a(e, w).$$

this is an equivalence relation in $P(V) \times W$.

Check!

$$W_\rho = P(V) \times W / \underset{\rho}{\sim}$$

space of objects of type ρ

One can introduce a structure of (a) vector space in W_ρ

$[(e, w)]$, $[(e', w')] \sim [(e, \tilde{w})]$

$\alpha [(e, w)] + \beta [(e', w')] = [(e, \alpha w + \beta \tilde{w})]$

Check that this does not depend on the choice of representatives

$\dim W_\rho = \dim W$

Examples

① $W = \mathbb{R}^{n(r+s)}$, $V = \mathbb{R}^n$,

$$\rho(a) K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = a^{\mu_1}{}_{\alpha_1} \dots a^{\mu_r}{}_{\alpha_r} K^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s} a^{-1\beta_1}{}_{\nu_1} \dots a^{-1\beta_s}{}_{\nu_s}$$

$W_\rho = V^r_s$ tensors of type $\binom{r}{s}$ $\quad = \rho^r_s(a) K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}$

② $W = \mathbb{R}^{n(r+s)}$, $V = \mathbb{R}^n$

$$\rho(a) = (\det a)^\omega \rho^r_s(a)$$

W_ρ - tensor densities of weight ω.

e.g. $\boxed{A}$ Levi-Civita symbol defined by a) b) c):

a) $\varepsilon_{\mu_1 \dots \mu_n} = \epsilon_{[\mu_1 \dots \mu_n]}$; b) $\epsilon_{1 \dots n} = 1$

$\uparrow$ totally skew symmetric in $\underline{\underline{n}}$ indices

c) $\epsilon'_{\mu_1 \dots \mu_n} = \epsilon_{\mu_1 \dots \mu_n}$

$$a^{-1\mu_1}{}_{\nu_1} \dots a^{-1\mu_n}{}_{\nu_n} \epsilon_{\mu_1 \dots \mu_n} = (\det a)^{-1} \epsilon_{\nu_1 \dots \nu_n} =$$

$$= (\det a)^{-1} \epsilon'_{\nu_1 \dots \nu_n}$$

$$\Rightarrow \quad \epsilon'_{\nu_1 \dots \nu_n} = (\det a) \rho_n(a) \epsilon_{\nu_1 \dots \nu_n}$$

~~[illegible]~~ density of covariant n-tensor of weight $+1$.

$\boxed{B}$ $\det(g_{\mu\nu})$

$$\det(g'_{\mu\nu}) = (\det a)^{-2} \det(g_{\mu\nu})$$

scalar density of weight -2

$\boxed{C}$ $W = \mathbb{R}^{n(r+s)}$, $\rho(a) = \operatorname{sgn}(\det a) \rho^r_s(a)$

W_ρ - pseudotensors

e.g.

$$\eta_{\mu_1 \dots \mu_n} = \sqrt{|\det g|}\, \epsilon_{\mu_1 \dots \mu_n}$$

Tensor fields

$$T_p(M) \rightsquigarrow T_p(M)^r_s$$

$$M \ni p \longrightarrow K_p \in T_p(M)^r_s$$

locally $(\mathcal{U}, x)$:

$$K = \underbrace{K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}}_{} \frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_r}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_s}$$

smooth $\Longrightarrow$ K is smooth

$\mathcal{X}(M)^r_s$ – module of smooth tensor fields on M over $\mathcal{F}(M)$

vector space of smooth tensor fields on M over $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

$$\mathcal{X}(M)^0_0 = \mathcal{F}(M)$$

$$\mathcal{T}(M) = \left(\bigoplus_{\substack{r \geq 0 \\ s \geq 0}}^{\infty} \mathcal{X}(M)^r_s \, , \, \otimes \right)$$

algebra of smooth tensor fields over M.

① Local frames.

Set of smooth vector fields $(X_\mu)_{\mu=1,\dots,n}$ in $\mathcal{U} \subset M$ (open) is a frame in $\mathcal{U}$

if $(X_\mu|_p)$ is a basis in $T_p(M)$ $\forall p \in \mathcal{U}$.

Holonomic frame (X_μ) $\Leftrightarrow$ $[X_\mu, X_\nu] = 0$ $\forall \mu, \nu = 1, \dots n$

Holonomic frame (X_μ) in $\mathcal{U}$ $\Leftrightarrow$ $\exists\, x^\mu$ in $\mathcal{U}$ s.t.

$$X_\mu = \frac{\partial}{\partial x^\mu}$$

$\frac{\partial}{\partial x^\mu}$ and $A^\mu{}_\nu = A^\mu{}_\nu(x)$ invertible matrix-valued functions in $\mathcal{U}$

$\Rightarrow$ $X_\nu = A^\mu{}_\nu \frac{\partial}{\partial x^\mu}$ is in general nonholonomic.

② $\Lambda^s M$ - skew-symmetric smooth tensor fields of type $\binom{0}{s}$

$$\Lambda^s M \ni \overset{s}{\omega} \Leftrightarrow \overset{s}{\omega}(X_1, \dots X_i, \dots X_j, \dots X_s) = -\omega(X_1, \dots X_j, \dots X_i, \dots X_s) \quad \forall i \neq j$$

Exterior differential

$$\boxed{d : \Lambda^s M \longrightarrow \Lambda^{s+1} M}$$

$$\boxed{\begin{aligned} d\overset{s}{\omega}(X_0, \dots X_s) = &\sum_{i=0}^{s} (-1)^i X_i (\overset{s}{\omega}(X_0, \dots \hat{X_i} \dots, X_s)) + \\ &+ \sum_{0 \leq i < j \leq s} (-1)^{i+j} \overset{s}{\omega}([X_i, X_j], X_0, \dots \hat{i} \dots \hat{j} \dots, X_s) \end{aligned}}$$

Check that $d\omega$ is f-linear!

In particular:

$$\boxed{d\overset{1}{\omega}(X,Y) = X(\overset{1}{\omega}(Y)) - Y(\overset{1}{\omega}(X)) - \overset{1}{\omega}([X,Y])}$$

③ Wedge product:

antisymmetrization

$$\mathrm{Alt}_s : \mathcal{X}(M)^0_s \xrightarrow{f\text{-linear}} \mathcal{X}(M)^0_s$$

$$\mathrm{Alt}_s(\omega_1 \otimes \dots \otimes \omega_s) = \frac{1}{s!} \sum_{\sigma \in S_s} \mathrm{sgn}(\sigma)\, \omega_{\sigma(1)} \otimes \dots \otimes \omega_{\sigma(s)}$$

in particular:

$$\Lambda^s M \ni \omega : \quad \mathrm{Alt}_s(\omega) = \omega.$$

$$\overset{s}{\omega} \wedge \overset{t}{\omega} = \frac{(s+t)!}{s!\,t!} \mathrm{Alt}_{s+t}(\overset{s}{\omega} \otimes \overset{t}{\omega})$$

Ex

$$dx^\mu \wedge dx^\nu = \frac{(1+1)!}{1!\,1!} \mathrm{Alt}_{1+1}(dx^\mu \otimes dx^\nu)$$

$$= \frac{2!}{2!}(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu)$$

④ Cartan algebra $(\Lambda M, \wedge, d)$

$$\Lambda M = \bigoplus_{s=0}^{n} \Lambda^s M \quad , \quad \Lambda^0 M = \mathcal{F}(M)$$

⑤ Derivations of ΛM of degree k.

$$\mathcal{D} : \Lambda M \longrightarrow \Lambda M \quad \text{s.t.}$$

$$\mathcal{D} : \Lambda^s M \longrightarrow \Lambda^{s+k} M$$

$$\mathcal{D}(\overset{s}{\omega} \wedge \overset{p}{\omega}) = \mathcal{D}\overset{s}{\omega} \wedge \overset{p}{\omega} + (-1)^{sk}\, \overset{s}{\omega} \wedge \mathcal{D}\overset{p}{\omega}.$$

Example 1: $d : \Lambda^{\bullet} M \longrightarrow \Lambda M$ – derivation of degree $+1$

Example 2

$\mathfrak{X}(M) \ni X$ defines derivation of degree -1. by

$$\begin{cases} X \lrcorner f = 0 \\ X \lrcorner df = X(f). \end{cases}$$

$$(X \lrcorner \overset{s}{\omega})(X_2, \dots X_s) = \overset{s}{\omega}(X, X_2, \dots X_s)$$

~~Ex~~ locally $\omega = \frac{1}{s!} \omega_{\mu_1 \dots \mu_s} dx^{\mu_1} \wedge \dots dx^{\mu_s}$

$X \lrcorner \omega$ from the Leibnitz rule!

Example 3

$$\underset{X}{\mathcal{L}} = X \lrcorner d + d X \lrcorner \qquad \Lambda^s M \longrightarrow \Lambda^s M$$

$$\underset{X}{\mathcal{L}}(\overset{s}{\omega} \wedge \overset{t}{\omega}) = \dots \qquad = \underset{X}{\mathcal{L}} \overset{s}{\omega} \wedge \overset{t}{\omega} + \overset{s}{\omega} \wedge \underset{X}{\mathcal{L}} \overset{t}{\omega}.$$

Example 4

Locally $\overset{2}{\omega} = \frac{1}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} \omega_{\mu\nu} (dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu)$

$$\overset{2}{\omega}(X, Y) = \frac{1}{2} \omega_{\mu\nu} (X^\mu Y^\nu - X^\nu Y^\mu)$$

$$\boxed{\overset{2}{\omega}(X,Y) = (X \lrcorner \overset{2}{\omega})(Y) = Y \lrcorner X \lrcorner \overset{2}{\omega}}$$

$$Y \lrcorner X \lrcorner \overset{2}{\omega} = Y \lrcorner X \lrcorner \left(\frac{1}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu\right) =$$

$$= \frac{1}{2} \omega_{\mu\nu} Y \lrcorner (X^\mu dx^\nu - X^\nu dx^\mu) =$$

$$= \frac{1}{2} \omega_{\mu\nu} (X^\mu Y^\nu - X^\nu Y^\mu)$$

$$(X \lrcorner \overset{2}{\omega})(Y) =$$

$$= \frac{1}{2} \omega_{\mu\nu} (X^\mu dx^\nu - X^\nu dx^\mu)(Y)$$

$$= \frac{1}{2} \omega_{\mu\nu} (X^\mu Y^\nu - X^\nu Y^\mu)$$

OK

$$\boxed{Y \lrcorner X \lrcorner \overset{2}{\omega} = \overset{2}{\omega}(X, Y)}$$

$$X_k \lrcorner \dots X_1 \lrcorner \overset{k}{\omega} = \overset{k}{\omega}(X_1, \dots X_k)$$

(6) Maurer-Cartan theorem

X_μ - frame in $\mathcal{U}$

$$[X_\mu, X_\nu] = c^\rho{}_{\mu\nu} X_\rho$$

$c^\rho{}_{\mu\nu}$ are smooth functions in $\mathcal{U}$.

$$c^\rho{}_{\mu\nu} = -c^\rho{}_{\nu\mu}$$

$c^\rho{}_{\mu\nu} = 0$ in $\mathcal{U}$ $\iff$ $\exists\, x^\mu$ local coord. system s.t. $X_\mu = \frac{\partial}{\partial x^\mu}$

ω^μ is a coframe dual to X_μ in $\mathcal{U}$ iff

$$X_\mu \lrcorner\, \omega^\nu = \omega^\nu(X_\mu) = \delta^\nu{}_\mu .$$

Thm

$$\boxed{[X_\mu, X_\nu] = c^\rho{}_{\mu\nu} \iff d\omega^\mu = -\frac{1}{2} c^\mu{}_{\nu\rho}\, \omega^\nu \wedge \omega^\rho}$$

Proof

$$X_\rho \lrcorner X_\nu \lrcorner d\omega^\mu = d\omega^\mu(X_\nu, X_\rho) = X_\nu(\cancel{\omega^\mu(X_\rho)}) - X_\rho(\cancel{\omega^\mu(X_\nu)}) - \omega^\mu([X_\nu, X_\rho]) =$$

$$= -c^\mu{}_{\nu\rho}$$

$$X_\rho \lrcorner X_\nu \lrcorner \left(-\frac{1}{2} c^\mu{}_{\alpha\beta}\, \omega^\alpha \wedge \omega^\beta\right) = \frac{1}{2} X_\rho \lrcorner \left(-c^\mu{}_{\nu\beta}\, \omega^\beta + c^\mu{}_{\alpha\nu}\, \omega^\alpha\right) =$$

$$= -\frac{1}{2}\left(c^\mu{}_{\nu\rho} + c^\mu{}_{\rho\nu}\right) = -c^\mu{}_{\nu\rho}$$

$$\boxed{d\omega^\mu = -\frac{1}{2} c^\mu{}_{\alpha\beta}\, \omega^\alpha \wedge \omega^\beta}$$

□.

(7) Fröbenius revisited

S – distribution

$$S^* = \{ \omega \in \Lambda^1 M \; : \; \omega(X) = 0 \quad \forall X \in S \}$$

$(X_i)_{i=1,\dots,m}$ frame for S $\qquad i,j,k \dots = 1,\dots,m$

$(\omega^\alpha)_{\alpha=m+1,\dots,n}$ frame for S^* $\qquad \alpha,\beta,\gamma \dots = m+1,\dots,n$

The following conditions are equivalent:

1) Through every point $p \in M$ passes precisely one integral manifold of S

$\Updownarrow$

2) $[X_i, X_j] = c^k{}_{ij} X_k$

3) $X_i = A^j{}_i (x^k, x^\gamma) \frac{\partial}{\partial x^j}$

4) $d\omega^\alpha \wedge \omega^{m+1} \wedge \dots \wedge \omega^n = 0 \quad \forall \alpha = m+1,\dots,n$

5) $\omega^\alpha = B^\alpha{}_\beta (x^k, x^\gamma)\, dx^\beta$

Proof

1) $\Leftrightarrow$ 2) ✓

3) $\Rightarrow$ 5) since $X_i \lrcorner\, \omega^\alpha = 0$ and $\dim S^* = n-m$

5) $\Rightarrow$ 4) obvious

4) $\Rightarrow$ 2) $d\omega^\alpha = -\frac{1}{2} c^\alpha{}_{\beta\gamma}\, \omega^\beta \wedge \omega^\gamma - \frac{1}{2} \overset{(4)\,\Downarrow}{\cancel{c^\alpha{}_{ij}\, \omega^i \wedge \omega^j}} - \frac{1}{2} c^\alpha{}_{i\beta}\, \omega^i \wedge \omega^\beta$

$$[X_i, X_j] = c^k{}_{ij} X_k + \cancel{c^\alpha{}_{ij} X_\alpha}$$

$\square$.

Rank q of a 2-form α i defined by:

$$\underbrace{\alpha\wedge\dots\wedge\alpha}_{q\text{-times}} \neq 0 \qquad \underbrace{\alpha\wedge\dots\wedge\alpha}_{(q+1)\text{-times}} = 0 \qquad 2q \leq n$$

Darboux theorem

(1) σ be a 1-form s.t. $d\sigma$ has rank $2q$.

Then there exist local coordinates $x^1, \dots x^q, y^1, \dots y^{n-q}$ s.t.

$$\sigma = \begin{cases} x^1 dy^1 + \dots + x^q dy^q & \text{if } \sigma\wedge \underbrace{d\sigma\wedge\dots\wedge d\sigma}_{q\text{-times}} = 0 \\ x^1 dy^1 + \dots + x^q dy^q + dy^{q+1} & \text{if } \sigma\wedge \underbrace{d\sigma\wedge\dots\wedge d\sigma}_{q\text{-tims.}} \neq 0 \end{cases}$$

(2) For any 2-form α of rank q there exists a basis (ω^a) such that

$$\alpha = \omega^1\wedge\omega^2 + \omega^3\wedge\omega^4 + \dots + \omega^{2q-1}\wedge\omega^{2q}$$

If $d\alpha = 0$ then there exists coordinates $x^1, \dots x^q, y^1, \dots y^{n-q}$

$$\alpha = dx^1\wedge dy^1 + \dots + dx^q \wedge dy^q$$

Proof Sternberg S (1964)

Lectures on differential geometry (Prentice-Hall, Englewood Cliffs, NJ)

Affine connection ∇ is a map:

$$\mathfrak{X}(M) \times \mathcal{T}(M) \ni (X, K) \longrightarrow \nabla_X K \in \mathcal{T}(M)$$

s.t.

1° $\nabla_X : \mathcal{T}(M) \to \mathcal{T}(M)$ preserves the type of tensor

2° $\nabla_{fX+gY} K = f \nabla_X K + g \nabla_Y K$

3° ∇ is $\mathbb{R}$-linear in K $\quad \nabla_X (\alpha K_1 + \beta K_2) = \alpha \nabla_X K_1 + \beta \nabla_X K_2$, $\alpha, \beta \in \mathbb{R}$

4° $\nabla_X (K \otimes L) = \nabla_X K \otimes L + K \otimes \nabla_X L$

5° ∇_X commutes with contractions

6° $\nabla_X f = X(f)$

$\nabla_X K$ - is called covariant derivative of K w.r.t. X

Rmk 1 ∇ defines a map

$$\mathfrak{X}(M)^r_s \longrightarrow \mathfrak{X}(M)^r_{s+1}$$
$$K \longmapsto DK \quad \text{s.b.}$$

$$DK(X_1, \dots X_s, X) = (\nabla_X K)(X_1, \dots, X_s)$$

Rmk 2 How to introduce connections on mfds?

Connection is a local notion in the following sense:

Let (X_μ) be a frame in $\mathcal{U}$ $\Rightarrow$

$$\boxed{\nabla_{X_\mu} X_\nu = \Gamma^\rho_{\nu\mu} X_\rho}$$

functions on $\mathcal{U}$. (Smooth since X and ∇ is smooth)

$\Gamma^\rho_{\nu\mu}$ — these functions determine connection in $\mathcal{U}$.

Let (ω^μ) be a coframe dual to (X_μ):

$$X_\mu \lrcorner\, \omega^\nu = \omega^\nu(X_\mu) = \delta^\nu_\mu$$

then:

contraction

$$\left[\nabla_{X_\mu}(\omega^\nu)\right](X_\varrho) = C^1_1(\nabla_{X_\mu}\omega^\nu \otimes X_\varrho) \overset{4^\circ}{=} C^1_1\,\nabla_{X_\mu}(\omega^\nu \otimes X_\varrho) - C^1_1(\omega^\nu \otimes \nabla_{X_\mu} X_\varrho)$$

$$\overset{5^\circ}{=} \nabla_{X_\mu} C^1_1(\omega^\nu \otimes X_\varrho) - C^1_1(\omega^\nu \otimes \Gamma^\alpha{}_{\varrho\mu} X_\alpha) = \cancel{\nabla_{X_\mu}\delta^\nu_\varrho} - \Gamma^\alpha{}_{\varrho\mu}\, C^1_1(\omega^\nu \otimes X_\alpha) =$$

$$= -\Gamma^\nu{}_{\varrho\mu}$$

$$\Longrightarrow \quad \boxed{\nabla_{X_\mu}\omega^\nu = -\Gamma^\nu{}_{\varrho\mu}\,\omega^\varrho}$$

Now: useful notation:
(X_μ) - a frame in U (not necessarily holonomic!)

$$\nabla_{X_\mu} K := \nabla_\mu K$$

Then, for a general tensor field

$$K = K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\, X_{\mu_1} \otimes \dots \otimes X_{\mu_r} \otimes \omega^{\nu_1} \otimes \dots \otimes \omega^{\nu_s}$$

if Z is a vector field $Z = Z^\mu X_\mu$ then

$$\boxed{\nabla_Z K \overset{2^\circ}{=} Z_\varrho \nabla_\varrho K = Z_\varrho \underline{(\nabla_\varrho K)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}}\, X_{\mu_1} \otimes \dots \otimes X_{\mu_r} \otimes \omega^{\nu_1} \otimes \dots \otimes \omega^{\nu_s}}$$

Where:

(CD)
$$\boxed{\begin{aligned}(\nabla_\varrho K)^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} &= X_\varrho(K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}) + \\ &+ \sum_{i=1}^{N} \Gamma^{\mu_i}{}_{\sigma\varrho}\, K^{\mu_1 \dots \underset{\substack{\uparrow \\ \text{ith place}}}{\sigma} \dots \mu_r}{}_{\nu_1 \dots \nu_s} - \sum_{i=1}^{s} \Gamma^\sigma{}_{\nu_i\varrho}\, K^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \overset{\substack{\text{ith place} \\ \downarrow}}{\sigma} \dots \nu_s}\end{aligned}}$$

Formula (CD), once $\Gamma^{\mu}{}_{\nu\rho}$ are specified, determines ∇ in $\mathcal{U}$.

- change of basis

$$X'_\mu = X_\nu a^{-1\nu}{}_\mu \iff X_\mu = a^\nu{}_\mu X'_\nu$$

thus

$$\Gamma^\rho{}_{\nu\mu} X_\rho = \nabla_{X_\mu} X_\nu = \nabla_{a^\alpha{}_\mu X'_\alpha}\left(a^\beta{}_\nu X'_\beta\right) =$$

$$= a^\alpha{}_\mu \nabla_{X'_\alpha}\left(a^\beta{}_\nu X'_\beta\right) = a^\alpha{}_\mu \left[X'_\alpha(a^\beta{}_\nu) X'_\beta + a^\beta{}_\nu \Gamma'^\sigma{}_{\beta\alpha} X'_\sigma\right]$$

$$= X_\mu(a^\beta{}_\nu) X'_\beta + a^\alpha{}_\mu a^\beta{}_\nu \Gamma'^\sigma{}_{\beta\alpha} X_\rho a^{-1\rho}{}_\sigma =$$

$$= X_\mu \lrcorner da^\beta{}_\nu \, a^{-1\rho}{}_\beta X_\rho + a^\alpha{}_\mu a^\beta{}_\nu \Gamma'^\sigma{}_{\beta\alpha} a^{-1\rho}{}_\sigma X_\rho$$

$$\boxed{\Gamma^\rho{}_{\nu\mu} = a^\alpha{}_\mu a^\beta{}_\nu \Gamma'^\sigma{}_{\beta\alpha} a^{-1\rho}{}_\sigma + a^{-1\rho}{}_\beta X_\mu(a^\beta{}_\nu)}$$

Trick: Connection 1-forms in frame (X_μ)

$$\Gamma^\rho{}_{\mu\nu} \rightsquigarrow \Gamma^\rho{}_{\mu\nu}\omega^\nu =: \Gamma^\rho{}_\mu$$

$$\boxed{\Gamma^\rho{}_\nu = a^\beta{}_\nu \Gamma'^\sigma{}_\beta a^{-1\rho}{}_\sigma + a^{-1\rho}{}_\beta d(a^\beta{}_\nu)}$$

in matrix notation

$$\Gamma = a^{-1}\Gamma' a + a^{-1} da$$

$$\Rightarrow \quad \Gamma' = a\Gamma a^{-1} - da \cdot a^{-1}$$

but $\underset{\overset{\shortparallel}{0}}{d(a a^{-1})} = da\, a^{-1} + a\, da^{-1}$

$$\boxed{\Gamma' = a\Gamma a^{-1} + a\, da^{-1}} \quad \text{or}$$

$$\boxed{\Gamma'^{\mu}{}_{\nu} = a^{\mu}{}_{\alpha}\Gamma^{\alpha}{}_{\beta}a^{-1\beta}{}_{\nu} + a^{\mu}{}_{\alpha}\, da^{-1\alpha}{}_{\nu}} \quad (TC)$$

Connections transform differently then tensors! These are different kind of objects!

Now having two frames X_μ and X'_μ on two open sets $\mathcal{U}$ and $\mathcal{U}'$ with $\mathcal{U}\cap\mathcal{U}' \neq \phi$ we can take $\Gamma^{\mu}{}_{\nu}$ in $\mathcal{U}$ and $\Gamma'^{\mu}{}_{\nu}$ in $\mathcal{U}'$. They define <u>the same</u> connection in $\mathcal{U}\cup\mathcal{U}'$ provided the there exists $GL(n,\mathbb{R})$-valued function a on $\mathcal{U}\cap\mathcal{U}'$ so that Γ' and Γ are related by (TC).

<u>Rmk 3</u> How the notion of connection was obstracted?

Digresion àla Cartan

$G = GL(n,\mathbb{R})$ acts on the space of all local frames in U

$$(X_\mu) \xmapsto{a} (X'_\mu) = (X_\nu a^{-1\nu}{}_\mu)$$

It also acts on the space of all coframes in U

$$(\omega^\mu) \xmapsto{a} (\omega'^\mu) = (a^\mu{}_\nu \omega^\nu)$$

$$\omega \longmapsto \omega' = a\omega .$$

If we have $W = \mathbb{R}^N$ and representation

$$\rho : GL(n,\mathbb{R}) \to GL(N,\mathbb{R})$$

we define:

$$\boxed{\begin{array}{l} \underline{k\text{-form of type } \rho \text{ in } U} \quad \text{is an assignment:} \\[1em] \qquad \alpha : \omega \longmapsto \alpha(\omega) \in W \otimes \Lambda^k U \\[1em] \text{s.t.} \qquad \alpha(a\omega) = \rho(a)\alpha(\omega). \end{array}}$$

Example

1) $W = \mathbb{R}^n$, $\rho = id$, i.e. $\rho(a) = a$, $k=1$

'moving coframe'

$\theta = (\theta^\mu)$ $\quad \mu = 1, \dots, n$ $\qquad\qquad$ θ – 1-form of type id.

$$\theta^\mu(\omega) := \omega^\mu$$

$$\theta^\mu(a\omega) = a^\mu{}_\nu \omega^\nu .$$

2) $W = \mathbb{R}^n$, $\rho = id$, $k = 0$

X - vector field.

$$X^\mu(\omega) = X \lrcorner \omega^\mu$$

$$X^\mu(a\omega) = X \lrcorner (a^\mu{}_\nu \omega^\nu) = a^\mu{}_\nu (X \lrcorner \omega^\nu) = a^\mu{}_\nu X^\nu(\omega)$$

$\Rightarrow$ (components of vector-fields) $\rightsquigarrow$ (0-forms of type id)

3) tensors $\rightsquigarrow$ 0-forms of type $\rho^r{}_s$.

4) scalar forms $\equiv$ forms

$W = \mathbb{R}^1$, $\rho(a) = 1$.

Differentiation of forms of type ρ.

$X^\mu(\omega)$. $\rightsquigarrow$ $dX^\mu(\omega)$

what object is this?

$$dX^\mu(a\omega) = d(a^\mu{}_r X^r(\omega)) =$$

$$= a^\mu{}_r \, d(X^r(\omega)) + \underbrace{da^\mu{}_r \, X^r(\omega)}$$

this term makes $dX^\mu(a\omega)$ an object beyond the class of forms of types ρ.

In order to define differentiation that transforms objects of type ρ into objects of type ρ one introduces

$$\Gamma^\mu{}_\nu(\omega).$$

We want that

$$dX^\mu(a\omega) + \underbrace{\Gamma^\mu{}_\nu(a\omega)}_{\substack{\text{matrix-valued}\\ 1\text{-form}}} X^\nu(a\omega) = a^\mu{}_\nu\left(dX^\nu(\omega) + \Gamma^\nu{}_\rho(\omega)X^\rho(\omega)\right)$$

$$\|$$

$$da^\mu{}_\nu X^\nu(\omega) + a^\mu{}_\nu dX^\nu(\omega) + \Gamma^\mu{}_\nu(a\omega)a^\nu{}_\rho X^\rho(\omega)$$

$$\Gamma(a\omega)a + da = a\Gamma(\omega)$$

$$\boxed{\Gamma(a\omega) = a\Gamma(\omega)a^{-1} - da\,a^{-1}} \quad (\mathrm{TR})$$

Affine connection on M is an assigment

$$\Gamma : \omega \longrightarrow \Gamma(\omega) \in \mathrm{End}(\mathbb{R}^n) \otimes \Lambda^1 M$$

in such a way that if $\omega \to a\omega$ then (TR) for Γ.

$$(DX^\mu)(\omega) = dX^\mu(\omega) + \Gamma^\mu{}_\nu(\omega)X^\nu(\omega)$$

$$\boxed{DX^\mu = dX^\mu + \Gamma^\mu{}_\nu X^\nu} \leftarrow \text{covariant differential.}$$

Two definitions of connections

(M, ∇) $\quad \nabla: \mathfrak{X}(M) \times \mathcal{T}(M) \to \mathcal{T}(M)$

$\quad (X, K) \longmapsto \nabla_X K$

- ∇_X - $\mathbb{R}$ linear in K and preserves type
- $\nabla_{fX+gY} K = f \nabla_X K + g \nabla_Y K$
- $\nabla_X (K \otimes L) = \nabla_X K \otimes L + K \otimes \nabla_X L$
- commutes with contractions
- $\nabla_X (f) = X(f)$.

Second definition in terms of charts.

In particular

$$\nabla_{X_\mu} X_\nu = \Gamma^{\rho}{}_{\nu\mu}(\omega) X_\rho$$

connection coefficients

Connections and parallelism:

Proposition

$$\left(\begin{array}{l} M\text{-manifold with } \nabla \\ \gamma : I \longrightarrow M \text{ differentiable curve} \\ V\text{- vector field along } \gamma \end{array}\right) \Longrightarrow$$

$\exists! \frac{DV}{dt}$ another vector field along γ with the following properties:

$$1^\circ \quad \frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$$

$$2^\circ \quad \frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$$

3° If V is induced by a vector field $\tilde{V}$ on M

then $$\frac{DV}{dt} = \nabla_{\frac{d\gamma}{dt}} \tilde{V}$$

Proof

First let us assume that such $\frac{D}{dt}$ exists.

$$(\mathcal{U}, x), \quad X_\mu = \frac{\partial}{\partial x^\mu} \implies V = V^\mu X_\mu$$

$$\frac{DV}{dt} = \frac{D}{dt}(V^\mu X_\mu) = \frac{dV^\mu}{dt} X_\mu + V^\mu \frac{DX_\mu}{dt} =$$

$$= \frac{dV^\mu}{dt} X_\mu + V^\mu \nabla_{\frac{d\gamma}{dt}} X_\mu = \frac{dV^\mu}{dt} + V^\mu \frac{dx^\alpha}{dt} \nabla_{X_\alpha} X_\mu =$$

$$= \frac{dV^\mu}{dt} X_\mu + V^\mu \frac{dx^\alpha}{dt} \Gamma^\beta_{\mu\alpha} X_\beta =$$

$$= \underbrace{\left(\frac{dV^\beta}{dt} + \Gamma^\beta_{\mu\alpha} V^\mu \frac{dx^\alpha}{dt}\right)}_{\text{this are local components of } \frac{DV}{dt}.} X_\beta \qquad (*)$$

$\boxed{\text{So if } \frac{DV}{dt} \text{ exists it is unique}}$

Now <u>locally</u> we define $\frac{DV}{dt}$ by $(*)$.

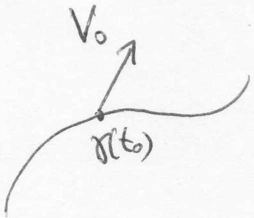
If we take another $(\mathcal{U}', x')$ and define $\frac{DV}{dt}$ by $(*)$. The def. agree on the overlap by uniqueness.

Definition

(M,∇). Vector field V along $\gamma: I \to M$ is called parallel if $\frac{DV}{dt} = 0$, $\forall t \in I$.

Proposition

(M,∇). $\gamma: I \to M$ and let V_0 be a vector tangent to M at $\gamma(t_0)$, $t_0 \in I$.



Then there exists a unique vector field V which is parallel along γ ~~and~~ such that

$$V(t_0) = V_0.$$

Proof

In one coordinate chart ~~along~~ around $C(t_0)$ $V(t)$ satisfies

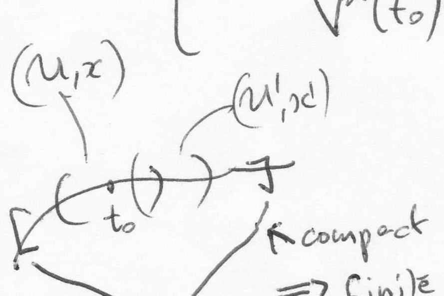
$$0 = \frac{DV}{dt} = \left[\frac{dV^\mu}{dt} + \Gamma^\mu{}_{\nu\rho} V^\nu \frac{dx^\rho}{dt}\right] X_\mu$$

Thus coordinates of V should satisfy

$$\begin{cases} \frac{dV^\mu}{dt} + \Gamma^\mu{}_{\nu\rho} V^\nu \frac{dx^\rho}{dt} = 0 \\ V^\mu(t_0) = V^\mu_0 \end{cases}$$

linear differential equation for V^μ with initial cond $V^\mu(t_0) = V^\mu_0$.

has a unique solution for all t



$\Rightarrow$ finite number of $(\mathcal{U},x)$ is enough. $\square$.

Second definition

$$\nabla \rightsquigarrow \Gamma^{\mu}{}_{\nu\rho} \rightsquigarrow \Gamma^{\mu}{}_{\nu}(\omega)$$

$\mathcal{U}$ and $\mathcal{U}'$ two open sets s.t. $\mathcal{U}\cap\mathcal{U}' \neq \emptyset$.

ω^{μ} coframe in $\mathcal{U}$

ω'^{μ} coframe in $\mathcal{U}'$

We define connection in $\mathcal{U}$ by connection 1-forms $\Gamma^{\mu}{}_{\nu}$ in $\mathcal{U}$.

Connection 1-forms $\Gamma'^{\mu}{}_{\nu}(\omega)$ ~~define~~ in $\mathcal{U}'$ define the same connection in $\mathcal{U}\cup\mathcal{U}'$ iff there exists a function $a : \mathcal{U}\cap\mathcal{U}' \to GL(n,\mathbb{R})$ s.t.

$$\Gamma'^{\mu}{}_{\nu}(\omega) = a^{\mu}{}_{\alpha}\,\Gamma^{\alpha}{}_{\beta}(\omega)\,a^{-1\beta}{}_{\nu} + da^{\mu}{}_{\alpha}\,a^{-1\alpha}{}_{\nu}$$

<u>k-form of type ρ</u> : $\rho : GL(n,\mathbb{R}) \xrightarrow{\text{hom.}} GL(N,\mathbb{R})$, $W=\mathbb{R}^N$.

$$\overset{k}{\alpha} : \omega \longmapsto \alpha^{k}(\omega) \in W \otimes \Lambda^{k} M$$

$$\overset{k}{\alpha}(a\omega) = \rho(a)\,\overset{k}{\alpha}(\omega).$$

$d\overset{k}{\alpha}(a\omega) \neq \rho(a)\, d\overset{k}{\alpha}(\omega)$ (except for scalar forms $\rho(a)=1$, $N=1$)

D extension of d s.t.

$$D\overset{k}{\alpha}(a\omega) = \rho(a)\, d\overset{k}{\alpha}(\omega).$$

ex

$DX^{\mu} = dX^{\mu} + \Gamma^{\mu}{}_{\nu} \wedge X^{\nu}$ $\quad X^{\mu}$ - k-form of type id // $\rho(a)=a$

$D\Omega^{\mu}{}_{\nu} = d\Omega^{\mu}{}_{\nu} + \Gamma^{\mu}{}_{\alpha}\wedge\Omega^{\alpha}{}_{\nu} - \Gamma^{\alpha}{}_{\nu}\wedge\Omega^{\mu}{}_{\alpha}$ $\quad \Omega^{\mu}{}_{\nu}$ k-form of type Ad $\quad [\mathrm{Ad}(a)\Omega]^{\mu}{}_{\nu} = a^{\mu}{}_{\alpha}\Omega^{\alpha}{}_{\beta}a^{-1\beta}{}_{\nu}$

⋮

e.t.c.

Check: $\Omega^{\mu}{}_{\nu}(a\omega) = a^{\mu}{}_{\alpha}\,\Omega^{\alpha}{}_{\beta}(\omega)\,a^{-1\beta}{}_{\nu}$

$\Rightarrow D\Omega^{\mu}{}_{\nu}(a\omega) = a^{\mu}{}_{\alpha}\,D\Omega^{\alpha}{}_{\beta}(\omega)\,a^{-1\beta}{}_{\nu}$, because $\Gamma'(a\omega) = a\Gamma(\omega)a^{-1} + da\,a^{-1}$.

$\rho : GL(n,\mathbb{R}) \xrightarrow{\text{homo}} GL(N,\mathbb{R})$

$\rho' : \operatorname{End}\mathbb{R}^n \xrightarrow{\text{homo}} \operatorname{End}\mathbb{R}^N$

$A \in \operatorname{End}\mathbb{R}^n \Rightarrow a = \exp(tA) \in GL(n,\mathbb{R})$

$$\rho'(A) = \frac{d}{dt}\rho(\exp(tA))\Big|_{t=0}. \in \operatorname{End}(\mathbb{R}^N)$$

$\mathbb{R}^n \ni v^\mu \quad \mu = 1, \dots n \qquad a = (a^\mu{}_\nu) \in GL(n,\mathbb{R})$

$\mathbb{R}^N \ni V^A \quad A = 1, \dots N \qquad \rho(a)^A{}_B \in GL(N,\mathbb{R})$

$$\frac{d}{dt}\rho(\exp(tA))^A{}_B\Big|_{t=0} = \frac{\partial \rho^A{}_B}{\partial a^\mu{}_\nu}\Big|_{a=1} A^\mu{}_\nu$$

$$= \rho^A{}_B{}^\nu{}_\mu A^\mu{}_\nu =: \rho'(A)^A{}_B$$

$$\boxed{\rho'(A)^A{}_B = \rho^A{}_B{}^\nu{}_\mu A^\mu{}_\nu, \quad \rho^A{}_B{}^\nu{}_\mu = \frac{\partial \rho^A{}_B}{\partial a^\mu{}_\nu}\Big|_{a=1}}$$

α - k-form of type ρ

$\Rightarrow$

$$D\alpha = d\alpha + \rho'(\Gamma) \wedge \alpha$$

$$\boxed{(D\alpha)^A(\omega) = d\alpha^A(\omega) + \rho^A{}_B{}^\nu{}_\mu \Gamma^\mu{}_\nu(\omega) \wedge \alpha^B(\omega)}$$

Affine connection

$\Gamma(\omega)$ s.t. $\quad \Gamma(a\omega) = a\Gamma(\omega)a^{-1} + a\,da^{-1}$

note that $\quad \underbrace{d(aa^{-1})}_{=0} = da\,a^{-1} + a\,da^{-1}$

$$\Rightarrow \quad \Gamma(a\omega) = a\Gamma(\omega)a^{-1} - da\,a^{-1}$$

Prop

At every point $p \in M$ there exists ω s.t. $\Gamma(\omega)_p = 0$.

Proof

If $\Gamma(\omega)_p \neq 0$. then

$$\underbrace{\Gamma(a\omega)_p}_{=0} = a(p)\,\Gamma(\omega)_p\,a^{-1}(p) - (da)_p\,a^{-1}(p)$$

$$\Rightarrow a(p)\Gamma(\omega)_p = (da)_p.$$

Thus it is enough to take $a : U \to GL(n, \mathbb{R})$

s.t. $\quad a(p) = 1$

$\quad (da)_p = \Gamma(\omega)_p.$

$$\Rightarrow \quad \Gamma(a\omega)_p = 0. \quad \square$$

When we can gauge Γ to zero in a neighbourhood?

$$\underset{\overset{\|}{0}}{\Gamma(a\omega)} = a\Gamma(\omega)a^{-1} - da\, a^{-1}$$

$$\Leftrightarrow \quad da = a\Gamma(\omega).$$

$$\Rightarrow 0 = d^2 a = da \wedge \Gamma(\omega) + a\, d\Gamma(\omega) =$$
$$= a\left(\Gamma(\omega)\wedge\Gamma(\omega) + d\Gamma(\omega)\right)$$

$\Gamma(a\omega) = 0$ only if $\underline{\underline{\Omega(\omega) = d\Gamma(\omega) + \Gamma(\omega)\wedge\Gamma(\omega) = 0}}$

if these equations are satisfied then by Cauchy-Kowalewski

$da = a\Gamma(\omega)$ has a unique solution.

<u>Fact</u>

Check that

$$\Omega^\mu{}_\nu(\omega) = d\Gamma^\mu{}_\nu(\omega) + \Gamma^\mu{}_\rho(\omega)\wedge\Gamma^\rho{}_\nu(\omega)$$

is a 2-form of type Ad

$$\Omega^\mu{}_\nu(a\omega) = a^\mu{}_\alpha\, \Omega^\alpha{}_\beta(\omega)\, a^{-1\beta}{}_\nu\ .$$

<u>Curvature 2-form</u> !

Another cannonical form:

$$\theta^\mu(\omega) := \omega^\mu \qquad \text{— cannonical 1-form of type id.}$$

$$D\theta^\mu = d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu = \Theta^\mu$$

torsion 2-form.

$$\Theta^\mu(\omega) = d\omega^\mu + \Gamma^\mu{}_\nu(\omega)\wedge\omega^\nu.$$

If $d\omega^\mu|_p = 0$ and $\Gamma^\mu{}_\nu(\omega)_p = 0 \Rightarrow \Theta^\mu(\omega)_p = 0$

$$\Theta^\mu(\omega)_p = 0$$
$$\Downarrow$$
$$\omega' \text{ s.t. } \Gamma^\mu{}_\nu(\omega')_p = 0$$
$$\Downarrow$$
$$d\omega'^\mu|_p = 0$$

$$\boxed{\Theta(\omega)_p = 0 \Leftrightarrow \exists\,\omega \text{ s.t. } d\omega^\mu|_p = 0 \text{ and } \Gamma^\mu{}_\nu(\omega)_p = 0}$$

Ricci formula

$$\Gamma(a\omega) = a\Gamma(\omega)a^{-1} - da\,a^{-1}$$

α – k-form of type ρ. $\Rightarrow D\alpha = d\alpha + \rho'(\Gamma)\wedge\alpha$;

in addition at every point $p\in M$ we can find ω s.t. $\Gamma(\omega)_p = 0$. This is very useful during calclulations!

Fact

Let β be an ℓ-form of type σ on M (e.g. $\beta = D\alpha$)

$$\left(\Gamma(\omega)_p = 0 \text{ and } \beta(\omega)_p = 0\right) \Rightarrow \beta(a\omega)_p = 0$$

(because $\beta(a\omega) = \sigma(a)\beta(\omega)$).

Example:

$\underset{k_1,\ \rho_1}{\alpha_1}$; $\underset{k_2,\ \rho_2}{\alpha_2}$ $\Rightarrow$ $\underset{k_1+k_2,\ \rho_1\otimes\rho_2}{\alpha_1\wedge\alpha_2}$

$\Downarrow$

$$D(\alpha_1\wedge\alpha_2) = D\alpha_1\wedge\alpha_2 + (-1)^{k_1}\alpha_1\wedge D\alpha_2$$

because in a frame in which $\Gamma(\omega)_p = 0$ we have

$$\left[D(\alpha_1\wedge\alpha_2) - \left(D\alpha_1\wedge\alpha_2 + (-1)^{k_1}\alpha_1\wedge D\alpha_2\right)\right]_p \overset{\Gamma(\omega)_p=0}{=} d(\alpha_1\wedge\alpha_2)_p + {}- \left(d\alpha_1\wedge\alpha_2 + (-1)^{k_1}\alpha_1\wedge d\alpha_2\right)_p = 0.$$

Better example: Ricci formula

$$\boxed{D^2\alpha = \rho'(\Omega)\wedge\alpha}$$

$$\begin{aligned} D\left[d\alpha + \rho'(\Gamma)\wedge\alpha\right] &= \rho'(d\Gamma)\wedge\alpha + \text{terms linear in } \Gamma \\ &= \rho'(\Omega)\wedge\alpha + \text{terms linear in } \Gamma \\ &\underset{\Gamma(\omega)_p=0}{=} \rho'(\Omega)\wedge\alpha \quad \text{in this frame;} \end{aligned}$$

hence, since

$D^2\alpha - \rho'(\Omega)\wedge\alpha$ is $k+2$ form of type ρ

and $(D^2\alpha - \rho'(\Omega)\wedge\alpha)_p = 0$ in this frame

$\Rightarrow$ in every frame!

Bianchi identities

II: $D\Omega^\mu{}_\nu = d\Omega^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Omega^\rho{}_\nu - \Gamma^\rho{}_\nu \wedge \Omega^\mu{}_\rho =$

$$= d\left(d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu\right) + \text{terms linear in } \Gamma =$$

$$= d^2\Gamma^\mu{}_\nu + \text{terms linear in } \Gamma = 0 + \text{terms linear in } \Gamma$$

$$\underset{\Gamma(\omega)_p = 0}{\overset{\uparrow}{=}} 0 \quad \Rightarrow \quad \boxed{D\Omega^\mu{}_\nu = 0} \qquad \text{II}^{nd} \text{ B.I.}$$

I: $D\Theta^\mu = D^2\theta^\mu \underset{\text{Ricci formula}}{\overset{\uparrow}{=}} \Omega^\mu{}_\nu \wedge \theta^\nu$

$$\boxed{D\Theta^\mu = \Omega^\mu{}_\nu \wedge \theta^\nu} \qquad \text{I}^{st} \text{ B.I.}$$

If α is a 0-form of type ρ:

$$D\alpha^A(\omega) = \omega^\mu \nabla_{X_\mu} \alpha^A$$

in other words:

$$\nabla_{X_\mu} \alpha^A = X_\mu \lrcorner\, D\alpha^A(\omega)$$

$$\boxed{\nabla_X \alpha^A = X \lrcorner\, D\alpha^A(\omega)}$$

Exercise: calculate

$$\nabla_{[\mu} \nabla_{\nu]} \alpha^A = ?$$

How torsion and curvature look in terms of ∇.?

$\Theta^\mu = d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu$; by Maurer-Cartan:

$$d\theta^\mu = -\tfrac{1}{2} C^\mu{}_{\nu\rho}\, \theta^\nu \wedge \theta^\rho$$

$$\boxed{X_\alpha \lrcorner X_\beta \lrcorner \Theta^\mu} = X_\alpha \lrcorner X_\beta \lrcorner \left(-\tfrac{1}{2} C^\mu{}_{\nu\rho}\, \theta^\nu \wedge \theta^\rho + \Gamma^\mu{}_\nu \wedge \theta^\nu\right) =$$

$$= \boxed{-C^\mu{}_{\beta\alpha} + \Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha}} \quad (1)$$

$X_\alpha \lrcorner \theta^\mu = \delta_\alpha{}^\mu$

Observe that

$$\boxed{X_\alpha \lrcorner \theta^\mu(\omega) = X_\alpha \lrcorner \omega^\mu = \delta_\alpha{}^\mu}$$

introduce

$\delta_\alpha{}^\mu(\omega) = \delta_\alpha{}^\mu$

scalar 0-form

Define

$T : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ by:

$$\boxed{T(Y,Z) = \nabla_Y Z - \nabla_Z Y - [Y,Z].}$$

Properties

1) $T(Y,Z) = -\ (Z,Y)$ ✓

2) T is f-linear: e.g.

$$T(fY,Z) = f\nabla_Y Z - \nabla_Z(fY) - [fY,Z] =$$

$$= f\nabla_Y Z - \cancel{Z(f)Y} - f\nabla_Z Y - f[Y,Z] + \cancel{Z(f)Y}$$ ✓

$$T(Y,Z) = T^\mu(Y,Z)\, X_\mu$$

$$\boxed{X_\alpha \lrcorner X_\beta \lrcorner T^\mu} = T^\mu(X_\beta, X_\alpha) = \left(\nabla_{X_\beta} X_\alpha - \nabla_{X_\alpha} X_\beta - [X_\beta, X_\alpha]\right)^\mu =$$

$$= \boxed{\Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha} - C^\mu{}_{\beta\alpha}} \quad (2)$$

$$\Longrightarrow T^\mu = \Theta^\mu$$

or $\boxed{T = \Theta^\mu X_\mu}$

In a similar way:

$$\Omega^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu$$ – curvature

Define:

$$R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

$$\boxed{R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z}$$

Properties

1) $R(X,Y) = -R(Y,X)$

2) R is f-linear in each argument

$$R(X,Y)Z = R^\alpha{}_\beta(X,Y)\,\theta^\beta(Z)\,X_\alpha$$

and $\mathrm{End}(\mathbb{R}^n)$-valued 2-forms $R^\alpha{}_\beta$ coincide with $\Omega^\alpha{}_\beta$

$$R^\alpha{}_\beta = \Omega^\alpha{}_\beta$$

$$\Longrightarrow \quad \boxed{R = \Omega^\alpha{}_\beta\, \theta^\beta \otimes X_\alpha}$$

or $$\boxed{R(\cdot,\cdot)X_\mu = \Omega^\alpha{}_\mu X_\alpha}$$

Manifolds with affine connection ∇

$\boxed{\text{Two languages:}}$

$(M, \nabla, + \text{axioms})$ | $M, \Gamma(\omega), \Gamma(a\omega) = a\Gamma(\omega)a^{-1} - da\,a^{-1})$

$$\left.\begin{aligned} \Gamma^\mu{}_\nu(\omega) &= \Gamma^\mu{}_{\nu\rho}\,\omega^\rho \\ \nabla_{X_\mu}\omega^\nu &= -\Gamma^\nu{}_{\rho\mu}\,\omega^\rho \end{aligned}\right\} \text{ in LOCAL FRAME } X_\mu \longleftrightarrow \omega^\mu$$

Torsion:

$T \in \mathfrak{X}(M)^1_2$

Cannonical 1-form of type id

$\theta^\mu(\omega) = \omega^\mu$

$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ | $\Theta^\mu = D\theta^\mu = d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu$

$$T(X,Y) = \Theta^\mu(X,Y)X_\mu$$

Curvature:

$R \in \mathfrak{X}(M)^1_3$:

$\Omega = d\Gamma + \Gamma \wedge \Gamma$

$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$

$$R(X,Y)Z = \Omega^\alpha{}_\beta(X,Y)\theta^\beta(Z)X_\alpha$$

Bianchi identity

$D\Theta^\mu = \Omega^\mu{}_\nu \wedge \theta^\nu$ $\quad \text{I}^{st}$

$D\Omega^\mu{}_\nu = 0$ $\quad \text{II}^{nd}$

If $T = 0$ then

II^{nd} is $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$

I^{st} is $(\nabla_X R)(Y,Z) + (\nabla_Z R)(X,Y) + (\nabla_Y R)(Z,X) = 0$

<u>Check</u>

0-form of type $\binom{1}{3}$

$$X_\alpha \lrcorner X_\beta \lrcorner \Omega^\mu{}_\nu(\omega) = X_\alpha \lrcorner X_\beta \lrcorner \left(d\Gamma^\mu{}_\nu(\omega) + \Gamma^\mu{}_\rho(\omega)\wedge\Gamma^\rho{}_\nu(\omega)\right) =$$

$$= X_\alpha \lrcorner X_\beta \lrcorner \left(d\Gamma^\mu{}_{\nu\rho}\,\omega^\rho + \Gamma^\mu{}_{\nu\rho}\, d\omega^\rho + \dots \right) =$$

$$= X_\alpha \lrcorner \left(X_\beta(\Gamma^\mu{}_{\nu\rho})\,\omega^\rho - d\Gamma^\mu{}_{\nu\beta} + \dots \right) =$$

$$= \underbrace{X_\beta(\Gamma^\mu{}_{\nu\alpha}) - X_\alpha(\Gamma^\mu{}_{\nu\beta}) + \dots}$$

can be written as:

0-form of type $\binom{1}{3}$

$$R(X_\beta, X_\alpha)X_\nu \overset{?}{=} \left(X_\alpha \lrcorner X_\beta \lrcorner \tilde{\Omega}^\mu{}_\nu(\omega) \right) X_\mu$$

$$\|$$

$$\nabla_{X_\beta}\nabla_{X_\alpha}X_\nu - \nabla_{X_\alpha}\nabla_{X_\beta}X_\nu - \nabla_{[X_\beta, X_\alpha]}X_\nu =$$

$$= \nabla_{X_\beta}\left(\Gamma^\rho{}_{\nu\alpha}X_\rho\right) - \nabla_{X_\alpha}\left(\Gamma^\rho{}_{\nu\beta}X_\rho\right) - C^\rho{}_{\beta\alpha}\nabla_{X_\rho}X_\nu =$$

$$= X_\beta(\Gamma^\rho{}_{\nu\alpha})X_\rho - X_\alpha(\Gamma^\rho{}_{\nu\beta})X_\rho - C^\rho{}_{\beta\alpha}\Gamma^\sigma{}_{\nu\rho}X_\sigma + \dots =$$

$$= \left(\underbrace{X_\beta(\Gamma^\mu{}_{\nu\alpha}) - X_\alpha(\Gamma^\mu{}_{\nu\beta}) + \dots} \right) X_\mu$$

✓

$$\Rightarrow \quad \Omega^\mu{}_\nu = \tilde{\Omega}^\mu{}_\nu$$

$$\Theta^\mu = \frac{1}{2} Q^\mu{}_{\nu\rho}\, \theta^\nu \wedge \theta^\rho$$

$$\Omega^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\sigma}\, \theta^\rho \wedge \theta^\sigma$$

$Q^\mu{}_{\nu\rho}$ - 0-form of type $\binom{1}{2}$

$R^\mu{}_{\nu\rho\sigma}$ - 0-form of type $\binom{1}{3}$

In general:

α^A – 0-form of type ρ:

$$\Rightarrow \quad \boxed{D\alpha^A = \nabla_\mu \alpha^A\, \theta^\mu} \;; \quad \nabla_\mu \alpha^A = \nabla_{X_\mu} \alpha^A.$$

$$D\Theta^\mu = \frac{1}{2} DQ^\mu{}_{\nu\rho} \wedge \theta^\nu \wedge \theta^\rho + \frac{1}{2} Q^\mu{}_{\nu\rho}\, D\theta^\nu \wedge \theta^\rho +$$
$$- \frac{1}{2} Q^\mu{}_{\nu\rho}\, \theta^\nu \wedge D\theta^\rho =$$

$$= \frac{1}{2} \nabla_\alpha Q^\mu{}_{\nu\rho}\, \theta^\alpha \wedge \theta^\nu \wedge \theta^\rho + \frac{1}{2} Q^\mu{}_{\nu\rho} \frac{1}{2} Q^\nu{}_{\alpha\beta}\, \theta^\alpha \wedge \theta^\beta \wedge \theta^\rho +$$
$$- \frac{1}{2} Q^\mu{}_{\nu\rho}\, \theta^\nu \wedge \frac{1}{2} Q^\rho{}_{\alpha\beta}\, \theta^\alpha \wedge \theta^\beta =$$

$$= \frac{1}{2} \left[\nabla_\alpha Q^\mu{}_{\nu\rho} + \frac{1}{2} Q^\mu{}_{\beta\rho} Q^\beta{}_{\alpha\nu} + \frac{1}{2} Q^\mu{}_{\nu\beta} Q^\beta{}_{\alpha\rho} \right] \theta^\alpha \wedge \theta^\nu \wedge \theta^\rho$$
$$- \frac{1}{2} Q^\mu{}_{\beta\nu} Q^\beta{}_{\alpha\rho}$$

$$\Omega^\mu{}_\nu \wedge \theta^\nu = \frac{1}{2} R^\mu{}_{\nu\alpha\rho}\, \theta^\alpha \wedge \theta^\rho \wedge \theta^\nu = -\frac{1}{2} R^\mu{}_{\nu\alpha\rho}\, \theta^\alpha \wedge \theta^\nu \wedge \theta^\rho$$

$$\boxed{\nabla_{[\alpha} Q^\mu{}_{\nu\rho]} + \frac{1}{2} Q^\mu{}_{\beta[\rho} Q^\beta{}_{\alpha\nu]} - \frac{1}{2} Q^\mu{}_{\beta[\nu} Q^\beta{}_{\alpha\rho]} = -R^\mu{}_{[\nu\alpha\rho]}}$$

If $\Theta^\mu = 0 \iff \boxed{R^\mu{}_{[\nu\alpha\rho]} = 0}$

$$D\Omega^\mu{}_\nu = \frac{1}{2} D\left(R^\mu{}_{\nu\rho\sigma}\,\theta^\rho \wedge \theta^\sigma\right) =$$

$$= \nabla_\alpha R^\mu{}_{\nu\rho\sigma}\,\theta^\alpha \wedge \theta^\rho \wedge \theta^\sigma + \ldots$$

If $\Theta^\mu = 0$

$$D\Omega^\mu{}_\nu = 0 \Longleftrightarrow \boxed{R^\mu{}_{\nu[\rho\sigma;\alpha]} = 0}$$

<u>Thm</u>

If $\Theta^\mu = 0$ $(T \equiv 0)$ then

$$\begin{cases} R^\mu{}_{[\nu\alpha\rho]} = 0 & \text{II}^{\text{nd}} \text{ Bianchi} \\ R^\mu{}_{\nu[\rho\sigma;\alpha]} = 0 & \text{I}^{\text{st}} \text{ Bianchi} \end{cases}$$

Assume that $T = 0$ or $\Theta^\mu = 0$.

$$\Longrightarrow \boxed{\Omega^\mu{}_\nu \wedge \theta^\nu = 0}$$

Let's check:

$$R(V,Y)Z + R(Z,V)Y + R(Y,Z)V =$$

$$= \Omega^\alpha{}_\beta(V,Y)\theta^\beta(Z)X_\alpha + \Omega^\alpha{}_\beta(Z,V)\theta^\beta(Y)X_\alpha + \Omega^\alpha{}_\beta(Y,Z)\theta^\beta(V)X_\alpha =$$

$$= \left(R^\alpha{}_{\beta\mu\nu}\underline{V^\mu Y^\nu Z^\beta} + R^\alpha{}_{\beta\mu\nu} Z^\mu V^\nu Y^\beta + R^\alpha{}_{\beta\mu\nu} Y^\mu Z^\nu V^\beta\right) X_\alpha =$$

$$= \left[R^\alpha{}_{\beta\nu\mu} + R^\alpha{}_{\nu\beta\mu} + R^\alpha{}_{\mu\nu\beta}\right] V^\mu Y^\nu Z^\beta X_\alpha$$

$$= 2\, R^\alpha{}_{[\beta\nu\mu]} V^\mu Y^\nu Z^\beta X_\alpha = 0$$

$$\nabla_X R = X^\mu \left(\nabla_\mu R^\alpha{}_{\beta\gamma\delta}\right) X_\alpha \otimes \omega^\beta \otimes \omega^\gamma \otimes \omega^\delta$$

$$\left(\nabla_X R\right)(Y,Z) = X^\mu \, \nabla_\mu R^\alpha{}_{\beta\gamma\delta} Y^\gamma Z^\delta \; X_\alpha \otimes \omega^\beta =$$

$$= X^\mu Y^\gamma Z^\delta \, \nabla_\mu R^\alpha{}_{\beta\gamma\delta} \; X_\alpha \otimes \omega^\beta$$

$$\left(\nabla_Z R\right)(X,Y) = X^\mu Y^\gamma Z^\delta \, \nabla_\delta R^\alpha{}_{\beta\mu\gamma} \; X_\alpha \otimes \omega^\beta$$

$$\left(\nabla_Y R\right)(Z,X) = X^\mu Y^\gamma Z^\delta \, \nabla_\gamma R^\alpha{}_{\beta\delta\mu} \; X_\alpha \otimes \omega^\beta$$

$$\Rightarrow \left(\nabla_X R\right)(Y,Z) + \left(\nabla_Z R\right)(X,Y) + \left(\nabla_Y R\right)(Z,X) =$$

$$= X^\mu Y^\gamma Z^\delta \left[\underbrace{\nabla_\mu R^\alpha{}_{\beta\gamma\delta} + \nabla_\delta R^\alpha{}_{\beta\mu\gamma} + \nabla_\gamma R^\alpha{}_{\beta\delta\mu}}_{= \, 0}\right] X_\alpha \otimes \omega^\beta$$

Riemannian manifolds

(M,g) : M - n-dimensional manifold equipped with a tensor field $g \in \mathfrak{X}^0_2(M)$ s.t.

$$\left.\begin{array}{ll} 1^\circ & g \text{ is symmetric } g(X,Y)=g(Y,X) \\ 2^\circ & g(X,X)=0 \iff X=0 \\ 3^\circ & g(X,X)\geq 0 \end{array}\right\} \forall X,Y\in\mathfrak{X}(M)$$

is called <u>Riemannian manifold</u>

<u>Rmk 1</u> In physics a weaker structure is usually used for which 2° is replaced by

$$2^{\circ\prime} \quad \big(g(X,Y)=0 \quad \forall Y\in\mathfrak{X}(M)\big) \Rightarrow X=0$$

and 3° is abandoned.

(M,g) with $1^\circ, 2^{\circ\prime}$ is called <u>pseudo-Riemannian manifold</u>

<u>Rmk 2</u> of course 2° implies $2^{\circ\prime}$ since if $2^{\circ\prime}$ was not satisfied we would have $X\neq 0$ s.t. $g(X,Y)=0 \quad \forall Y\in\mathfrak{X}(M)$. In particular $g(X,X)=0$ with $X\neq 0$, contradicting 2°.

<u>Rmk 3</u> Locally $g, (X_\mu)$: $g(X_\mu,X_\nu)=g_{\mu\nu}$ - functions in U s.t. $g_{\mu\nu}=g_{\nu\mu}$. At every point $p\in U$ by a $GL(n,\mathbb{R})$ transformation can be brought to $g_{\mu\nu}=(\underbrace{1,\dots,1}_{p},\underbrace{-1,\dots,-1}_{q})$

$\boxed{\text{In Riemannian case } q=0}$ (p,q) is called signature of g at p.

Can not change from point to point without violation of continuity.

Definition

differentiable $\equiv$ class C^∞
M^n - manifold of dim n.

1) (M,g), (M',g') two (pseudo) Riemannian manifolds

$$\phi: M \xrightarrow{\text{diffeo}} M'$$

is called <u>isometry</u> iff

$$g_p(X_p, Y_p) = g'_{\phi(p)}(\phi_{*p} X_p, \phi_{*p} Y_p) \quad \forall X_p, Y_p \in T_p M \quad \forall p \in M$$

2) $\phi: M \to M'$ is called a <u>local isometry</u> at $p \in M$ if there exists $U \subset M$ of p s.t.

neighbourhood $\phi: U \to \phi(U)$ is an isometry between (U, g) and $(\phi(U), g')$.

3) (M,g) is <u>locally isometric</u> to (M',g') if for every $p \in M$ there exists U of p and a local isometry $\phi: U \to \phi(U) \subset M'$.

Examples

1) $M = \mathbb{R}^n$. with cartesian coordinates $(x^1, \ldots, x^n)$

$$\Rightarrow \quad g = dx^{1^2} + \ldots dx^{n^2}$$

$$dx^{\mu^2} = dx^\mu \otimes dx^\mu \quad \left(\begin{array}{l} \text{notation} \\ dx^\mu dx^\nu = \frac{1}{2}\, dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu \end{array} \right)$$

$(\mathbb{R}^n, g)$ – Euclidean space of dimension n

2) Immersed manifolds

$$\phi : M^n \xrightarrow[\text{immersion}]{} N^{n+k} \qquad \left(\begin{array}{l} \phi\text{-differentiable} + \\ \phi_{*p} \text{ injective } \forall p \in M^n \end{array} \right)$$

If N^{n+k} is Riemannian with metric g' we define g by:

$$g_p(X_p, Y_p) = g'_{\phi(p)}(\phi_{*p} X_p, \phi_{*p} Y_p) \qquad \begin{array}{l} \forall p \in M^n \\ \forall X_p, Y_p \in T_p M^n \end{array}$$

Since ϕ is immersion g is positive definite.

In particular if $M \subset (N, g')$ is a submanifold we have

$$\imath : M \xrightarrow[\text{embedding}]{} N$$

$$\Rightarrow \quad g = \imath^* g' \quad \text{is a Riemannian metric on } M$$

$$S^n = h^{-1}(0), \quad h : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$$

$$h(x^\mu) = x^{1^2} + \ldots + x^{n+1^2} - 1$$

is a submanifold of $\mathbb{R}^{n+1}$.

Pullback of Euclidean metric g from $\mathbb{R}^{n+1}$ to S^n is a canonical metric on S^n.

Lie Groups

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G – a group which is a manifold s.t. the map

$$G \times G \ni (a,b) \longmapsto ab^{-1} \in G \quad \text{is differentiable}$$

is a Lie Group

Left translations: $\forall a \in G : L_a : G \to G$

$$L_a(b) = a \cdot b$$

Right translations: $R_a : G \to G$

$$R_a(b) = b \cdot a$$

are diffeomorphisms

Def A Riemannian metric g on G is left invariant iff

$$g_b(X_b, Y_b) = g_{ab}(L_{a*_b} X_b, L_{a*_b} Y_b)$$

$\forall a, b \in G$
$\forall X_b, Y_b \in T_b G$

(similarly right invariant)

g is biinvariant iff it is left and right invariant.

Def

A vector field X on G is left invariant iff

$$\forall a \in G \quad L_{a*} X = X \qquad (L_{a*_b} X_b = X_{ab} \quad \forall a, b \in G)$$

(note that since L_a is diffeomorphism we can pushforward vector fields!)

Left invariant vector fields are completely determined by their values at e – identity element in G.

This enables to introduce additional structure in T_eG.

Take any vector $X_e \in T_eG$.

Define a left invariant vector field X by

$$X_a = L_{a*_e} X_e$$

Taking another $Y_e \in T_eG$ we have also Y s.t.

$$Y_a = L_{a*_e} X_e$$

We equipp T_eG with a structure of Lie algebra by setting

$$[X_e, Y_e] = [X, Y]_e.$$

Exercise: check that $L_{a*}[X,Y] = [L_{a*}X, L_{a*}Y]$

This is ok, since $L_{a*}[X,Y] = [L_{a*}X, L_{a*}Y] = [X,Y]$

How to define a left invariant metric on G?

Take <u>any</u> scalar product $\langle\, , \rangle_e$ in T_eG.

Define

$$(LI) \quad \boxed{g_a(X_a, Y_a) = \langle L_{a^{-1}*_a} X_a, L_{a^{-1}*_a} Y_a \rangle_e} \qquad \forall a\, G,\ \forall X_a \in T_aG$$

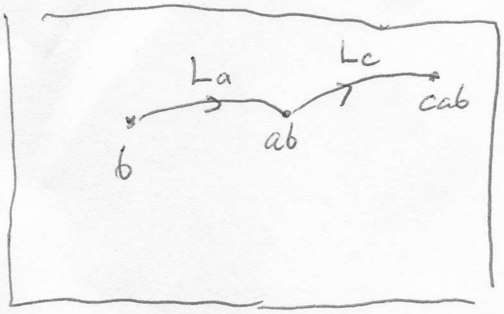
This is clearly left invariant because:

$$L_a \circ L_b = L_{ab}, \quad L_{c*_{ab}} \circ L_{a*_b} = L_{ca*_b}$$

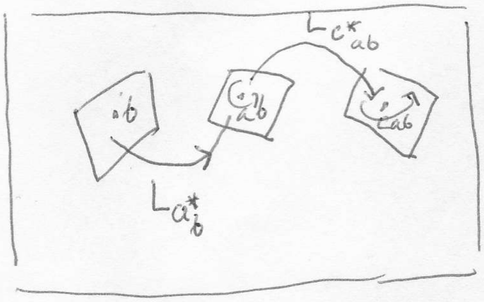
In the same way we define right invariant metric on G.

<u>Thm</u> see exercise 7 p. 46 in DoCarmo

Let G be a compact connected Lie group G.
Then G admits a bi-invariant Riemannian metric.



$$L_c \circ L_a = L_{ca}$$



$$L_{c^*_{ab}} \circ L_{a^*_b} = L_{ca^*_b}$$

$$g_{ab}\left(L_{a^*_b} X_b, L_{a^*_b} Y_b\right) =$$

$$= \left\langle L_{ab^{-1}{}^*_{ab}} L_{a^*_b} X_b, L_{(ab)^{-1}{}^*_{ab}} L_{a^*_b} Y_b \right\rangle_e =$$

$$= \left\langle L_{(b^{-1}a^{-1})^*_{ab}} L_{a^*_b} X_b, L_{(b^{-1}a^{-1})^*_{ab}} L_{a^*_b} Y_b \right\rangle_e =$$

$$= \left\langle L_{b^{-1}{}^*_b} X_b, L_{b^{-1}{}^*_b} Y_b \right\rangle_e = g_b(X_b, Y_b).$$

Example Upper half plane

$$\mathbb{H}_+ = \{\mathbb{R}^2 \ni (x,y) : y > 0\}$$

Group structure: on the space of functions $f_{(x,y)} : \mathbb{R} \to \mathbb{R}$

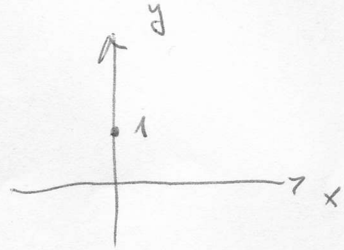
s.t. $f_{(x,y)}(t) = yt + x$ consider composition:

$$f_{(x',y')} \circ f_{(x,y)}$$

$$f_{(x',y')} \circ f_{(x,y)}(t) = f_{(x',y')}(f_{(x,y)}(t)) = f_{(x',y')}(yt + x) =$$

$$= y'yt + y'x + x' =$$

$$= f_{(y'x+x',\, y'y)}(t)$$



$$(x',y') \cdot (x,y) = (y'x + x', y'y) \in \mathbb{H}_+$$

Lie Group with $e = (0,1)$ and inverse $(x,y)^{-1} = \left(-\frac{x}{y}, \frac{1}{y}\right)$

Left invariant vector fields

Take ∂_x at e. $\quad L_{(a,b)}(x,y) = (bx + a, by)$

$$L_{(a,b)*}\, \partial_x|_e = b\, \partial_x|_{(a,b)} \qquad L_{(a,b)*} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$$

$$X_1 = y\,\partial_x \qquad L_{(a,b)*}\,\partial_y|_e = b\,\partial_y|_{(a,b)}$$

$$X_2 = y\,\partial_y$$

$$[X_1, X_2] = -y\,\partial_x = -X_1$$

Right invariant vector fields

$$R_{(a,b)}(x,y) = (x,y)\cdot(a,b) = (ya + x, yb)$$

$\partial_x \sim \gamma = (t,1)$; $R\gamma = (a+t, b) \Rightarrow R_*\partial_x = \partial_x$

$\partial_y \sim (0, t+1) \Rightarrow R(0,t+1) = (a(t+1), b(t+1))$

$$R_{(a,b)*}\,\partial_x|_e = \partial_x \qquad R_{(a,b)*}\,\partial_y|_e = a\,\partial_x + b\,\partial_y$$

$$Y_1 = \partial_x$$

$$Y_2 = x\,\partial_x + y\,\partial_y$$

$$[Y_1, Y_2] = Y_1$$

Left invariant metric which is δ_{ij} at $(0,1)$

$$\begin{cases} g(\partial_x, \partial_x) = \frac{1}{y}\cdot\frac{1}{y}\langle \partial_x, \partial_x\rangle_e = \frac{1}{y^2} \\ g(\partial_x, \partial_y) = \frac{1}{y}\cdot\frac{1}{y}\langle \partial_x, \partial_y\rangle_e = 0 \\ g(\partial_y, \partial_y) = \frac{1}{y}\cdot\frac{1}{y}\langle \partial_y, \partial_y\rangle_e = 0 \end{cases}$$

$$g = \frac{dx^2+dy^2}{y^2}$$

Right inv. metric which is δ_{ij} at $(0,1)$

$$g(\partial_x, \partial_x) = \langle \partial_x, \partial_x\rangle_e = 1$$

$$g(\partial_x, \partial_y) = \left\langle -\frac{x}{y}\partial_x + \frac{1}{y}\partial_y, \partial_x\right\rangle_e = -\frac{x}{y}$$

$$g(\partial_y, \partial_y) = \left\langle -\frac{x}{y}\partial_x + \frac{1}{y}\partial_y, -\frac{x}{y}\partial_x + \frac{1}{y}\partial_y\right\rangle = \frac{x^2}{y^2} + \frac{1}{y^2}$$

$$g = dx^2 - 2\frac{x}{y}\,dx\,dy + \frac{x^2+1}{y^2}dy^2 =$$

$$= \left(dx - \frac{x}{y}dy\right)^2 + \frac{1}{y^2}dy^2 =$$

$$= \frac{(y\,dx - x\,dy)^2 + dy^2}{y^2} = d\left(\frac{x}{y}\right)^2 + \left(\frac{dy}{y}\right)^2$$

$$\left(= dx'^2 + dy'^2 \qquad \begin{cases} x' = \frac{x}{y} \\ y' = \log y \end{cases} \right)$$

If g is a bi-invariant metric on G then the scalar product $\langle\, , \rangle$ induced by g in $T_e G$ satisfies

$$\langle [U, X], Y \rangle + \langle X, [U, Y] \rangle = 0. \qquad (*)$$

And the oposite is true:

If we have a scalar product in $T_e G$ such that $(*)$ holds then the metric defined by (LI) is biinvariant on G. (proof doCarmo p. 40-41)

4) Product metric

(M_1, g_1), (M_2, g_2)

Consider $M_1 \times M_2$ with projections $\pi_1 : M_1 \times M_2 \longrightarrow M_1$
$\pi_2 : M_1 \times M_2 \longrightarrow M_2$

$$g_1 \oplus g_2 (X, Y) := g_1(\pi_{1*}X, \pi_{1*}Y) + g_2(\pi_{2*}X, \pi_{2*}Y)$$

e.g. take a torus
$$\mathbb{T}^n = S^1 \times \ldots \times S^1$$

and take a metric g on S^1 as an induced metric that S^1 gets from euclidean metric in $\mathbb{R}^2$.

$$\Rightarrow \quad g_{\mathbb{T}^n} = \underbrace{g_1 \oplus \ldots \oplus g_1}_{n\text{-times.}}$$

flat torus

5) Every manifold (Hausdorff + countable basis) admits a Riemannian metric.

<u>Partition of unity:</u>

family of functions $f_\alpha : M \to \mathbb{R}$ s.t.

closure of the set of points where $f_\alpha \neq 0$.

1) $\forall \alpha \quad f_\alpha \geq 0$ and $\text{supp } f_\alpha \subset U_\alpha$

2) $\{U_\alpha\}$ is a locally finite cover of M i.e.

$$\bigcup_\alpha U_\alpha = M, \quad \text{and} \quad \forall p \in M \ \exists \underset{\text{neighb. of } p}{W} \text{ s.t } W \cap U_\alpha \neq \emptyset$$

for only finite number of α

3) $\sum_\alpha f_\alpha(p) = 1 \qquad \forall p \in M$

Partition of unity always exists on M which is Hausdorff and has countable basis (see do Carmo p. 30)

$\Rightarrow$ take such a partition on M

$\{f_\alpha\}$, $\{U_\alpha\}$ coordinate charts

In each U_α we define a metric g^α s.t. in coordinate basis

$$g^\alpha\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = \delta^\alpha_{\mu\nu}$$

$$\Rightarrow \quad g = \sum_\alpha f_\alpha(p)\, g^\alpha .$$

Geodesics

(M, ∇) manifold with an affine connection.

$\gamma : I \to M$ of class C^2 is called a geodesic for connection ∇ iff

$$\boxed{\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \lambda . \frac{d\gamma}{dt}}$$

for some function $\lambda = \lambda(t)$ along γ.

Recall
local expression
$$\frac{D}{dt}(V) = \left(\frac{dV^\mu}{dt} + \Gamma^\mu{}_{\nu\rho} V^\nu V^\rho\right) X_\rho$$

for $V = V^\mu(t) X_\rho$ ← vector fields along γ.

$$\frac{d\gamma}{dt} = \frac{dx^\mu}{dt} X_\mu \quad , \quad X_\mu = \frac{\partial}{\partial x^\mu}$$

$\Rightarrow \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \lambda \frac{d\gamma}{dt}$ can be locally written as

$$\boxed{\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = \lambda \frac{dx^\mu}{dt}}$$

Reparametrization

$$t \to t' = f(t) \quad , \quad \underset{\displaystyle \overset{\circ}{f}}{f'(t)} \neq 0.$$

$$\frac{d}{dt} = \overset{\circ}{f} \frac{d}{dt'}$$

$$\frac{d^2x^\mu}{dt^2} = \frac{d}{dt}\left(\overset{\circ}{f} \frac{dx^\mu}{dt'}\right) = \overset{\circ\circ}{f} \frac{dx^\mu}{dt'} + \overset{\circ}{f}{}^2 \frac{d^2x^\mu}{dt'^2}$$

$$\ddot{f}\frac{dx^\mu}{dt'} + \dot{f}^2\frac{d^2x^\mu}{dt'^2} + \Gamma^\mu{}_{\nu\rho}\dot{f}^2\frac{dx^\nu}{dt'}\frac{dx^\rho}{dt'} = \lambda\dot{f}\frac{dx^\mu}{dt'}$$

$$\frac{d^2x^\mu}{dt'^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt'}\frac{dx^\rho}{dt'} = \frac{\lambda\dot{f}-\ddot{f}}{\dot{f}^2}\frac{dx^\mu}{dt'}$$

$$\boxed{\frac{D}{dt'}\left(\frac{d\gamma}{dt'}\right) = \lambda'\frac{d\gamma}{dt'}, \quad \lambda' = \frac{\lambda\dot{f}-\ddot{f}}{\dot{f}^2}}$$

Note:

1) Definition does not depend on the choice of paramehization. If $t \to t' = f(t)$ then $\lambda \to \lambda' = \frac{\lambda\dot{f}-\ddot{f}}{\dot{f}^2}$

2) Given a geodesic with a parametrization for which we have function λ there exists a paramehization for which $\lambda' = 0$ ($\lambda\dot{f} = \ddot{f}$ has always solution). This paramehization is called <u>affine</u>.

3) Affine paramehization is defined up to an affine transformation $t' \to at' + b$ $\underset{\text{const}}{a} \neq 0$, $b = \text{const}$.

$$\left(\lambda = 0 \Rightarrow \lambda' = \frac{-\ddot{f}}{\dot{f}^2} = 0 \Rightarrow f = at + b\right)$$

In affine paramehization the geodesic equation is

$$\boxed{\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0}$$

Observe that to determine a geodesic for a connection one needs only to know $\Gamma^\mu{}_{(\nu\rho)}$.

Indeed:

$$\Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = \Big(\Gamma^\mu{}_{(\nu\rho)} + \cancel{\underset{\text{antisym}}{\Gamma^\mu{}_{[\nu\rho]}}}\Big)\underbrace{\frac{dx^\nu}{dt}\frac{dx^\rho}{dt}}_{\text{sym.}}$$

$$\Gamma^\mu{}_{\nu\rho} = \underbrace{\tfrac{1}{2}\left(\Gamma^\mu{}_{\nu\rho}+\Gamma^\mu{}_{\rho\nu}\right)}_{\Gamma^\mu{}_{(\nu\rho)}} + \underbrace{\tfrac{1}{2}\left(\Gamma^\mu{}_{\nu\rho}-\tfrac{1}{2}\Gamma^\mu{}_{\rho\nu}\right)}_{\Gamma^\mu{}_{[\nu\rho]}}$$

Corollary

There are different connections that have the same geodesics!

(add such antisymmetric (in $\nu\rho$) to $\Gamma^\mu{}_{\nu\rho}$)

How to determine connection?

Recall: $\nabla_\mu X_\nu = \Gamma^\rho{}_{\nu\mu}X_\rho$, $\quad [X_\rho, X_\sigma] = C^\mu{}_{\rho\sigma}X_\mu$

$$T(X_\mu, X_\nu) = \nabla_\mu X_\nu - \nabla_\nu X_\mu - [X_\mu, X_\nu] =$$
$$= \left(\Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu} - C^\rho{}_{\mu\nu}\right)X_\rho = Q^\rho{}_{\mu\nu}X_\rho$$

$$\Rightarrow \quad \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu} = C^\rho{}_{\mu\nu} + Q^\rho{}_{\mu\nu}$$

$$\boxed{\Gamma^\rho{}_{[\nu\mu]} = \tfrac{1}{2}C^\rho{}_{\mu\nu} + \tfrac{1}{2}Q^\rho{}_{\mu\nu}} \quad \text{(AS)}$$

Corollary

1) Antisymmetric part of the connection is determined by the torsion and the anholonomy coefficients.

$$\boxed{\Gamma^\rho{}_{\nu\mu} = \Gamma^\rho{}_{(\nu\mu)} + \frac{1}{2} C^\rho{}_{\mu\nu} + \frac{1}{2} Q^\rho{}_{\mu\nu}}$$

2) Torsionless connection in holonomic frame is symmetric.

(Pseudo) Riemannian connections

(M,g). It is natural to look for a connection ∇ which preserves the metric:

$$\nabla_X g = 0 \qquad \forall X \in \mathfrak{X}(M)$$

$\rightsquigarrow$ $g = g_{\mu\nu}\,\theta^\mu\theta^\nu$ where $\theta^\mu\theta^\nu = \frac{1}{2}(\theta^\mu\otimes\theta^\nu + \theta^\nu\otimes\theta^\mu)$

$X_\mu \lrcorner \theta^\nu = \delta_\mu{}^\nu$

$$0 = \nabla_\mu g_{\nu\rho} = X_\mu(g_{\nu\rho}) - \Gamma^\alpha{}_{\nu\mu}\, g_{\alpha\rho} - \Gamma^\alpha{}_{\rho\mu}\, g_{\nu\alpha} \qquad \Big|\ \theta^\mu$$

$$\boxed{0 = D g_{\nu\rho} = d g_{\nu\rho} - g_{\rho\alpha}\Gamma^\alpha{}_\nu - g_{\nu\alpha}\Gamma^\alpha{}_\rho} \qquad (MC)$$

Digression

Assumption about the nondegeneracy of g

$$g(X,Y)=0 \quad \forall Y \implies X=0 \quad \text{means that}$$

at every point the map:

$$T_pM \ni X_p \longrightarrow g_p(X_p, \cdot) \in T_p^*M$$

is isomorphism of vector spaces T_pM and T_p^*M.

Using the metric (even is ΨRiemannian case!) we can identify TM and T^*M.

In the old tensorial language:

(•) V^μ — coefficients of a vector field V

$$V^\mu \xmapsto{g_{\mu\nu}} g_{\mu\nu}V^\nu = V_\nu \leftarrow \text{coefficients of a 1-form}$$

(••) λ_μ - coefficients of a 1-form λ

$g_{\mu\nu}$ is invertible $\implies$ $g^{\alpha\beta}$ s.t. $g^{\alpha\beta}g_{\beta\mu} = \delta^\alpha_\mu$ is uniquely defined by g.

$$\lambda_\mu \xmapsto{g^{\mu\nu}} g^{\mu\nu}\lambda_\nu = \lambda^\nu \leftarrow \text{coefficients of a vector field.}$$

Our condition (MC) for a connection that preserves the metric

$$dg_{\nu\varrho} = \Gamma_{\varrho\nu} + \Gamma_{\nu\varrho} \qquad \text{or}$$

$$\boxed{\Gamma_{\varrho\nu\mu} + \Gamma_{\nu\varrho\mu} = X_\mu(g_{\nu\varrho})} \qquad (1)$$

We also have:

$$\boxed{\Gamma_{\varrho\nu\mu} - \Gamma_{\varrho\mu\nu} = c^\varrho_{\mu\nu} + Q^\varrho_{\mu\nu}} \qquad (2)$$

(1) − (2):

$$\Gamma_{\nu\varrho\mu} + \Gamma_{\varrho\mu\nu} = X_\mu(g_{\nu\varrho}) - c^\varrho_{\mu\nu} - Q^\varrho_{\mu\nu} =: H_{\nu\varrho\mu}$$

$$\begin{array}{r|l}
 & \underline{\Gamma_{\nu\varrho\mu}} + \Gamma_{\varrho\mu\nu} = H_{\nu\varrho\mu} \\
+ & \Gamma_{\mu\nu\varrho} + \underline{\Gamma_{\nu\varrho\mu}} = H_{\mu\nu\varrho} \\
- & \Gamma_{\varrho\mu\nu} + \Gamma_{\mu\nu\varrho} = H_{\varrho\mu\nu} \\
\hline
\end{array}$$

$$2\Gamma_{\nu\varrho\mu} = H_{\nu\varrho\mu} + H_{\mu\nu\varrho} - H_{\varrho\mu\nu}$$

$$\left\{ \begin{array}{l}
\boxed{\Gamma_{\nu\varrho\mu} = \frac{1}{2}\left(H_{\nu\varrho\mu} + H_{\mu\nu\varrho} - H_{\varrho\mu\nu}\right)} \quad \text{where} \\
\boxed{H_{\nu\varrho\mu} = X_\mu(g_{\nu\varrho}) - c^\varrho_{\mu\nu} - Q^\varrho_{\mu\nu}}
\end{array} \right.$$

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$$(Dg)_{\mu\nu} = dg_{\mu\nu} - \Gamma_{\mu\nu} - \Gamma_{\nu\mu}$$

$$\Rightarrow \boxed{(\Gamma_{\mu\nu\rho} + \Gamma_{\nu\mu\rho})\theta^\rho = dg_{\mu\nu} - (Dg)_{\mu\nu}}$$

$$d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu = \Theta^\mu$$

$$d\theta^\mu = -\frac{1}{2} C^\mu{}_{\nu\rho}\, \theta^\nu \wedge \theta^\rho$$

$$\Gamma_{\mu\nu\rho}\, \theta^\rho \wedge \theta^\nu = \Theta_\mu + \frac{1}{2} C_{\mu\nu\rho}\, \theta^\nu \wedge \theta^\rho$$

$X_\nu \lrcorner X_\rho \lrcorner$:

$$\Theta_\mu = \frac{1}{2} Q_{\mu\nu\rho}\, \theta^\nu \wedge \theta^\rho$$

$$(2) \quad \Gamma_{\mu\nu\rho} - \Gamma_{\mu\rho\nu} = -Q_{\mu\nu\rho} - C_{\mu\nu\rho}$$

$$(Dg)_{\mu\nu} = 0$$

$$(1) \quad \Gamma_{\nu\mu\rho} + \Gamma_{\mu\nu\rho} = g_{\mu\nu|\rho}$$

$$df = f_{|\rho}\theta^\rho = X_\rho(f)\theta^\rho$$

$$(1)-(2) \quad \Gamma_{\nu\mu\rho} + \Gamma_{\mu\rho\nu} = g_{\mu\nu|\rho} + Q_{\mu\nu\rho} + C_{\mu\nu\rho} = H_{\nu\mu\rho}$$

$$\boxed{H_{\nu\mu\rho} = g_{\mu\nu|\rho} + Q_{\mu\nu\rho} + C_{\mu\nu\rho}}$$

$$\Gamma_{\nu\mu\rho} + \Gamma_{\mu\rho\nu} = H_{\nu\mu\rho}$$

$$+ \quad \Gamma_{\rho\nu\mu} + \Gamma_{\nu\mu\rho} = H_{\rho\nu\mu}$$

$$- \quad \Gamma_{\mu\rho\nu} + \Gamma_{\rho\nu\mu} = H_{\mu\rho\nu}$$

$$H_{\rho\nu\mu} = g_{\nu\rho|\mu} + Q_{\nu\rho\mu} + C_{\nu\rho\mu}$$

$$H_{\mu\rho\nu} = g_{\rho\mu|\nu} + Q_{\rho\mu\nu} + C_{\rho\mu\nu}$$

$$2\Gamma_{\nu\mu\rho} = H_{\nu\mu\rho} + H_{\rho\nu\mu} - H_{\mu\rho\nu}$$

$$\boxed{\Gamma_{\nu\mu\rho} = \frac{1}{2}\Big(g_{\nu\mu|\rho} + g_{\rho\nu|\mu} - g_{\mu\rho|\nu} + C_{\mu\nu\rho} + C_{\nu\rho\mu} - C_{\rho\mu\nu} + Q_{\mu\nu\rho} + Q_{\nu\rho\mu} - Q_{\rho\mu\nu} \Big)}$$

Given a torsion $\Theta^\rho = \frac{1}{2} Q^\rho{}_{\mu\nu}\theta^\mu \wedge \theta^\nu$ there is a UNIQUE connection ∇ s.t. it has Θ^ρ as a torsion form and which satisfies $Dg_{\mu\nu} = 0$.

(Extend this then to $Dg_{\mu\nu} = B_{\mu\nu}$ – given 1-form of type $\binom{0}{2}$)

Assumption | No TORSION: $Q \equiv 0$ | LEVI-CIVITA CONNECTION

(1) Coordinate frame $X_\mu = \frac{\partial}{\partial x^\mu}$, $[X_\mu, X_\nu] = 0$

$\Rightarrow C_{\mu\nu\rho} \equiv 0$

$$\Rightarrow \boxed{\Gamma_{\nu\mu\rho} = \frac{1}{2}\left(g_{\nu\mu,\rho} + g_{\rho\nu,\mu} - g_{\mu\rho,\nu}\right)}$$

$\Rightarrow \Gamma^\nu{}_{\mu\rho} = g^{\nu\alpha}\,\Gamma_{\alpha\mu\rho} =: \left\{ {\nu \atop \mu\rho} \right\} \leftarrow$ Christoffell symbols

(2) $g_{\mu\nu} = (\text{const})_{\mu\nu}$

e.g. $\boxed{\text{Orthonormal frame}}$ $\quad g_{\mu\nu} = \pm\delta_{\mu\nu}$

$$\Rightarrow \boxed{\Gamma_{\nu\mu\rho} = \frac{1}{2}\left(C_{\mu\nu\rho} + C_{\nu\rho\mu} - C_{\rho\mu\nu}\right)}$$

↑

in an orthonormal frame (or other frame in which $g_{\mu\nu} = \text{const}_{\mu\nu}$) $\Gamma_{\nu\mu\rho}$ are determined by the anholonomy coefficients.

No Torsion

<u>Note</u>

$$\begin{cases} d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu = 0 \\ dg_{\mu\nu} - \Gamma_{\mu\nu} - \Gamma_{\nu\mu} = 0 \end{cases}$$

in the orthonormal (or constant coefficient frame) we have

$$\begin{cases} d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu = 0 \\ \Gamma_{\mu\nu} + \Gamma_{\nu\mu} = 0 \end{cases}$$

These determine connection.

Arc length

$t \to \gamma(t)$, $X = \frac{d\gamma}{dt}$

$$s := \int_{t_0}^{t} \sqrt{g(X,X)}\, dt$$

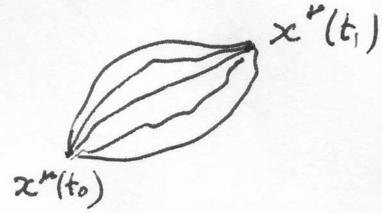
$t \to t' = f(t)$, $f' > 0 \Rightarrow$ s does not change.

s is itself a good parameter : $\frac{ds}{dt} = \sqrt{g(X,X)} > 0$

Def

γ of class C^2 is a geodesic iff it is a critical point for the functional

$$\gamma \longmapsto s[\gamma]$$



Locally

$$\delta \int_{t_0}^{t} \sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\; dt = 0$$

$\delta x^\mu(t_0) = 0$
$\delta x^\nu(t_1) = 0$

Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}^\mu}\sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} - \frac{\partial}{\partial x^\mu}\sqrt{g_{\mu\nu}\dot{x}^\mu x^\nu} = 0$$

$$\frac{d}{dt}\frac{g_{\mu\nu}\dot{x}^\nu}{\sqrt{}} - \frac{1}{2}\frac{g_{\nu\rho,\mu}\dot{x}^\nu \dot{x}^\rho}{\sqrt{}} = 0 \quad \Big| : \sqrt{}$$

$$\frac{d}{ds} = \frac{d}{\sqrt{}\, dt}$$

$$\frac{d}{ds} g_{\mu\nu}\frac{dx^\nu}{ds} - \frac{1}{2} g_{\nu\rho,\mu}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

$$g_{\mu\nu}\frac{d^2x^\nu}{ds^2} + \underbrace{\left(g_{\mu\nu,\rho} - \tfrac{1}{2}g_{\nu\rho,\mu}\right)}_{\frac{1}{2}\left(g_{\mu\nu,\rho} + g_{\mu\rho,\nu}\right)}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

$$\frac{d^2x^\sigma}{ds^2} + \frac{1}{2}g^{\sigma\mu}\left(g_{\mu\nu,\rho} + g_{\rho\mu,\nu} - g_{\nu\rho,\mu}\right)\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0$$

$$\boxed{\frac{d^2x^\sigma}{ds^2} + \left\{ {\sigma \atop \nu\rho} \right\}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0}$$

christoffel symbols

<u>Corollary</u>

Geodesics $\equiv$ self parallels for the Levi-Civita connection in affine paramentrization.

<u>What about pseudoriemannian situation?</u>

for null vectors X $\quad s = 0$!

Energy functional

$$E[\gamma] = \frac{1}{2}\int_{t_0}^{t} g(X,X)\,dt = \int_{t_0}^{t} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\,dt$$

does depend on parametrization

$$\delta E = 0$$

$$\frac{1}{2}\frac{d}{dt}\frac{\partial}{\partial \dot{x}^\mu} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - \frac{1}{2}\frac{\partial}{\partial x^\mu} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$$

$$\frac{d}{dt} g_{\mu\nu}\dot{x}^\nu - \frac{1}{2} g_{\nu\rho,\mu}\dot{x}^\nu \dot{x}^\rho = 0$$

$$g_{\mu\nu}\ddot{x}^\nu + \left(g_{\mu\nu,\rho} - \frac{1}{2} g_{\nu\rho,\mu}\right)\dot{x}^\nu \dot{x}^\rho = 0$$

$$\boxed{\frac{d^2 x^\sigma}{dt^2} + \left\{ {\sigma \atop \nu\rho} \right\} \dot{x}^\nu \dot{x}^\rho = 0}$$

1) in this parametrization we again get geodesics equation with t as an affine parameter.

2) derivation is also good for $\frac{d\gamma}{dt} = X$ s.t. $g(X,X) = 0$.

TODAY: Every statement is valid also in pseudo riemannian case

L. 12

SB 16.10.2008

Metric connections (Why?)

$\forall X, Y, Z \in \mathfrak{X}(M)$

$$g(X,Y) = (C_1^1 \circ C_1^1)(g \otimes X \otimes Y)$$

$$\Rightarrow \quad \nabla_Z (g(X,Y)) = C_1^1\left(C_1^1\left(\nabla_Z g \otimes X \otimes Y + g \otimes \nabla_Z X \otimes Y + g \otimes X \otimes \nabla_Z Y\right)\right)$$

$$= (\nabla_Z g)(X,Y) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

$$\boxed{\nabla_Z (g(X,Y)) = (\nabla_Z g)(X,Y) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y)}$$

Def

∇ is metric iff $\forall Z \in \mathfrak{X}(M)$ $\quad \nabla_Z g \equiv 0$.

Thus for metric connections we have

$$\nabla_Z (g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

Corollary

∇ is metric $\Longleftrightarrow$ Parallel transport preserves scalar product of vectors.

<u>Proof</u>

$\gamma \rightsquigarrow Z = \frac{d\gamma}{dt}$

Let X, Y be two vectors parallely transported along γ, s.t.

$X(0) = X_0$, $Y(0) = Y_0$. We have $\frac{DX}{dt} = 0$, $\frac{DY}{dt} = 0$

$$\frac{d}{dt}\left(g(X(t), Y(t))\right) = \left(\nabla_Z g\right)(X(t), Y(t)) + \cancel{g\left(\frac{DX}{dt}, Y\right)} + \cancel{g\left(X, \frac{DY}{dt}\right)}$$

$\Rightarrow$: if $\nabla_Z g \equiv 0 \;\; \forall Z \Rightarrow \frac{d}{dt} g(X(t), Y(t)) = 0$

$\Rightarrow g(X(t), Y(t)) = \text{const}$ along γ.

$\Leftarrow$: if $g(X(t), Y(t)) = \text{const}$ along $\gamma \Rightarrow$

$\nabla_Z g \equiv 0$ along γ

but we want that $g(X(t), Y(t)) = \text{const}$ along all γs

$\Rightarrow \nabla_Z g \equiv 0$ along any curve

$\Rightarrow \nabla_Z g \equiv 0 \;\; \forall Z \in \mathfrak{X}(M)$.

$\square$.

This in particular means that if we calculate in local frames:

$$\nabla_\mu X_\nu = \nabla_\mu (g_{\nu\rho} X^\rho) = \cancel{(\nabla_\mu g)_{\nu\rho} X^\rho} + g_{\nu\rho} \nabla_\mu X^\rho$$

if connection is metric we may commute $g_{\nu\rho}$ with ∇_μ!

But <u>only if</u> ∇ is metric!

(M, g)

<u>Riemann tensor</u> $\equiv$ curvature tensor for the Levi-Civita connection.

$$\left.\begin{array}{l}(Dg)_{\mu\nu} \equiv 0 \\ \Theta^{\mu} \equiv 0\end{array}\right\} \iff \nabla \text{ - Levi-Civita}$$

$$\Updownarrow$$

$$\begin{cases}\nabla_X g \equiv 0 & \forall X \in \mathfrak{X}(M) \\ \nabla_X Y - \nabla_Y X = [X,Y] & \forall X, Y \in \mathfrak{X}(M)\end{cases}$$

$$\boxed{R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.} \qquad \forall X, Y, Z \in \mathfrak{X}(M)$$

$\uparrow$

Riemann tensor.

<u>Symmetries</u>

1) $R(X,Y) = -R(Y,X)$

2) $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$ — I^{st} Bianchi (no torsion!)

3) $g(R(X,Y)Z, T) = -g(R(X,Y)T, Z)$

4) $g(R(X,Y)Z, T) = g(R(Z,T)X, Y)$

or:

$$R(X_\mu, X_\nu)Y_\rho = R^{\sigma}{}_{\rho\mu\nu} Y_\sigma \xleftrightarrow{g_{\alpha\beta}} R_{\sigma\rho\mu\nu}$$

1) $R_{\sigma\rho\mu\nu} = -R_{\sigma\rho\nu\mu}$

2) $R_{\sigma\rho\mu\nu} + R_{\sigma\nu\rho\mu} + R_{\sigma\mu\nu\rho} = 0$

3) $R_{\sigma\rho\mu\nu} = -R_{\rho\sigma\mu\nu}$

4) $R_{\sigma\rho\mu\nu} = R_{\mu\nu\sigma\rho}$

Comments

1) holds for any connection

2) holds if torsion vanishes (I^{st} Bianchi + $T \equiv 0$)

3) holds if connection is metric ($Dg \equiv 0$)

4) holds for Levi-Civita. (i.e $T \equiv 0$ & $Dg \equiv 0$)

(homework: Thursday 23 Oct. prove 3) and 4) using any of the three languages.)

Fact # of independent components of Riemann: $\frac{1}{12} n^2(n^2-1)$

Thm

Riemann $\equiv 0$ $\Longleftrightarrow$ there exists a local coord system (x^i) s.t.

$$g = dx^{1^2} + \dots + dx^{p^2} - dx^{p+1^2} - \dots - dx^{n^2}$$

$$(g_{\mu\nu} = \mathrm{diag}(1, \dots 1, -1, \dots -1))$$

Proof

$$\Gamma^\mu{}_\nu = \frac{1}{2} g^{\mu\alpha} (g_{\alpha\nu,\rho} + g_{\rho\alpha,\nu} - g_{\nu\rho,\alpha}) dx^\rho$$

$\Leftarrow$ Hence: $\Gamma^\mu{}_\nu \equiv 0 \Rightarrow \Omega^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu \equiv 0$ ✓

$\Rightarrow R^\mu{}_{\nu\rho\sigma} \equiv 0$, we showed before that this means that one can make $\Gamma^\mu{}_{\nu\rho} \equiv 0$ in a neighbourhood in holonomic frame x^μ.
And because $T^\mu{}_{\nu\rho} \equiv 0$ this can be made in a neighbourhood

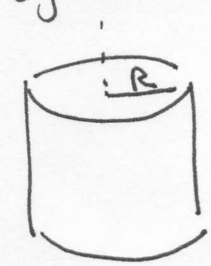
$\Rightarrow g_{\alpha\nu,\rho} = 0$ in a neighbourhood

$\Rightarrow g_{\alpha\nu} = (\mathrm{const})_{\alpha\nu}$

$\Rightarrow$ Linear transf. of coordinates brings $g_{\alpha\nu} = \mathrm{diag}(1, \dots 1, -1, \dots -1)$ $\square$

Example

Cylinder



$$g\Big|_{\text{cylinder}} = dx^2+dy^2+dz^2 = dz^2 + R^2 d\varphi^2$$

const

$$\Longrightarrow \quad R^{\mu}{}_{\nu\rho\sigma} = 0 \quad \Longrightarrow \quad \text{cylinder is } \underline{\text{flat}}.$$

Decomposition of Riemann onto irreducible bits

Example

$V = \mathbb{R}^n$,

$A_{\mu\nu} \in (\mathbb{R}^{n*}) \otimes (\mathbb{R}^{n*})$

$G = GL(n,\mathbb{R})$ acts on $V^* \otimes V^*$ via:

$$V^*\otimes V^* \ni A_{\alpha\beta} \xmapsto{a} \rho(a)^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} A_{\alpha\beta} = A_{\alpha\beta}\, a^{-1\alpha}{}_{\mu}\, a^{-1\beta}{}_{\nu} \in V^*\otimes V^*$$

Hence we have representation $\rho^{\alpha\beta}{}_{\mu\nu}$ of $GL(n,\mathbb{R})$ in $V^*\otimes V^*$.

But this representation is reducible. Indeed

$V^*\wedge V^*$ and $V^*\odot V^*$ are $GL(n,\mathbb{R})$ invariant subspaces.

$$A_{\alpha\beta} = A_{(\alpha\beta)} + A_{[\alpha\beta]}$$

$$A_{(\alpha\beta)} = \tfrac{1}{2}(A_{\alpha\beta} + A_{\beta\alpha}) \in V^*\odot V^*$$
$$A_{[\alpha\beta]} = \tfrac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha}) \in V^*\wedge V^*$$

$$V^*\otimes V^* = V^*\odot V^* \oplus V^*\wedge V^*$$

± 1 eigenspaces of S s.t.

$S A_{\alpha\beta} = A_{\beta\alpha}$.

$\rho(a)^\alpha{}_\mu{}^\beta{}_\nu A_{(\alpha\beta)} = A'_{\mu\nu}$ is symmetric

$\rho(a)^\alpha{}_\mu{}^\beta{}_\nu A_{[\alpha\beta]} = A'_{\mu\nu}$ is antisymmetric !

This is the end! Spaces $V^* \odot V^*$ and $V^* \wedge V^*$ are irreducible w.r.t. $GL(n,\mathbb{R})$.

Suppose now that we in addition have (pseudo) riemannian <u>metric</u> $g_{\mu\nu}$ in $V = \mathbb{R}^n$.

We restrict the group $GL(n,\mathbb{R})$ to its subgroup $O(g)$ preserving metric:

$$O(g) = \{ a \in GL(n,\mathbb{R}) \text{ s.t. } g_{\mu\nu} a^{-1\mu}{}_\alpha a^{-1\nu}{}_\beta = g_{\alpha\beta} \}.$$

What about decomposition of $V^* \otimes V^*$ on irreducibles w.r.t. the <u>restricted</u> group $O(g)$?

We have

$$V^* \otimes V^* \underset{\substack{\uparrow \\ O(g)}}{=} V^* \wedge V^* \oplus \underbrace{V^* \odot V^*}$$

but now this is reducible!

It has an invariant subspace

$$Tr = \{ A_{\mu\nu} = \lambda g_{\mu\nu} , \lambda \in \mathbb{R} \} \subset V^* \odot V^*$$

$$V^* \odot V^* = (V^* \odot V^*)_0 \oplus Tr(V^* \odot V^*)$$

We have $A_{\mu\nu}$, $g_{\mu\nu}$ and its inverse $g^{\alpha\beta}$ s.t.

$$g_{\alpha\beta}\, g^{\beta\nu} = \delta_\alpha{}^\nu .$$

$$A_{(\mu\nu)} \rightsquigarrow A = g^{\mu\nu} A_{\mu\nu}$$

$$A_{(\mu\nu)} \rightsquigarrow \check{A}_{(\mu\nu)} = A_{(\mu\nu)} - \frac{1}{n} A\, g_{\mu\nu}$$

is traceless.

$$\boxed{A_{\mu\nu} = \underset{\substack{\uparrow \\ V^* \wedge V^*}}{A_{[\mu\nu]}} + \underset{\substack{\uparrow \\ (V^* \odot V^*)_0}}{\check{A}_{(\mu\nu)}} + \underset{\substack{\uparrow \\ \mathrm{Tr}\,(V^* \odot V^*)}}{\frac{1}{n} A\, g_{\mu\nu}}}$$

With the exception of $\dim V = 4$ (+ orientability) this is decomposition onto irreducibles w.r.t. $O(g)$.

The same for Riemann:

$$R^\mu{}_{\nu\rho\sigma} \longrightarrow \underset{\substack{\uparrow \\ \text{is symmetric}}}{R_{\nu\sigma}} = R^\mu{}_{\nu\mu\sigma}$$

|| This is called RICCI tensor

$$R_{\nu\sigma} \xrightarrow{\mathrm{Tr}} \mathbf{R} = g^{\nu\sigma} R_{\nu\sigma}$$

|| This is called RICCI scalar / SCALAR CURVATURE

$$\check{R}_{\nu\sigma} = R_{\nu\sigma} - \frac{1}{n} R\, g_{\nu\sigma}$$

$$R^{\mu\nu}{}_{\rho\sigma} = \underset{\text{totally traceless.}}{\boxed{C^{\mu\nu}{}_{\rho\sigma}}} + a\, \delta^{[\mu}_{[\rho}\, \check{\mathbf{R}}^{\nu]}_{\sigma]} + b\, \delta^{[\mu}_{\rho}\, \delta^{\nu]}_{\sigma}\, R$$

Calculate a and b.

1) contraction over μ, ρ:

$$R^\nu{}_\sigma = \frac{a}{4}\left(n \check{R}^\nu{}_\sigma - \check{R}^\nu{}_\sigma - \check{R}^\nu{}_\sigma + \cancel{\delta^\nu{}_\sigma \check{R}}\right)$$
$$+\frac{b}{2}(n-1)\,\delta^\nu{}_\sigma R$$

but always:

$$R^\nu{}_\sigma = \check{R}^\nu{}_\sigma + \frac{1}{n}\delta^\nu{}_\sigma R$$

hence:

$$\frac{a}{4}(n-2) = 1 \quad \Rightarrow \quad a = \frac{4}{n-2}$$
$$\frac{b}{2}(n-1) = +\frac{1}{n} \quad \Rightarrow \quad b = \frac{-2}{n(1-n)}$$

$$\boxed{R^{\mu\nu}{}_{\rho\sigma} = C^{\mu\nu}{}_{\rho\sigma} + \frac{4}{n-2}\,\delta^{[\mu}_{[\rho}\,\check{R}^{\nu]}{}_{\sigma]} + \frac{2}{n(n-1)}\,\delta^{[\mu}_{\rho}\,\delta^{\nu]}_{\sigma}\,R}$$

Weyl tensor.

Low dimensions

$$\frac{1}{12}n^2(n^2-1) \xrightarrow{n=1} 0, \quad \xrightarrow{n=2} 1, \quad \xrightarrow{n=3} 6, \quad \xrightarrow{n=4} 20$$

$n=1 \Rightarrow R_{\mu\nu\rho\sigma} \equiv 0$

$n=2 \Rightarrow R_{\mu\nu\rho\sigma}$ has only one component

$\Rightarrow C_{\mu\nu\rho\sigma} \equiv 0$, $\check{R}_{\mu\nu} \equiv 0 \Rightarrow$ only $R \neq 0$.

$n=3 \Rightarrow$ six components $\Rightarrow C_{\mu\nu\rho\sigma} \equiv 0$. All curvature in $R_{\mu\nu}$

$n \geqslant 4$ in general $C_{\mu\nu\rho\sigma} \not\equiv 0$.

Def

Two metrics g and $\hat{g}$ are conformally equivalent iff there exists $r : M \to \mathbb{R}$ s.t.

$$\hat{g} = e^{2r} g.$$

Thm

1) $\hat{C}^{\mu}{}_{\nu\rho\sigma} = C^{\mu}{}_{\nu\rho\sigma}$ for conformally equivalent metrics.

2) $n \geq 4$ a metric g is conformally equivalent to a flat metric iff $C^{\mu}{}_{\nu\rho\sigma} \equiv 0$.

Analogous fact for Riemann:

2) $R^{\mu}{}_{\nu\rho\sigma} \equiv 0 \iff$ metric g is isometric to flat metric

Isometry

$$(M,g) \xrightarrow{\varphi} (M',g')$$

$$\varphi^* g' = g$$

1) $\varphi^* R' = R$

L. 13

SB. 21.10.2008

Examples

① Canonical metric on quadrics & their curvature

$\tilde{M} = \underset{k}{\mathbb{R}} \times \underset{g}{\mathbb{R}^n}$ $\qquad \tilde{g} = k \oplus g$

$$\tilde{g} = k\, dx^{0\,2} + g_{\mu\nu}\, dx^\mu dx^\nu \quad ; \quad k = \pm 1$$

(signature of $g_{\mu\nu}$ can be arbitrary)

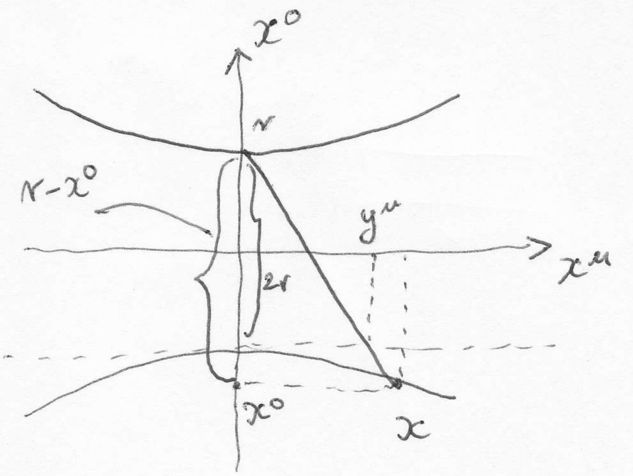
$\Sigma_i^1 = \{ (x^0, x^\mu) \in \tilde{M} :$

$$k(x^0)^2 + g_{\mu\nu} x^\mu x^\nu = k \cdot r^2 \}$$

$\boxed{g_{\mu\nu} = \mathrm{diag}(1, \ldots 1, -1, \ldots -1)}$

$$K := \frac{1}{k r^2}$$

Stereographic projection: (you know it for S^n, so we do it for hyperboloids)



$$\frac{y^\mu}{2r} = \frac{x^\mu}{r - x^0}$$

$$\Downarrow$$

$$y^\mu = \frac{2x^\mu}{1 - \frac{x^0}{r}}$$

Claim

$$\boxed{\tilde{g}|_\Sigma = \frac{g_{\mu\nu}\, dy^\mu dy^\nu}{\left(1 + \frac{K}{4} g_{\mu\nu} y^\mu y^\nu\right)^2}}$$

$$\begin{cases} g(x,x') = x x' = g_{\mu\nu} x^\mu x'^\nu \\ g(x,x) = |x|^2 \end{cases}$$

<u>Proof</u> of the claim:

$$|y|^2 = \frac{4|x|^2}{\left(1-\frac{x^0}{r}\right)^2}$$

$$k x^{0\,2} + |x|^2 = k r^2 \implies |x|^2 = k r^2 \left(1 - \frac{x^{0\,2}}{r^2}\right)$$

$$\frac{K}{4}|y|^2 = \frac{1-\frac{x^{0\,2}}{r^2}}{\left(1-\frac{x^0}{r}\right)^2} = \frac{1+\frac{x^0}{r}}{1-\frac{x^0}{r}}$$

$$\implies \frac{x^0}{r} = \frac{\frac{K}{4}|y|^2 - 1}{\frac{K}{4}|y|^2+1} \implies 1-\frac{x^0}{r} = \frac{2}{\frac{K}{4}|y|^2+1}$$

$$\implies \boxed{\frac{dx^0}{r} = \frac{K\, y\, dy}{\left(1+\frac{K}{4}|y|^2\right)^2}}$$

$$\boxed{x^\mu = \frac{y^\mu}{1+\frac{K}{4}|y|^2}}$$

$$dx^\mu = \frac{dy^\mu}{1+\frac{K}{4}|y|^2} - \frac{y^\mu \frac{K}{2}\, y\, dy}{\left(1+\frac{K}{4}|y|^2\right)^2}$$

$$\tilde{g}|_\Sigma = k\, dx^{0\,2} + |dx|^2 =$$

$$= \frac{k}{r^2}\frac{(y\,dy)^2}{\left(1+\frac{K}{4}|y|^2\right)^4} + \frac{|dy|^2}{\left(1+\frac{K}{4}|y|^2\right)^2} - K\frac{(y\,dy)^2}{\left(1+\frac{K}{4}|y|^2\right)^3}$$

$$+ \frac{K^2}{4}\frac{|y|^2 (y\,dy)^2}{\left(1+\frac{K}{4}|y|^2\right)^4}$$

$$= \frac{|dy|^2}{\left(1+\frac{K}{4}|y|^2\right)^2} + \frac{(y\,dy)^2}{\left(1+\frac{K}{4}|y|^2\right)^4}\left[\cancel{K} - K\left(\cancel{1} + \cancel{\tfrac{K}{4}|y|^2}\right) + \cancel{\tfrac{K^2}{4}|y|^2}\right]$$

$$= \frac{|dy|^2}{\left(1+\frac{K}{4}|y|^2\right)^2}$$

$\square$

Orthonormal frame:

$$\omega^\mu = \frac{dy^\mu}{1+\frac{K}{4} g_{\mu\nu} y^\mu y^\nu}$$

$$\tilde{g}_{|\Sigma} = g_{\mu\nu}\,\omega^\mu \omega^\nu$$

Structure equations:

(1) $d\omega^\mu + \Gamma^\mu{}_\rho \wedge \omega^\rho = 0$ no torsion

(2) $\cancel{dg_{\mu\nu}} - \Gamma_{\mu\nu} - \Gamma_{\nu\mu} = 0$ metricity

$$d\omega^\mu = -\frac{K}{2}\, y_\mu\, \omega^\nu \wedge \omega^\mu = \frac{K}{2}\left(y_\nu \omega^\mu - y^\mu \omega_\nu\right)\wedge \omega^\nu$$

$$\Downarrow$$

$$\boxed{\Gamma^\mu{}_\nu = -\frac{K}{2}\left(y_\nu \omega^\mu - y^\mu \omega_\nu\right)}$$

because it satisfies (1) ; $\Gamma_{\mu\nu} + \Gamma_{\nu\mu} = 0$ hence it also satisfies (2) + uniqueness of Levi-Civita!

Curvature

$$\Omega^\mu{}_\nu = d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\rho \wedge \Gamma^\rho{}_\nu =$$

$$= -\frac{K}{2}\left(dy_\nu \wedge \omega^\mu - dy^\mu \wedge \omega_\nu\right) + \frac{K^2}{4}\left(+\underline{y_\nu y_\rho\, \omega^\rho \wedge \omega^\mu} + y^\mu y_\rho\, \omega^\rho \wedge \omega_\nu\right)$$

$$+ \frac{K^2}{4}\left(\underline{y_\rho}\, \omega^\mu - y^\mu \omega_\rho\right)\wedge\left(\underline{y_\nu}\, \omega^\rho - y^\rho \omega_\nu\right) =$$

$$= \frac{K}{\cancel{2}}\left(1+\frac{K}{4}|y|^2\right)\left(\omega^\mu \wedge \omega_\nu - \cancel{\omega_\nu \wedge \omega^\mu}\right) + \frac{K^2}{4}\left(-|y|^2\, \omega^\mu \wedge \omega_\nu\right)$$

$$= K\, \omega^\mu \wedge \omega_\nu$$

$$\Omega^\mu{}_\nu = K\, g^\mu{}_\alpha\, g_{\nu\beta}\, \omega^\alpha \wedge \omega^\beta =$$

$$= K\, g^\mu{}_{[\alpha}\, g_{\beta]\nu}\, \omega^\alpha \wedge \omega^\beta = \frac{1}{2} R^\mu{}_{\nu\alpha\beta}\, \omega^\alpha \wedge \omega^\beta$$

$$\Rightarrow \boxed{R^\mu{}_{\nu\alpha\beta} = K\left(g^\mu{}_\alpha\, g_{\beta\nu} - g^\mu{}_\beta\, g_{\alpha\nu}\right)}$$

$$\boxed{R_{\nu\beta} = K(n-1)\, g_{\nu\beta}}$$

$$\boxed{R = K(n-1)\cdot n}$$

$$\xRightarrow{n=2}$$

$$\Downarrow$$

$$R = 2K$$

$\uparrow$

Gauss curvature!

$\Downarrow$

note that

$$\underline{C^\mu{}_{\nu\rho\sigma} \equiv 0\ !}$$

Also $\underline{\nabla_\rho R^\mu{}_{\nu\alpha\beta} = 0!}$

Decomposition of Riemann

$$\boxed{R^{\mu\nu}{}_{\rho\sigma} = C^{\mu\nu}{}_{\rho\sigma} + \frac{4}{n-2}\,\delta^{[\mu}_{[\rho}\check{R}^{\nu]}{}_{\sigma]} + \frac{2}{n(n-1)}\,R\,\delta^{\mu}_{[\rho}\delta^{\nu}_{\sigma]}}$$

$\boxed{n \geq 3}$

this, when solved for $C^{\mu\nu}{}_{\rho\sigma}$ gives a definition of Weyl.

$$\Sigma:\begin{cases} k\,x^{0\,2} + g_{\mu\nu}x^{\mu}x^{\nu} = k r^2 \\ K = \frac{1}{kr^2} \end{cases} ; \quad (x^0, x^{\mu}) \in \mathbb{R}\times\mathbb{R}^n$$

$$\tilde{g} = k\,dx^{0\,2} + g_{\mu\nu}dx^{\mu}dx^{\nu}$$

$$y^{\mu} = \frac{2x^{\mu}}{1 - \frac{x^0}{r}}$$

Canonical metric on a hyperquadric Σ: $\Longrightarrow$

$$\boxed{\tilde{g}|_{\Sigma} = \frac{g_{\mu\nu}\,dy^{\mu}dy^{\nu}}{\left(1 + \frac{K}{4}g_{\mu\nu}y^{\mu}y^{\nu}\right)^2}} = g_{\mu\nu}\omega^{\mu}\omega^{\nu}$$

Its curvature is

$$\boxed{R^{\mu\nu}{}_{\rho\sigma} = 2K\,\delta^{\mu}_{[\rho}\delta^{\nu}_{\sigma]}}$$ in the coframe $\omega^{\mu} = \dfrac{dy^{\mu}}{1 + \frac{K}{4}g_{\mu\nu}y^{\mu}y^{\nu}}$

$$\Longrightarrow C^{\mu\nu}{}_{\rho\sigma} \equiv 0, \quad \check{R}^{\nu}{}_{\sigma} \equiv 0, \quad \frac{R}{n(n-1)} = K \quad \Longrightarrow \boxed{R = n(n-1)K}$$

Since $C^{\mu\nu}{}_{\rho\sigma}$, $\check{R}^{\nu}{}_{\sigma}$ vanish and K is const, this spaces have maximal group of symmetries (otherwise the group should preserve $\check{R}^{\nu}{}_{\sigma}$, $C^{\mu\nu}{}_{\rho\sigma}$ and would be reduced.)

(M,g) is Einstein $\equiv$ $\boxed{\begin{array}{c} \check{R}_{\mu\nu} \equiv 0 \\ \Updownarrow \text{Thm} \\ R_{\mu\nu} = \tilde{\Lambda} g_{\mu\nu} \\ \text{and } \tilde{\Lambda} = \text{const} \end{array}}$ or

$$\check{R}_{\mu\nu} = 0 \iff R_{\mu\nu} - \frac{1}{n} R g_{\mu\nu} = 0 \iff$$

$$R_{\mu\nu} = \frac{1}{n} R g_{\mu\nu}$$

note that 2^{nd} Bianchi ietentity gives:

$$\nabla_{[\mu} R^{\alpha}{}_{|\beta|\gamma\delta]} = 0$$

$$\nabla_\mu R^\alpha{}_{\beta\gamma\delta} + \nabla_\delta R^\alpha{}_{\beta\mu\gamma} + \nabla_\gamma R^\alpha{}_{\beta\delta\mu} = 0 \qquad \Big|_{\mu \nearrow \alpha}$$

$$\nabla_\mu R^\mu{}_{\beta\gamma\delta} + \nabla_\delta R_{\beta\gamma} - \nabla_\gamma R_{\beta\delta} = 0 \qquad \Big|_{\beta \nearrow \delta}$$

$$\nabla_\mu R^\mu{}_\gamma + \nabla_\mu R^\mu{}_\gamma - \nabla_\gamma R = 0.$$

$$\boxed{\nabla^\mu \left(R_{\mu\gamma} - \frac{1}{2} g_{\mu\gamma} R\right) = 0}$$

Key identity in the theory of Relativity

$$\boxed{G_{\mu\gamma} = R_{\mu\gamma} - \frac{1}{2} g_{\mu\gamma} R}$$ ← Einstein tensor.

If (M,g) is Einstein, then

$$\frac{1}{2}\nabla_\nu R = \nabla^\mu R_{\mu\nu} = \frac{1}{n} \nabla^\mu (R g_{\mu\nu}) = \frac{1}{n} \nabla_\nu R \qquad n \neq 2$$

$$\Rightarrow \nabla_\nu R = 0 \Rightarrow R = \text{const}.$$

$$R_{\mu\nu} = \tilde{\Lambda} g_{\mu\nu} \Rightarrow R = n\tilde{\Lambda} \Rightarrow \tilde{\Lambda} = \frac{1}{n}R \Rightarrow \check{R}_{\mu\nu} = 0.$$

$\square$.

Einstein equations

$$\boxed{G_{\mu\nu} = T_{\mu\nu}} \leftarrow \text{energy momentum of matter fields.}$$

Bianchi identity $\nabla^\mu G_{\mu\nu} \equiv 0$ gives

$\nabla^\mu T_{\mu\nu} \equiv 0$ — conservation of energy.

Einstein theory

(M, g), $\dim M = 4$, $p = 1$, $q = 3$

g satisfies $\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = T_{\mu\nu}}$

Einstein equations with cosmological constant.

cosmological constant. (dark energy)

Example of solutions:

take Σ_1 when $n = 4$, $g_{\mu\nu}$ of sign. $p=1$, $q=3$.

$$0 = \check{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R = G_{\mu\nu} + \frac{1}{4} R g_{\mu\nu}$$
$$= G_{\mu\nu} + 3K g_{\mu\nu}$$

$$\Lambda = 3K, \quad T_{\mu\nu} = 0.$$

$K = 0 \Rightarrow$ Minkowski space-time; $K \gtrless 0$ De Sitter spacetime / anti De Sitter spacetime.

Axioms for the Lie derivative

$X \in \mathfrak{X}(M)$

$$\mathcal{L}_X : \mathcal{T}(M) \longrightarrow \mathcal{T}(M)$$

such that

1) $\mathcal{L}_X$ is $\mathbb{R}$-linear
2) $\mathcal{L}_X$ preserves the type of tensor
3) $\mathcal{L}_X(K \otimes L) = \mathcal{L}_X K \otimes L + K \otimes \mathcal{L}_X L$
4) $\mathcal{L}_X$ commutes with contractions
5) $\mathcal{L}_X$ commutes with Alt.
6) on forms: • $\mathcal{L}_X$ is a derivation of degree 0.

$$\mathcal{L}_X(\tilde{\omega} \wedge \tilde{\tilde{\omega}}) = \mathcal{L}_X \tilde{\omega} \wedge \tilde{\tilde{\omega}} + \tilde{\omega} \wedge \mathcal{L}_X \tilde{\tilde{\omega}}$$

• $\mathcal{L}_X \alpha = X \lrcorner d\alpha + d(X \lrcorner \alpha)$

• $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$

in particular on functions: $\mathcal{L}_X f = X(f)$.

7) $\mathcal{L}_{X_1} \circ \mathcal{L}_{X_2} - \mathcal{L}_{X_2} \circ \mathcal{L}_{X_1} = \mathcal{L}_{[X_1, X_2]}$.

Isometries ; Killing equation

$$\varphi : (M,g) \xrightarrow{\text{diff}} (M,g) \quad \text{s.t.}$$

$\varphi^* g = g$ is called an isometry of (M,g).

φ_1, φ_2 two isometries, $\varphi_1 \circ \varphi_2$ is also an isometry, as well as φ_1^{-1}.

$\uparrow$

they form an isometry group G of (M,g).

(local) 1-parameter group of isometries

$$\varphi_t : (M,g) \to (M,g) \quad \text{isometries s.t.}$$

$$\varphi_t \circ \varphi_{t'} = \varphi_{t+t'}$$

We have

$$\varphi_t^* g = g \Rightarrow \mathcal{L}_X g = 0 \quad \text{where}$$

X a vector field with flow φ_t.

X_1, X_2 s.t. $\mathcal{L}_{X_1} g = 0 = \mathcal{L}_{X_2} g = 0 \Rightarrow \mathcal{L}_{[X_1,X_2]} g = 0$

since $\mathcal{L}_{[X_1,X_2]} = \mathcal{L}_{X_1} \circ \mathcal{L}_{X_2} - \mathcal{L}_{X_2} \circ \mathcal{L}_{X_1}$.

X s.t. $\mathcal{L}_X g = 0$ is called infinitesimal symmetry for g or Killing field.

$\boxed{\mathcal{L}_X g = 0}$ $\leftarrow$ Killing equation

Example

$$g = g_{\mu\nu}\, dx^\mu dx^\nu \qquad g_{\mu\mu} = \mathrm{diag}(1, -1, -1, \dots -1)$$

$$X = a^\rho \partial_\rho$$

$$\underset{X}{\mathcal{L}} g = \cancel{X(g_{\mu\nu})}\, dx^\mu dx^\nu + 2 g_{\mu\nu}\, \underset{X}{\mathcal{L}}(dx^\mu)\, dx^\nu =$$

$$= 2 g_{\mu\nu}\, d\, \underset{X}{\mathcal{L}}(x^\mu)\, dx^\nu = 2 g_{\mu\nu}\, d(a^\mu)\, dx^\nu =$$

$$= 2 g_{\mu\nu}\, a^\mu{}_{,\rho}\, dx^\rho dx^\nu =$$

$$= 2\, a_{\nu,\rho}\, dx^\rho dx^\nu = 0$$

$$a_{(\nu,\rho)} = 0 \qquad \text{or}$$

$$\partial_{(\rho} a_{\nu)} = 0$$

$$\Rightarrow \begin{cases} \partial_\sigma \partial_{(\rho} a_{\nu)} = 0 \\ \partial_{[\sigma} \partial_{\rho]} a_\nu = 0 \end{cases}$$

$$\Rightarrow \quad \partial_\sigma \partial_\rho a_\nu = 0 \Rightarrow$$

$$\partial_\rho a_\nu = B_{\rho\nu} = \text{const}$$

$$\partial_{(\rho} a_{\nu)} = 0 \Rightarrow \boxed{B_{\rho\nu} = -B_{\nu\rho}}$$

$$\underline{\underline{G = O(g) \rtimes \mathbb{R}^n}}$$

$$\boxed{a_\nu = \underset{\substack{\uparrow \\ \text{Lorentz transf}}}{B_{\rho\nu}} x^\rho + \underset{\substack{\uparrow \\ \text{translations}}}{C_\nu}}$$

$$\dim G = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$$

Maximal symmetry group $\Longrightarrow$

$$C^{\mu}{}_{\nu\rho\sigma} \equiv 0, \quad \check{R}^{\mu}{}_{\nu} = 0$$

$$\Longrightarrow \quad R^{\mu\nu}{}_{\rho\sigma} = \underbrace{\frac{R}{n(n-1)}}_{K} \left(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}\right)$$

$$\Omega^{\mu\nu} = K\,\theta^{\mu}\wedge\theta^{\nu}$$

Bianchi identity:

$$0 = D\Omega^{\mu\nu} = DK\wedge\theta^{\mu}\wedge\theta^{\nu} = dK\wedge\theta^{\mu}\wedge\theta^{\nu}$$

and if $n \geq 3 \Longrightarrow \underline{K = \text{const}}$

<u>Can we find all such (M,g)?</u>

First Cartan structure eqs:

$$\begin{aligned} &(A) \\ &(B) \end{aligned} \begin{cases} d\theta^{\mu} + \Gamma^{\mu}{}_{\nu}\wedge\theta^{\nu} = 0 \\ d\Gamma^{\mu\nu} + \Gamma^{\mu}{}_{\rho}\wedge\Gamma^{\rho\nu} = K\,\theta^{\mu}\wedge\theta^{\nu} \end{cases} \quad \underline{\underline{g_{\mu\nu}}}$$

these are satisfied on $\underline{M}$ so $\Gamma^{\mu}{}_{\rho}$ are linearly dependent on (θ^{μ}).

<u>Trick</u>: consider (A) (B) on an abstract $n + \frac{n(n-1)}{2}$ dimensional mfd P where θ^{μ} and $\Gamma^{\mu}{}_{\nu}$ are linearly independent.

assume that we have P

Set. $\dim P = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$

and $\quad \sigma^A = (\underbrace{\theta^\mu}_{n}, \underbrace{\Gamma^\mu{}_\nu}_{\frac{n(n-1)}{2}})$ is a coframe on P

satisfying (A) and (B)

note that if σ^A satisfies (A) and (B) then

$$d\sigma^A = -\frac{1}{2} C^A{}_{BC}\, \sigma^B \wedge \sigma^C$$

where $C^A{}_{BC}$ are <u>all</u> constants.

$$\Rightarrow \boxed{C^A{}_{BC} = -C^A{}_{CB}} \quad \text{and}$$

$d^2\sigma^A \equiv 0$ is equivalent to

$$\frac{1}{2}\left(C^A{}_{BC}\, C^B{}_{DE}\, \sigma^D \wedge \sigma^E \wedge \sigma^C - C^A{}_{BC}\, C^C{}_{DE}\, \sigma^B \wedge \sigma^D \wedge \sigma^E\right) = 0$$

$$\boxed{C^A{}_{B[C}\, C^B{}_{DE]} = 0}$$

$$[X_A, X_B] = C^C{}_{AB} X_C$$

$$[[X_A, X_B], X_C] + [[X_C, X_A], X_B] + [[X_B, X_C], X_A] =$$

$$C^E{}_{AB}\, C^D{}_{EC} + C^E{}_{CA}\, C^D{}_{EB} + C^E{}_{BC}\, C^D{}_{EA} = C^D{}_{EC}\, C^E{}_{AB} + C^D{}_{EB}\, C^E{}_{CA} + C^D{}_{EA}\, C^E{}_{BC} = 2\, C^D{}_{E[C}\, C^E{}_{AB]} = 0$$

Thus $C^A{}_{BC}$ are structure constants of a certain Lie algebra $\mathfrak{g}$ of dimension $\frac{n(n+1)}{2}$.

Lie algebra is totally determined by $C^A{}_{BC}$. Hence by K.

If K
- > 0 all such $\mathfrak{g}$ are isomorphic to $\underline{o}(p+1, q)$
- $= 0$ all such $\mathfrak{g}$ are isomorphic to $\underline{o}(p,q) \oplus \mathbb{R}^n$
- < 0 all such $\mathfrak{g}$ are isomorphic to $\underline{o}(p, q+1)$

where $p+q=n$, and g has sign. $(\underset{\text{plus}}{p}, \underset{\text{minus}}{q})$

P is locally a Lie group $G = \begin{cases} SO(p+1,q) & K>0 \\ SO(p,q) \rtimes \mathbb{R}^n & K=0 \\ SO(p,q+1) & K<0 \end{cases}$

and σ^A are left invariant forms on G.

Left invariant forms on Lie groups are easy to find so we have <u>all</u> solutions to (A), (B) on $P = G$.

How to reconstruct (M,g) having $P=G$?

Maurer - Cartan form

$$\theta_{MC} = g^{-1}dg \qquad\qquad g \in G$$

is left invariant, and decomposing it onto the basis of Lie algebra $\mathfrak{g}$ of G

$(e_\mu, e_{\alpha\beta})$ we have

$$\theta_{MC} = g^{-1}dg = \theta^\mu e_\mu + \Gamma^{\mu\nu} e_{\mu\nu}$$

which gives us

$$\sigma^A = (\theta^\mu, \Gamma^{\mu\nu}) \quad \text{which solve (A), (B) on } P=G$$

<u>Now</u>: Let $(X_\mu, Y_{\mu\nu})$ be a basis of vector fields on P dual to $(\theta^\mu, \Gamma^{\mu\nu})$.

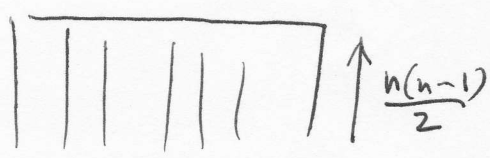
Observe that

$$d\theta^\mu \wedge \theta^1 \wedge \dots \wedge \theta^n = 0 \qquad \forall \mu = 1, \dots n$$

$\Rightarrow$ Frobenius theorem says that

$G=P$ is foliated by the leaves of INTEGRBLE distribution spanned by $Y_{\mu\nu}$

P



Consider

$$\hat{g} = g_{\mu\nu}\theta^\mu\theta^\nu \quad \text{where}$$

$$g_{\mu\nu} = \mathrm{diag}(\underbrace{1, \dots 1}_{p}, \underbrace{-1, \dots -1}_{q}).$$

This is a <u>degenerate</u> symmetric bilinear form on P and degeneracy occurs precisely in $\frac{n(n-1)}{2}$ directions spanned by $Y_{\mu\nu}$.

$$\underset{Y_{\mu\nu}}{\mathcal{L}} \tilde{g} = ?$$

$$\underset{Y_{\mu\nu}}{\mathcal{L}} \left(g_{\alpha\beta} \theta^\alpha \theta^\beta \right) = 2 g_{\alpha\beta} \theta^\alpha \underset{Y_{\mu\nu}}{\mathcal{L}} \theta^\beta$$

$$\underset{Y_{\mu\nu}}{\mathcal{L}} \theta^\beta = Y_{\mu\nu} \lrcorner d\theta^\beta + d(\cancel{Y_{\mu\nu} \lrcorner \theta^\beta})$$

$$\overset{(A)}{=} - Y_{\mu\nu} \lrcorner \left(\Gamma^{\beta\rho} \wedge \theta_\rho \right) = \frac{1}{2} Y_{\mu\nu} \lrcorner \left[\left(\Gamma^{\rho\beta} - \Gamma^{\beta\rho} \right) \wedge \theta_\rho \right] =$$

$$= \frac{1}{2} \left(\delta^\rho_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\rho_\nu \right) \theta_\rho =$$

$$= \delta^{[\rho}_\mu \delta^{\beta]}_\nu \theta_\rho$$

$$\underset{Y_{\mu\nu}}{\mathcal{L}} \left(g_{\alpha\beta} \theta^\alpha \theta^\beta \right) = 2 \theta_\beta \theta_\rho \ \delta^{[\rho}_\mu \delta^{\beta]}_\nu = 0 .$$

symmetric.

Thus $\tilde{g}$ descends to the leaf space of the foliation $M = P/\sim$. and g is nondegenerate there.

$$M \overset{\imath}{\hookrightarrow} P, \qquad g = \imath^* \tilde{g}$$

$\Rightarrow g$ satisfies (A) (B) on M. !

Spherically symmetric vacuum Ricci flat spacetimes.

- If $\dim M = 4$, and g has Lorentzian signature $(1,3)$ then (M,g) is called spacetime.
- If Ricci tensor of such g vanishes the spacetime is called vacuum.
- Spacetime is stationary if (M,g) has a timelike Killing vector (field).

1) $g(X,X)>0$

2) $\underset{X}{\mathcal{L}} g = 0,$

Condition 2) locally

if X is a Killing vector, introduce a coordinate system around $p \in M$ s.t. $X = \frac{\partial}{\partial x^0}$ (for timelike)

So around p we have coordinates $(x^0, x^i) = (x^\mu)$ and the metric is

$$g = g_{\mu\nu} dx^\mu dx^\nu$$

$$\underset{X}{\mathcal{L}} g = X(g_{\mu\nu}) dx^\mu dx^\nu + 2 g_{\mu\nu} \underset{X}{\mathcal{L}}(dx^\mu) dx^\nu =$$

$$= \frac{\partial g_{\mu\nu}}{\partial x^0} dx^\mu dx^\nu + 0 = 0 \Rightarrow \boxed{\frac{\partial g_{\mu\nu}}{\partial x^0} = 0}$$

Corollary if X is a timelike Killing vector, then around each point $p \in M$ one can introduce a coord. system s.t. $X = \partial_0$, $g = g_{\mu\nu} dx^\mu dx^\nu$ and $g_{\mu\nu} = g_{\mu\nu}(x^i)$ and do not depend on x^0.

- Spacetime is static iff it is stationary and the orthogonal complement $X^\perp = \{ Y \in TM : g(X,Y) = 0 \}$ of the timelike Killing vector is integrable as a vector distribution.

In the static case we can choose coordinates (x^0, x^i) s.t. $x^0 = \text{const}$ gives the leaves of the foliation of $X^\perp$.

Thus, in such coordinate system,

$$g\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^i}\right) = 0 \text{. But } g\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^i}\right) = g_{0i} \text{, hence:}$$

$$\boxed{\begin{array}{l} g = g_{00}(dx^0)^2 + g_{ij}\, dx^i dx^j \\ g_{00} > 0, \quad g_{ij} \text{ - negative-definite}, \quad \frac{\partial g_{00}}{\partial x^0} = 0, \quad \frac{\partial g_{ij}}{\partial x^0} = 0. \end{array}}$$

local form of the metric for a static spacetime.

- Spacetime is spherically symmetric if $SO(3)$ is an isometry group, with orbits being 2-dimensional submanifolds with topology of a 2-sphere,

Now: Killing vectors are X_1, X_2, X_3 s.t.

$$[X_1, X_2] = X_3, \quad [X_3, X_1] = X_2, \quad [X_2, X_3] = X_1$$

Locally: there exists a coordinate system

$$\underbrace{x^0, x^1}_{\substack{\downarrow \\ \text{labels the orbits}}}, \underbrace{x^2, x^3}_{\substack{\downarrow \\ \text{coordinates on each orbit.}}}$$

On 2-dimensional orbits acts $SO(3)$ group.

$n=2 \Rightarrow \frac{n(n+1)}{2} = 3 \left(= \dim SO(3)\right)$ so these 2-dimensional orbits <u>must</u> be spaces of constant curvature. But the topology of orbits is a topology of $S^2 \Rightarrow K > 0$.

So we can take x^2, x^3 to be (θ, φ) so that the metric on each orbit is

$$-r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right).$$

Thus we have coordinates $(x^0, x^1, \theta, \varphi)$ and Killing vectors

$$X_1 = -\sin\varphi \frac{\partial}{\partial\theta} - \cos\varphi \operatorname{ctg}\theta \frac{\partial}{\partial\varphi}$$
$$X_2 = \cos\varphi \frac{\partial}{\partial\theta} - \sin\varphi \operatorname{ctg}\theta \frac{\partial}{\partial\varphi}$$
$$X_3 = \frac{\partial}{\partial\varphi}$$

Check $[X_i, X_j] = \epsilon_{ijk} X_k$!

Imposing $\underset{X_i}{\mathcal{L}} g = 0 \quad \forall i = 1, 2, 3$ on

$$g = g_{A\mu}\, dx^A dx^\mu - r^2(x^A, \theta, \varphi)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

we get the most general form of spherically symmetric metric in the form

$$\boxed{g = g_{AB}(x^C)\, dx^A dx^B - r^2(x^A)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)}$$
$$A, B, C = 0, 1.$$

Three cases:

1) $\partial_\mu r \partial^\mu r < 0$

2) $\partial_\mu r \partial^\mu r > 0$

3) $\partial_\mu r \partial^\mu r = 0$ $\begin{cases} \partial_\mu r \neq 0 \\ \partial_\mu r = 0 \end{cases}$ analyze these!

Ad 1

Let $x^1 = r$

$$g = g_{00}\, dx^{0\,2} + 2 g_{01}\, dx^0 dr + g_{11}\, dr^2 + \dots$$

$x^1 = r$

$x^0 = x^0(r, t)$ $\qquad x^0_t \neq 0$

$$dx^0 = x^0_r\, dr + x^0_t\, dt$$

$$g = g_{00}\, x^{0\,2}_t dt^2 + 2\underbrace{\left(g_{00} x^0_r x^0_t + g_{01} x^0_t\right)}_{\overset{!}{=}0} dr\,dt + \left(g_{11} + 2 g_{01} x^0_r\right) dr^2 + \dots$$

$g_{00} \neq 0$

$g_{00} x^0_r = -g_{01} \Rightarrow$ Can solve for $x^0 = x^0(r,t)$

but $0 > g^{\mu\nu} \partial_\mu r \partial_\nu r = g^{11} = \dfrac{g_{00}}{\det(g_{AB})}$

So I can make this 0 !

But $\det g_{AB} < 0$ since the signature is $+---$

$\Rightarrow$ $\underline{\underline{g_{00} > 0}}$

$\Rightarrow$ spacetime is spherically symmetric + case 1)
then locally there exists a coordinate system (t, r, θ, φ) such that

$$\boxed{g = e^{2\mu(r,t)}dt^2 - e^{2\nu(r,t)}dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\varphi^2)}$$

Curvature

Orthonormal frame:

$$(*) \begin{cases} \theta^0 = e^{\mu}dt \\ \theta^1 = e^{\nu}dr \\ \theta^2 = r\, d\theta \\ \theta^3 = r \sin\theta \, d\varphi \end{cases} \qquad g = g_{\mu\nu}\theta^\mu\theta^\nu \quad \text{and} \quad g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Homework 14 november,

1) Find connection 1-forms $\Gamma_{\mu\nu}$ s.t.

$$d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu = 0$$

$$\Gamma_{\mu\nu} + \Gamma_{\nu\mu} = 0 \qquad \Gamma_{\mu\nu} = g_{\mu\alpha}\Gamma^\alpha{}_\nu$$

2) Calculate Ricci tensor in the coframe θ^μ as in $(*)$

3) Find all μ, ν s.t. $R_{\mu\nu} = 0$.

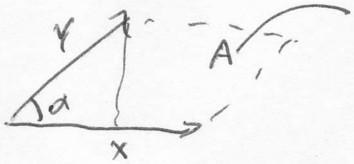
Hint. At the end of the integration procedure redefine t coordinate so that the metric does not depend on time!

Sectional curvature

(M,g) $\qquad g$ - Riemannian signature.

$p \in M$ and let $\sigma \subset T_pM$ ← 2-dimensional vector subspace.

$X, Y \in T_pM$ s.t. $\mathrm{Span}(X,Y) = \sigma$.



$$A(X,Y) = |X||Y|\sin\alpha = |X \wedge Y|$$

$$|X\wedge Y|^2 = |X|^2|Y|^2(1-\cos^2 a) =$$
$$= |X|^2|Y|^2 - |X|^2|Y|^2\cos a =$$
$$= g(X,X)g(Y,Y) - g(X,Y)^2$$

$$|X\wedge Y|^2 = g(X,X)g(Y,Y) - g(X,Y)^2 > 0 \text{ if } X, Y \text{ linearly indep.}$$

Def

Given $\sigma \subset T_pM$ and $X, Y \in T_pM$ s.t. $\sigma = \mathrm{span}(X,Y)$

a sectional curvature of σ at p is a reall number

$$K(X,Y) = \frac{g(R(Y,X)X,Y)}{|X\wedge Y|^2} = -\frac{g(R(X,Y)X,Y)}{|X\wedge Y|^2}$$

Fact K depends only on σ and <u>not</u> on the choice of the basis X, Y in it.

$$K(X,Y) = K(\sigma) \quad \text{where } \sigma = \mathrm{Span}(X,Y).$$

Proof

the following transformations:

$$(X,Y) \longmapsto (Y,X)$$
$$(X,Y) \longmapsto (\lambda X,Y)$$
$$(X,Y) \longmapsto (X+\lambda Y,Y)$$

by iterations generate the most general linear transformation in 2-dimensions.

But $K(X,Y)$ is unchanged by any of these $\square$.

Thm

Knowledge of $K(X,Y)$ for all $X,Y \in TM$ determines the curvature $R(X,Y)$.

Proof

Given $K(X,Y)$ we know that it is determined by R - which is the curvature of g.

Suppose that there exist R', a tensor

$$R' : V\times V\times V \longrightarrow V \quad \text{s.t.} \qquad R' \neq R$$

$$R'(X,Y)Z + R'(Z,X)Y + R'(Y,Z)X = 0$$
$$R'(X,Y) = -R'(Y,X)$$
$$g(R'(X,Y)Z,T) = -g(R'(X,Y)T,Z)$$
$$g(R'(X,Y)Z,T) = g(R'(Z,T)X,Y)$$

and for which $K(X,Y) = \dfrac{g(R(Y,X)X,Y)}{|X\wedge Y|^2} = \dfrac{g(R'(Y,X)X,Y)}{|X\wedge Y|^2}$

for all $X,Y \in TM$. This means that

$$g(R(X,Y)X,Y) = g(R'(X,Y)X,Y) \qquad \forall X,Y \in TM$$

We denote by $(X,Y,Z,T) = g(R(X,Y)Z,T)$

and $(X,Y,Z,T)' = g(R'(X,Y)Z,T)$.

and we have

$$(X,Y,X,Y) = (X,Y,X,Y)' \text{ by our assumption.}$$

$$\Rightarrow \quad (X+Z, Y, X+Z, Y) =$$

$$= (X,Y,X,Y) + 2(X,Y,Z,Y) + (Z,Y,Z,Y)$$

$$= (X,Y,X,Y)' + 2(X,Y,Z,Y)' + (Z,Y,Z,Y)'$$

hence $\quad (X,Y,Z,Y) = (X,Y,Z,Y)'$

$$\Rightarrow (X, Y+T, Z, Y+T) = (X,Y,Z,Y) + (X,Y,Z,T) + (X,T,Z,Y) + (X,T,Z,T)$$

$$= (X,Y,Z,Y)' + (X,Y,Z,T)' + (X,T,Z,Y)' + (X,T,Z,T)'$$

$$\Rightarrow \quad (X,Y,Z,T) + (X,T,Z,Y) = (X,Y,Z,T)' + (X,T,Z,Y)'$$

$$A = \boxed{(X,Y,Z,T) - (X,Y,Z,T)' = (Y,Z,X,T) - (Y,Z,X,T)'}$$

cyclic permutation

$$\sigma A = (Z,X,Y,T) - (Z,X,Y,T)' = (X,Y,Z,T) - (X,Y,Z,T)' = A$$

$$\sigma^2 A = (Y,Z,X,T) - (Y,Z,X,T)' = (Z,X,Y,T) - (Z,X,Y,T)' = \sigma A = A$$

$$0 = A + \sigma A + \sigma^2 A = 3A \Rightarrow A = 0 \Rightarrow \boxed{(X,Y,Z,T) = (X,Y,Z,T)'}$$

Which means that $R = R'$ $\square$.

What are spaces of constant $\overbrace{\text{curvature}}^{\text{SECTIONAL}}$?

$$K_0 = \frac{-(X,Y,X,Y)}{|X\wedge Y|^2} \implies (X,Y,X,Y) = -K_0 |X\wedge Y|^2$$

Define R' by:

$$\underbrace{g\left(R'(X,Y)Z,T\right)}_{\displaystyle \parallel \atop \displaystyle (X,Y,Z,T)'} = K_0\left[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)\right]$$

It satisfies all the four symmetries of Riemann.

Then:

$$(X,Y,X,Y)' = K_0\left(g(Y,X)g(X,Y) - g(X,X)g(Y,Y)\right) =$$

$$= -K_0 |X\wedge Y|^2$$

$$\implies (X,Y,X,Y)' = (X,Y,X,Y) \qquad \forall X,Y \in TM$$

$$\implies R = R'$$

$$R_{\mu\nu\rho\sigma} X^\rho Y^\sigma Z^\nu T^\mu = K_0\left[g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}\right]$$

$$\implies \boxed{R_{\mu\nu\rho\sigma} = K_0\left(g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}\right)}$$

$$\implies \begin{pmatrix}\text{spaces of constant} \\ \text{sectional} \\ \text{curvature}\end{pmatrix} \iff \begin{pmatrix}\text{Weyl} \equiv 0 \\ \text{Traceless Ricci} \equiv 0\end{pmatrix}$$

$\square$.

(1) Homogeneity of geodesics

$$\left.\begin{array}{l}\frac{DX}{dt}=0 \\ \\ p\in M\end{array}\right\}$$ then there exists an open set $V\subset M$ around p

and $\delta>0$, $\varepsilon_1>0$ ~~and~~ a map

$$E : \,]-\delta,\delta[\, \times \mathcal{U} \to M$$

with $\mathcal{U}=\{(q,v) : q\in V,\ v\in T_qM,\ |v|<\varepsilon_1\}$

s.t. a curve

$t \to E(t,q,v)$ $\quad t\in\,]-\delta,\delta[$ is

unique solution of $\frac{DX}{dt}=0$ passing through q at $t=0$ and for which $v\in T_qM$

$$\begin{cases} E(0,q,v)=q \\ \dot{E}(0,q,v)=v \end{cases}$$

Observe that if $\underline{E(t,q,v)}$ is a sol. for $\frac{DX}{dt}=0$,

then $\underline{E(st,q,v) \text{ also is.}}$

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)=0 \Rightarrow \frac{D}{d(st)}\frac{d\gamma}{d(st)}=0$$

$\underline{s\neq 0}$

Moreover $E(st,q,v)=E(t,q,sv)$

$$\dot{E}(st,q,v)\Big|_{t=0}=s\,\dot{E}(0,q,v)=sv$$

$$\Rightarrow \boxed{E(st,q,v)=E(t,q,sv).}$$

Using this we may make interval of definition for geodesic uniformly large.

For example $t \in]-2, 2[$:

$$(*) \quad \boxed{\left.\begin{array}{l} \dfrac{DX}{dt} = 0 \\ p \in M \end{array}\right\} \begin{array}{l} \text{there exists } V \subset M \text{ and } \varepsilon > 0 \text{ and a map} \\ \quad E : \underline{\underline{]-2,2[}} \times \mathcal{U}' \longrightarrow M \\ \mathcal{U} = \{(q, \omega) : q \in V, \omega \in T_q M, |\omega| < \varepsilon\} \end{array}}$$

<u>Indeed:</u>

$$E(t', q, v) \text{ was defined for } |t'| < \delta \text{ and } |v| < \varepsilon_1$$

$$\parallel$$

$$E\left(\frac{\delta t}{2}, q, v\right) = E\left(t, q, \frac{\delta}{2} v\right)$$

is defined for $|t| < 2$ and

$$\omega = \frac{\delta}{2} v \quad \text{s.t. } |\omega| < \frac{\delta \varepsilon_1}{2} = \varepsilon.$$

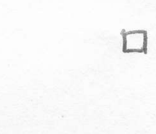
$\square$.

(2) <u>Exponential map</u>

$$\exp_q(v) = E(1, q, v) = E\left(|v|, q, \frac{v}{|v|}\right)$$

where $v \in \mathcal{U}$ as in $(*)$.

$$v \in B_\varepsilon(0) \subset T_q M$$



$$\exp_q : B_\varepsilon(0) \longrightarrow M$$

"Travel from q length equal to $|v|$ along geodesic with velocity $\frac{v}{|v|}$".

Calculate the differential of $\exp_q$:

$$d(\exp_q)_0(v) = (\exp_q)_{*0}(v) = \frac{d}{dt} \exp_q(tv)\Big|_{t=0} =$$

$$= \frac{d}{dt} E(1, q, tv)\Big|_{t=0} = \frac{d}{dt} E(t, q, v)\Big|_{t=0} = v$$

$$\Rightarrow \quad d(\exp_q)_0 = id\big|_{T_qM}$$

$\Rightarrow$ inverse function theorem says that $\exp_q$ is a diffeomorphism ~~in~~ in ~~a~~ a neighbourhood around 0 in T_qM

Prop

Given $q \in M$ there exists $\varepsilon > 0$ such that

$$\exp_q : B_\varepsilon(0) \to M \text{ is a diffeomorphism}$$

onto an open subset of M.

③ Terminology

If $\exp_q$ is a diffeomorphism of a neighbourhood V of 0 in T_qM, $\exp_q(V)$ is called normall neighbourhood of q in M.

If $B_\varepsilon(0)$ in T_qM is such $\overline{B_\varepsilon(0)} \subset V$ then $\exp_q B_\varepsilon(0) = B_\varepsilon(q)$ is called normal ball (or geodesic ball) with center in q and radius ε.

normal coordinates in $\exp_q V \subset M$

$e = (e_\mu)$ basis in $T_q(M)$

$$e : T_q(M) \longrightarrow \mathbb{R}^n$$
$$X = X^\mu e_\mu \longmapsto (X^\mu)$$

$$x_e : \exp_q(V) \longrightarrow \mathbb{R}^n$$

$$x_e = e \circ \exp_q^{-1} : p \longrightarrow x_e(p).$$

$$p(t) = E(t, q, X) = E(1, q, tX) = \exp_q(tX)$$

$$x_e(t) = tX^\mu$$

So in normal coordinates equation for geodesics read

$$x_e(t) = tX^\mu$$

$$\Gamma^\mu_{\nu\rho}(tX^\alpha)$$

$$0 = \frac{d^2 x_e^\mu}{dt^2} + \Gamma^\mu_{\nu\rho}\, \dot{x}_e^\nu \dot{x}_e^\rho = \Gamma^\mu_{\nu\rho}\, X^\nu X^\rho$$

$$\Rightarrow \boxed{\Gamma^\mu_{\nu\rho}(q) + \Gamma^\mu_{\rho\nu}(q) = 0}$$

4 <u>Example</u> Lobachevski metric

$$g = \frac{dx^2+dy^2}{y^2} = \theta^{1\,2} + \theta^{2\,2} = g_{\mu\nu}\theta^\mu\theta^\nu \qquad g_{\mu\nu} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\theta^1 = \frac{dx}{y}, \quad \theta^2 = \frac{dy}{y}$$

$$d\theta^1 = \frac{dx \wedge dy}{y^2} = \theta^1 \wedge \theta^2 = -\cancel{\Gamma^1{}_1\theta^1} - \Gamma^1{}_2\theta^2 \Rightarrow \Gamma^1{}_2 = -\theta^1 + \alpha\theta^2 = \Gamma_{12}$$

$$d\theta^2 = 0 = -\Gamma^2{}_1\theta^1 - \cancel{\Gamma^2{}_2\theta^2} \Rightarrow 0 = \Gamma_{12}\theta^1 = \alpha\theta^2\theta^1$$

$$\Rightarrow \alpha = 0$$

$$\Rightarrow \quad \Gamma_{ij}: \quad \text{only } \boxed{\Gamma_{12} = -\theta^1} \qquad \Gamma_{11} = \Gamma_{22} = 0$$

$$\Omega_{ij} = d\Gamma_{ij} + 0$$

$$\Rightarrow \text{only } \underline{\Omega_{12} = d\Gamma_{12} = -d\theta^1 = -\theta^1\wedge\theta^2}$$

$\Downarrow$ space of constant curvature

$$\underline{\underline{K=-1}} \quad !$$

<u>Geodesics:</u>

Tangent vector $V = V^\mu X_\mu$. The frame $(X_1, X_2) = (y\partial_x, y\partial_y)$.

The geodesic equation in this coframe is:

$$\frac{dV^\mu}{dt} + \Gamma^\mu{}_{\nu\rho}V^\nu V^\rho = 0$$

$$V^\mu = \begin{pmatrix}A\\B\end{pmatrix} \Rightarrow \left.\begin{aligned} &\frac{dA}{dt} + \Gamma^1{}_{21}BA = 0 \\ &\frac{dB}{dt} + \Gamma^2{}_{11}A^2 = 0 \end{aligned}\right\} \Rightarrow \begin{cases} \frac{dA}{dt} - AB = 0 \\ \frac{dB}{dt} + A^2 = 0 \end{cases}$$

Solving:

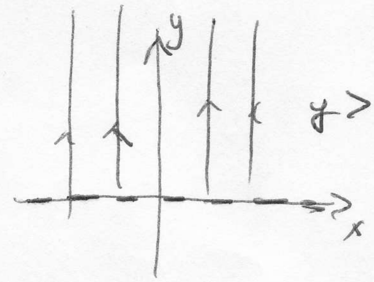
$$\begin{cases} \dfrac{dA}{dt} - AB = 0 \\[2ex] \dfrac{dB}{dt} + A^2 = 0 \end{cases}$$

(1) $A = 0$, $B = \text{const.}$

$$V = A\cancel{X_1} + BX_2 = \alpha y \partial_y$$

The curve tangent to V:

$$\left.\begin{aligned} \frac{dx}{dt} &= 0 \\ \frac{dy}{dt} &= \alpha y \end{aligned}\right\} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} x_0 \\ y_0 e^{\alpha t} \end{pmatrix}$$



$y > 0$

(2) $A \neq 0 \Rightarrow B = (\log A)^{\cdot}$

$$\boxed{(\log A)^{\cdot\cdot} + A^2 = 0} \xrightarrow[\substack{\text{equation} \\ C = \log A}]{\text{nice}} \boxed{\ddot{C} + e^{2C} = 0}$$

$$\begin{cases} A = \dfrac{\alpha}{\cosh \alpha(t - t_0)} \\[2ex] B = -\alpha \tanh(\alpha(t - t_0)) \end{cases}$$

$$V = AX_1 + BX_2 = \frac{\alpha}{\cosh \alpha(t-t_0)} y(t)\partial - \alpha \tanh(\alpha(t-t_0)) y(t) \partial_y$$

The curve:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\alpha y(t)}{\cosh \alpha(t-t_0)} \\[2ex] \dfrac{dy}{dt} = (-\alpha \tanh \alpha(t-t_0))\, y(t) \Rightarrow \boxed{y(t) = \dfrac{y_0'}{\cosh \alpha(t-t_0)}} \end{cases}$$

$$\frac{dx}{dt} = \frac{\alpha y_0'}{\cosh^2 \alpha(t-t_0)} \Rightarrow x(t) = y_0' \tanh \alpha(t-t_0) + x_0'$$

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} y_0' \tanh \alpha(t-t_0) + x_0' \\ \dfrac{y_0'}{\cosh \alpha(t-t_0)} \end{pmatrix}$$

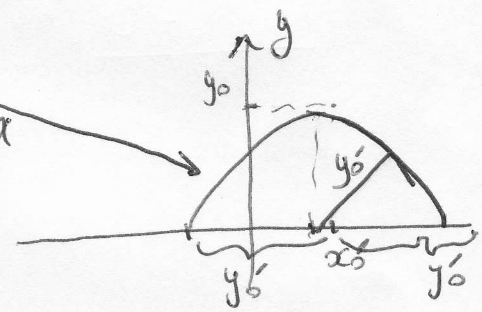
$$\begin{cases} x = y_0' \tanh \alpha(t-t_0) + x_0' \Rightarrow y_0'^2 \tanh^2 \alpha(t-t_0) = (x-x_0')^2 \\ y = \dfrac{y_0'}{\cosh \alpha(t-t_0)} \end{cases}$$

$$y^2 = \frac{y_0'^2}{\cosh^2 \alpha(t-t_0)} = y_0'^2 \frac{\cosh^2 \alpha(t-t_0) - \sinh^2 \alpha(t-t_0)}{\cosh^2 \alpha(t-t_0)} =$$

$$= y_0'^2 (1 - \tanh^2 \alpha(t-t_0))$$

$$\Rightarrow \boxed{y^2 + (x-x_0')^2 = y_0'^2}$$

$y > 0$



semicircles centered at $(x_0', 0)$ of radius y_0'

Two kinds of geodesics:

$$-y_0' \tanh\alpha t_0 + x_0' = x_0$$

$$\frac{y_0'}{\cosh\alpha t_0} = y_0$$

$$y_0' = y_0 \cosh\alpha t_0$$

$$x_0' = x_0 + y_0 \cosh\alpha t_0 \tanh\alpha t_0 =$$

$$= x_0 + y_0 \sinh\alpha t_0$$

$$\begin{cases} x(t) = y_0 \cosh\alpha t_0 \tanh\alpha(t-t_0) + y_0 \sinh\alpha t_0 + x_0 \\ y(t) = \dfrac{y_0 \cosh\alpha t_0}{\cosh\alpha(t-t_0)} \end{cases}$$

$$V_x = \frac{dx}{dt}\Big|_{t=0} = \frac{\alpha y_0}{\cosh\alpha t_0} \Rightarrow \boxed{\cosh\alpha t_0 = \frac{\alpha y_0}{V_x}}$$

$$V_y = \frac{dy}{dt}\Big|_{t=0} = \alpha y_0 \tanh\alpha t_0 \Rightarrow \tanh\alpha t_0 = \frac{v_y}{\alpha y_0}$$

$$\boxed{\sinh\alpha t_0 = \frac{v_y}{v_x}}$$

$$x(t) = y_0 \cosh\alpha t_0 \frac{\tanh\alpha t - \tanh\alpha t_0}{1 - \tanh\alpha t \tanh\alpha t_0} + y_0 \frac{v_y\, \alpha y_0}{\alpha y_0\, v_x} + x_0$$

$$\boxed{= y_0^2 \frac{\alpha}{v_x} \frac{\tanh\alpha t - \frac{v_y}{\alpha y_0}}{1 - \frac{v_y}{\alpha y_0}\tanh\alpha t} + y_0 \frac{v_y}{v_x} + x_0}$$

$$\alpha^2 y_0^2 - v_y^2 = v_x^2$$

$$\boxed{v_y^2 + v_x^2 = \alpha^2 y_0^2}$$

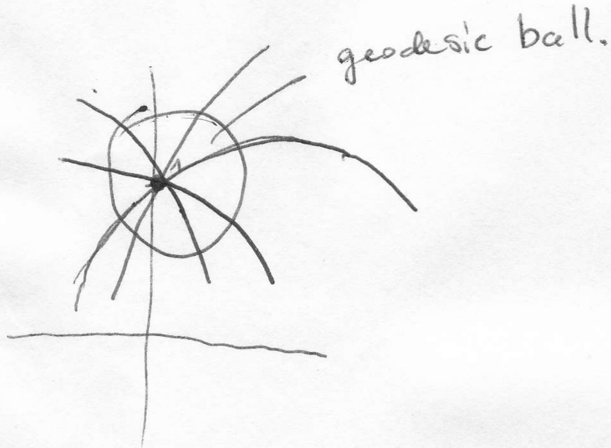
$$\frac{v_y}{\alpha y_0} = \sqrt{1 - \frac{v_x^2}{\alpha^2 y_0^2}}$$

$$y(t) = \frac{y_0^2 \frac{\alpha}{v_x}}{\cosh\alpha t \frac{\alpha y_0}{v_x} - \sinh\alpha t \frac{v_y}{v_x}} = \boxed{\frac{y_0}{\cosh\alpha t - \frac{v_y}{\alpha y_0}\sinh\alpha t}}$$

$$\cancel{\exp_{\binom{x_0}{y_0}}\binom{v_x}{v_y} = \left(y_0^2 \frac{\alpha}{v_x} \frac{\tanh\alpha - \frac{v_y}{\alpha y}}{1 - \frac{v_y}{\alpha y_0}\tanh\alpha} \right.}$$

$$\exp_{\binom{x_0}{y_0}}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{v_x^2+v_y^2}\, x_0 + (v_x y_0 - v_y x_0)\tanh \frac{\sqrt{v_x^2+v_y^2}}{y_0}}{\sqrt{v_x^2+v_y^2} - v_y \tanh \frac{\sqrt{v_x^2+v_y^2}}{y_0}} \\[2em] \dfrac{y_0 \sqrt{v_x^2+v_y^2}}{\sqrt{v_x^2+v_y^2}\cosh \frac{\sqrt{v_x^2+v_y^2}}{y_0} - v_y \sinh \frac{\sqrt{v_x^2+v_y^2}}{y_0}} \end{pmatrix}$$

$$\exp_{\binom{x_0}{y_0}}\begin{pmatrix} v_x \\ v_y \end{pmatrix} \simeq \begin{pmatrix} x_0 + v_x + \dfrac{v_x v_y}{y_0} + \dots \\[1em] y_0 + v_y + \dfrac{v_y^2}{2y_0} - \dfrac{v_x^2}{2y_0} + \dots \end{pmatrix}$$



Jacobi fields

$f(t,s)$ - a 1-parameter smooth family of geodesics in (M,g)

given $s=s_0$ $\quad t \to \gamma_{s_0}(t) = f(t,s_0)$ is an affinely parametrized geodesic in M

$$\Sigma = \{ M \ni p : \; p = f(t,s), \; 1 \le t \le 1, \; -\varepsilon \le s \le \varepsilon \}$$

Two vector fields: $u = \frac{\partial}{\partial t}$ and $J = \frac{\partial}{\partial s}$ on Σ!

We have

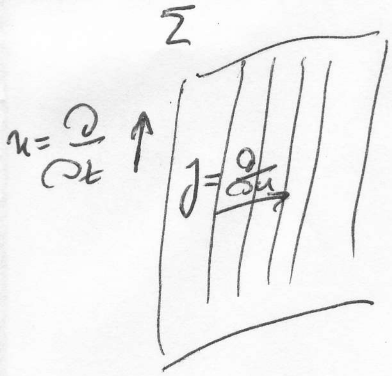
$$\nabla_u u = 0 \quad \text{since } \gamma_{s_0}(t) \text{ is geodesic for each } s=s_0$$

$$0 = \nabla_J \nabla_u u \underset{\uparrow}{=} \nabla_u \nabla_J u - R(u,J)u =$$

$$\nabla_u \nabla_J u - \nabla_J \nabla_u u - \cancel{\nabla_{[u,J]}} u = R(u,J)u$$

$$\underset{\uparrow}{=} \nabla_u^2 J - R(u,J)u$$

$$\nabla_J u - \nabla_u J - \cancel{[J,u]} = 0$$



$$\boxed{\nabla_u^2 J - R(u,J)u = 0}$$ Jacobi equation (geodesic deviation equation)

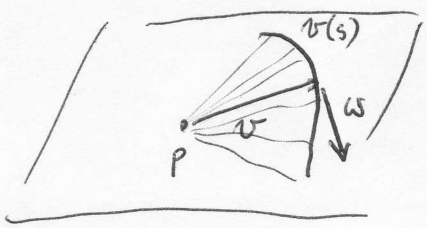
Def
A vector field J along a geodesic $\gamma : [0,a] \to M$ which satisfies

$$\frac{D^2 J}{dt^2} - R(\gamma', J)\gamma' = 0$$

is called a Jacobi field.

Let $v \in T_pM$ be such that $\exp_p(v)$ is defined.

$T_p(M)$



Let $w \in T_v(T_p(M))$

Consider a curve

$s \mapsto v(s) \in T_p(M)$ s.t.

$$\begin{cases} v(0) = v \\ \left.\dfrac{dv}{ds}\right|_{s=0} = w \end{cases} \qquad -\varepsilon < s \le \varepsilon$$

and a surface

$$\Sigma = \{ M \ni f(t,s) = \exp_p(t\,v(s)) \}_{0 \le t \le 1}$$

We are in the previous situation since $p(t, s_0)$ is a geodesic.

$$(\exp_p)_{*v}(w) = \left.\frac{d}{ds}\right|_{s=0} \exp_p(v(s)) = \frac{\partial f}{\partial s}(1,0)$$

$$\parallel$$

$$(d\exp_p)_v(w)$$

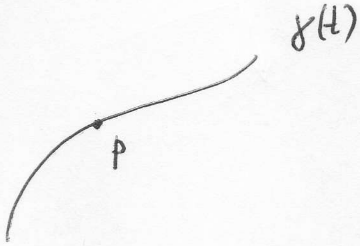
More generally:

$$(d\exp_p)_{tv}(tw) = \left.\frac{d}{ds}\right|_{s=0} \exp_p(t\,v(s)) = \frac{\partial f}{\partial s}(t,0)$$

$$\Rightarrow \boxed{\frac{D^2}{dt^2}\frac{\partial f}{\partial s} - R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t} = 0}$$

Local expression

$\gamma(t)$ – geodesic s.t. $\gamma(0)=p$



Choose an orthonormal frame $X_\mu(0)$ at p and propagate it parallelly along $\gamma(t)$.

$\Rightarrow$ frame $X_\mu(t)$ along $\gamma(t)$.

If $J(t)$ is a Jacobi field then

$$J(t) = J^\mu(t)\, X_\mu(t) \quad \text{and}$$

$$\frac{D^2 J}{dt^2} = \ddot{J}^\mu(t)\, X_\mu(t) \qquad \text{since} \quad \frac{DX_\mu}{dt} = 0.$$

$$\gamma'(t) = \frac{dx^\mu}{dt} X_\mu(t)$$

$$R(\gamma'(t), J(t))\gamma'(t) = R^\mu{}_{\nu\rho\sigma}\, \dot{x}^\rho J^\sigma \dot{x}^\mu X_\mu =$$

$$= a^\mu{}_\sigma(t)\, J^\sigma(t)\, X_\mu(t)$$

where $a^\mu{}_\sigma(t) = R^\mu{}_{\nu\rho\sigma}(t)\, \dot{x}^\nu(t)\, \dot{x}^\rho(t)$

$$\Rightarrow \qquad \boxed{\ddot{J}^\mu(t) = a^\mu{}_\sigma(t)\, J^\sigma(t)}$$

is a linear second order system for the unknowns $J^\mu(t)$.

$\Rightarrow$ $\left(J^\mu(0), \dot{J}^\mu(0)\right)$ initial conditions

$\Rightarrow$ <u>2n linearly independent solutions</u>! of class C^∞ on $[0,a]$.

Note that $J_1 = \gamma'(t)$ and $J_2 = t\gamma'(t)$ are Jacobi fields!

$$\begin{cases} J_1 \text{ is nonvanishing and } \frac{DJ_1}{dt} \equiv 0 \\ J_2(0) = 0 \qquad\qquad \frac{DJ_2}{dt}(0) = \gamma'(0) \end{cases} \Rightarrow \begin{array}{l} J_1 \text{ and } J_2 \\ \text{are linearly} \\ \text{independent.} \end{array}$$

It is sufficient to look for $2n-2$ linearly independent solutions which are orthogonal to $\gamma'(t)$.

<u>Example</u> Jacobi fields on manifolds of constant curvature.

We can always use an affine parameter such that $|\gamma'(t)| = 1$

$$\uparrow$$
<u>arc length</u>

Take J s.t. $g(\gamma'(t), J(t)) = 0$, $J(t) \neq 0$.

$$R(\gamma'(t), J(t))\gamma'(t) = R^{\mu}{}_{\nu\rho\sigma}(t)\dot{x}^{\nu}(t)\dot{x}^{\rho}(t)J^{\sigma}(t)X_{\mu}(t)$$

$$R^{\mu}{}_{\nu\rho\sigma}(t) = K\left(\delta^{\mu}_{\rho}\, g_{\nu\sigma} - \delta^{\mu}_{\sigma}\, g_{\nu\rho}\right)$$

$$R(\gamma'(t), J(t))\gamma'(t) = K\left(\dot{x}^{\mu} g(\gamma', J) - J^{\mu}|\gamma'|^2\right)X_{\mu} =$$

$$= -K \cdot J$$

$\Rightarrow$ Jacobi equation:

$$\boxed{\frac{D^2 J}{dt^2} + K J = 0} \qquad K = \text{const.}$$

Let $\omega(t)$ be parallel along $\gamma(t)$ and such that

$$g(\omega(t), \gamma'(t)) = 0, \qquad g(\omega(t), \omega(t)) = 1$$

$J(t) = j(t)\,\omega(t)$ and

$$\frac{d^2 j}{dt^2} + K j = 0 \quad \Rightarrow$$

$$J(t) = \begin{cases} \dfrac{\sin t\sqrt{K}}{\sqrt{K}}\,\omega(t) & \text{if } K>0 \\ t\,\omega(t) & \text{if } K=0 \\ \dfrac{\sinh t\sqrt{-K}}{\sqrt{-K}}\,\omega(t) & \text{if } K<0. \end{cases}$$

is a solution for the Jacobi equation satisfying

$$J(0) = 0, \quad J'(0) = \omega'(0).$$

Note that if $K>0$ there exists $t_0 = \frac{\pi}{\sqrt{K}}$ s.t.

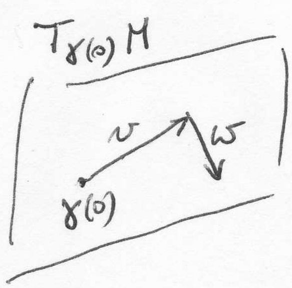
$$J(t_0) = J(0) = 0,$$

Such $t_0 \neq 0$ does not exist if $K \leq 0$!

Proposition

Let $\gamma : [0,a] \to M$ be a geodesic and let J be a Jacobi field along γ with $J(0)=0$. Let $\omega = \frac{DJ}{dt}\Big|_{t=0}$ and $v = \frac{d\gamma}{dt}\Big|_{t=0}$

Consider a curve $v(s)$ in $T_{\gamma(0)}$ s.t.



$$v(0) = av$$

$$\frac{dv}{ds}\Big|_{s=0} = w$$

and 2-dimensional surface in M given by

$$f(t,s) = \exp_{\gamma(0)}\left(\frac{t}{a} v(s)\right) \Rightarrow J(t) = \frac{\partial f}{\partial s}(t,0)$$

To prove it is enough to check the initial conditions.

γ - a geodesic s.t.

$\gamma(0)=p$

$\gamma'(0)=v$

Let $w \in T_v(T_pM)$ with $|w|=1$ and Jacobi field

$J(t) = (d\exp_{\gamma(0)})_{tv}(tw)$

Taylor expansion

$$\Rightarrow |J(t)|^2 = t^2 + \frac{1}{3} g(R(v,w)v,w)t^4 + \theta(t)$$

$$\frac{\theta(t)}{t^4} \xrightarrow[t\to 0]{} 0.$$

<u>Proof</u>

$J(0)=0$

$\frac{DJ}{dt}(0)=w$

$$\Rightarrow |J(0)|^2 = 0$$

$$(|J(0)|^2)' = g(J(t),J(t))'\big|_{t=0} = 2\,g(J(0),J'(0)) = 0$$

$$(|J(0)|^2)'' = 2\,g(J'(0),J'(0)) + 2\,g(J(0),J''(0)) = 2$$

$$(|J(0)|^2)''' = 6\,g(J'(0),J''(0)) + 2g(J(0),J'''(0)) = 0$$

$$J''(0) = R(\gamma',J)\gamma'(0) = 0$$

$$(\nabla^2_{\gamma'}J)(0) \; /\!/$$

Jacobi equation

$$J'''(0) = (\nabla^3_{\gamma'}J)(0) = \nabla_{\gamma'}(R(\gamma',J)\gamma')(0) = (R(\gamma',J')\gamma')(0)$$

since all the other terms from differentiation are zero since $J(0)=0$.

$$(J(0)^2)'''' = 8\,g(J'(0),J'''(0)) =$$

$$= 8\,g(R(v,w)v,w)$$

Taylor expansion $\Longrightarrow$

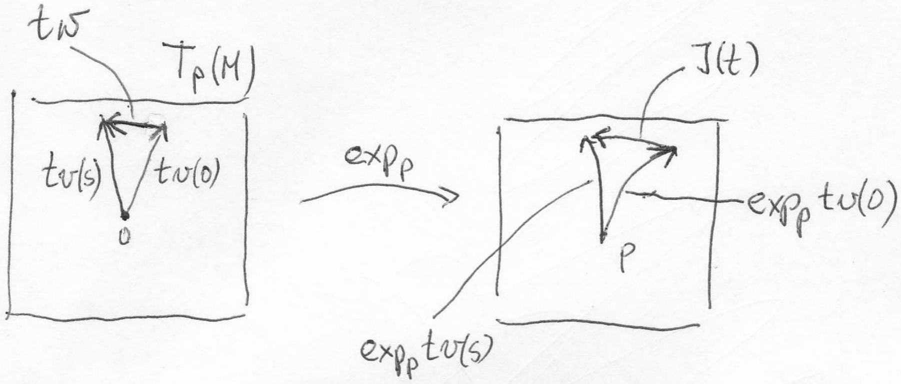
$$|J(t)|^2 = t^2 + \frac{8}{24}\,g(R(v,w)v,w)t^4 + \theta(t^5)$$

If γ is parametrized by arc length we have $|w|=1$ and g

$$\Rightarrow \quad |J(t)|^2 = t^2 - \frac{1}{3}K(p,\sigma)t^4 + O(t^5)$$

where $K(p,\sigma)$ is a sectional curvature at p w.r.t. the plane generated by v and w.

$$\Rightarrow \quad |J(t)| = t - \frac{1}{6}K(p,\sigma)t^3 + O(t^4)$$



$|tw| = t$ $\qquad$ $|J(t)| = t - \frac{1}{6}K(p,\sigma)t^3$

so if $K(p,\sigma) > 0$ geodesics are converging quicker than in $T_p(M)$

$K(p,\sigma) < 0$ — " - diverging —" — —" —

Conjugate points

Let $\gamma: [0,a] \to M$ be a geodesic.
The point $\gamma(t_0)$ is <u>conjugate</u> to $\gamma(0)$ along γ, $t_0 \in (0,a]$, iff there exists a Jacobi field J along γ, not identically equal to 0, with $J(0) = 0 = J(t_0)$. The moksimum number of such linearly independent Jacobi fields is called the <u>multiplicity</u> of the conjugate point $\gamma(t_0)$.

Corollary

If $\dim M = n$, the multiplicity of conjugate points never exceed number $n-1$.

Proof

Demanding $J(0) \overset{!}{=} 0$ we have n independent Jacobi fields $J_1, \dots J_n$ by setting initial condition for the first derivative $J_1'(0), J_2'(0), \dots, J_n'(0)$ to be linearly independent.

Among these n independent solutions there is $J(t) = t\gamma'(t)$ which satisfies $J(0) = 0$ but which never vanishes for $t \neq 0$

So we have <u>at most</u> $n-1$ independent solutions that may satisfy $J(0) = 0 = J(t_0)$ for some $t_0 \neq 0$ $\square$

Example

$\mathbb{S}^n = \{ x \in \mathbb{R}^{n+1} ; |x|^2 = 1 \}$

$\Rightarrow$ space of constant curvature with $K=1$.

$\Rightarrow$ $J(t) = (\sin t)\, w(t)$ where $w(t)$ is $\perp$ to geodesic (grate circle)

Conjugate point to $t=0$ is $t_0 = \pi$, which means that conjugate points are antipodal on $\mathbb{S}^n$.

Since on n-dimensional sphere we have $n-1$ linearly indep. vectors orthogonal to a given one (e.g. to the tangent vector to a geodesic) we see that on the sphere conjugate points has maximal multiplicity $= n-1$.

Conjugate locus

The set of first conjugate points to the point $p \in M$, for all the geodesics that start at p is called conjugate locus of p. We denote it by $C(p)$.

Example on $\mathbb{S}^n$ $C(p) =$ antipodal point to p on $\mathbb{S}^n$.

Usually, however, $C(p)$ is a curve on M with singular points on it. (See DoCarmo, p. 270, figure 4, for example)

Proposition

1) $q = \gamma(t_0)$ is a conjugate point to $p = \gamma(0)$ along a geodesic γ if and only if $v_0 = t_0\gamma'(0)$ is a critical point for $\exp_p$.

2) Moreover, the multiplicity of q is equal to the dimension of kernel of $(d\exp_p)_{v_0}$.

Proof

Ad 1) $J(0) = 0 = J(t_0)$. We set $v = \gamma'(0)$, $w = J'(0)$. Then

$$J(t) = (d\exp_p)_{tv}(tw).$$

If $w \neq 0$, what we assume, $J(t)$ is nonzero vector field along γ.

$$\Rightarrow 0 = J(t_0) = (d\exp_p)_{t_0 v}(t_0 w) \Leftrightarrow (d\exp_p)_{t_0 v} = 0$$

$\Rightarrow$ $t_0 v$ is critical point for $\exp_p$.

Ad 2) If we have k linearly independent $t_0 w$'s s.t.

$(d\exp_p)_{t_0 v}(t_0 w) = 0$ $\Rightarrow$ we can use each of them to get J s.t. $J(0) = 0 = J(t_0)$. In this way we obtain k linearly independent J's.

$\square$.

Prop

$$\left(\begin{array}{c} J \text{ - Jacobi field along } \gamma \\ \gamma \text{ - geodesic} \\ [a,a] \ni t \longrightarrow \gamma(t) \end{array}\right) \Rightarrow \left(\begin{array}{l} g(J(t), \gamma'(t)) = \\ = g(J'(0), \gamma'(0))\, t + g(J(0), \gamma'(0)) \\ \qquad\qquad \forall\, t \in [a,a] \end{array}\right)$$

Proof

$J'' = R(\gamma', J)\gamma'$. Note: $'$ on tensors mean $\nabla_{\gamma'}$. In particular on vectors it means $\frac{D}{dt}$.

$$g(J', \gamma')' = g(J'', \gamma') + \cancel{g(J', \tfrac{D\gamma'}{dt})} = g(R(\gamma', J)\gamma', \gamma') \underset{\uparrow}{=} 0$$

antisymmetry in $(\ldots \gamma', \gamma')$

$$\Rightarrow \quad g(J'(t), \gamma'(t)) = g(J'(0), \gamma'(0))$$

In addition:

$$\boxed{g(J, \gamma')' = g(J', \gamma') = g(J'(0), \gamma'(0))}$$

differential equation to $g(J, \gamma')$

$$\Rightarrow \quad g(J, \gamma') = g(J'(0), \gamma'(0))\,t + \underset{\substack{\parallel \\ g(J(0), \gamma'(0))}}{\text{const}}.$$

$\square$

In particular:

$$\boxed{\begin{array}{ll} \text{if} & g(J, \gamma')(t_1) = g(J, \gamma')(t_2) \\ \Rightarrow & g(J, \gamma') = \text{const.} \end{array}} \Rightarrow \boxed{\begin{array}{c} J(0) = J(a) = 0 \\ \Downarrow \\ g(J, \gamma') \equiv 0 \end{array}}$$

if a geodesic admit conjugate points $\Rightarrow$ the Jacobi field for which $J(0) = 0 = J(t_0)$ is always orthognal to γ.

Proposition

$\gamma : [0,a] \to M$ geodesic

$V_1 \in T_{\gamma(0)}M$, $V_2 \in T_{\gamma(a)}M$

$\gamma(a)$ is not conjugate to $\gamma(0)$

$\Longrightarrow$ there exists a unique Jacobi field along γ s.t. $J(0) = V_1$, $J(a) = V_2$.

Proof

Let $\mathcal{J}$ be the space of Jacobi fields J with $J(0) = 0$.

Define $\Theta : \mathcal{J} \to T_{\gamma(a)}M$ by

$\Theta(J) = J(a) \qquad J \in \mathcal{J}$

then Θ is injective

Since $\gamma(a)$ is not a conjugate point to $\gamma(0)$ (because $J(a) \neq 0$ for all $J \in \mathcal{J}$ s.t. $J \neq 0$ and Θ is linear).

$\dim \mathcal{J} = n = \dim T_{\gamma(a)}M \Rightarrow \Theta$ is an isomorphism.

$\Longrightarrow$ given $V_2 \in T_{\gamma(a)}M$ ~~there~~ exists unique $\bar{J}_1$ s.t. $\bar{J}_1(0) = 0$, $\bar{J}_1(a) = V_2$.

Reversing the argument, i.e. starting with a there exists unique $\bar{J}_2$ s.t. $\bar{J}_2(a) = 0$, $\bar{J}_2(0) = V_1$

take $J = \bar{J}_1 + \bar{J}_2$

$\square$

SO(1,2)

$$E_1 = \begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0&0&1\\ 0&0&0\\ 1&0&0 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 0&1&0\\ -1&0&0\\ 0&0&0 \end{pmatrix}$$

$$[E_1 E_2] = E_3, \quad [E_3 E_1] = E_2, \quad [E_2 E_3] = E_1$$

$$g = \exp(t_1 E_1)\exp(t_2 E_2)\exp(t_3 E_3)$$

$$\vartheta_{MC} = g^{-1} dg =$$

$$\overbrace{(dt_1 - \sinh t_2\, dt_3)}^{A_1} E_1 + \overbrace{(\cosh t_1\, dt_2 + \cosh t_2 \sinh t_1\, dt_3)}^{A_2} E_2$$

$$+ \overbrace{(\cosh t_1 \cosh t_2\, dt_3 + \sinh t_1\, dt_2)}^{A_3} E_3$$

$$A_3^2 - A_2^2 = \cosh^2 t_2\, dt_3^2 - dt_2^2$$

$$dA_1 = -A_2 A_3$$
$$dA_2 = A_1 A_3$$
$$dA_3 = A_1 A_2$$

$$\theta^1 = A_3, \quad \theta^2 = A_2 \qquad g = \theta^{1 2} - \theta^{2 2}$$

$$d\theta^1 = A_1 \theta^2 = -\Gamma^1{}_2 \theta^2 = -\Gamma_{12}\theta^2$$
$$d\theta^2 = A_1 \theta^1 = -\Gamma^2{}_1 \theta^1 = \Gamma_{21}\theta^1 = -\Gamma_{12}\theta^1 \implies -\Gamma_{12} = A_1$$

$$\Omega_{12} = d\Gamma_{12} = -dA_1 = A_2 A_3 = \theta^2 \theta^1 = -\theta^1 \wedge \theta^2$$

$$\underline{K = -1} \; !$$

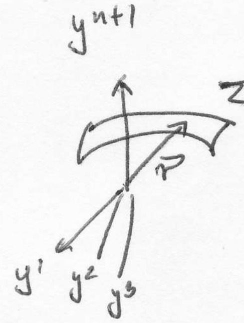
But also

$$g = A_1^2 + A_2^2 \qquad \text{is such that} \quad \mathcal{L}_{X_3} g = 0$$

$(X_3\ X_2\ X_1)$ dual to $A_3\ A_2\ A_1$

integrate X_3!

Hypersurfaces isometrically immersed in $\mathbb{R}^{n+1}$.



$$g = dy^{1^2} + dy^{2^2} + \dots + (dy^{n+1})^2 = (d\vec{r})^2$$

$$d\vec{r} = (dy^1, dy^2, \dots, dy^{n+1})$$

$$\Sigma = \{ \vec{r} = (y^1, \dots, y^{n+1}) \in \mathbb{R}^{n+1} : \vec{r} = \vec{r}(x^1, \dots x^n)$$
$$dx^1 \wedge \dots dx^n \neq 0 \}.$$

$$g|_\Sigma = |d\vec{r}(x^1, \dots x^n)|^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu}\theta^\mu\theta^\nu = \underbrace{\theta^{1^2} + \dots + \theta^{n^2}}_{\text{orthonormal frame}}$$

$$g_{\mu\nu} = \delta_{\mu\nu}$$

$(\theta^1, \dots, \theta^n)$ orthonormal frame on Σ

$(e_1, \dots e_n)$ dual frame on Σ.

We tautologically have:

$$(2) \quad \boxed{d\vec{r} = e_1(\vec{r})\theta^1 + e_2(\vec{r})\theta^2 + \dots + e_n(\vec{r})\theta^n} \quad (\text{e.g. } e_3 \lrcorner d\vec{r} = e_3(\vec{r}))$$

and let us denote $e_\mu(\vec{r})$ by $\vec{e}_\mu$:

$$\boxed{\vec{e}_\mu = e_\mu(\vec{r}).} \text{ — Vectors in } \mathbb{R}^{n+1} \quad !!!$$

Fact

$$\boxed{\vec{e}_\mu \cdot \vec{e}_\nu = \delta_{\mu\nu}.} \quad (1)$$

Proof

$$\theta^{1^2} + \dots + \theta^{n^2} = \delta_{\mu\nu}\theta^\mu\theta^\nu = (d\vec{r})^2 = (\vec{e}_\mu\theta^\mu)\cdot(\vec{e}_\nu\theta^\nu) = (\vec{e}_\mu \cdot \vec{e}_\nu)\theta^\mu\theta^\nu$$

$\square$.

Let $\vec{n} \in \mathbb{R}^{n+1}$ s.t. $\forall (x^1, \dots x^n) = \vec{r}$ $\quad \vec{n}\cdot\vec{e}_\mu = 0$ and $\vec{n}^2 = 1$.

Thus

$(\vec{e}_\mu, \vec{n})$ is an orthonormal basis in $\mathbb{R}^{n+1}$ at each point of Σ!

Cartan's lemma

Let $A_{\mu\nu}$, $\mu,\nu=1,\dots n$ be 1-forms on n-dimensional manifold M s.t. $A_{\mu\nu}=-A_{\nu\mu}$, Let θ^μ be a aframe on M.

If $\quad A_{\mu\nu}\wedge\theta^\nu=0\quad$ then $\quad A_{\mu\nu}=0$.

Proof

$$0=A_{\mu\nu}\wedge\theta^\nu=A_{\mu\nu\rho}\theta^\rho\wedge\theta^\nu=0 \implies A_{\mu[\nu\rho]}=0 \implies$$

$$-\begin{cases}\cancel{A_{\mu\nu\rho}}-A_{\mu\rho\nu}=0\\ A_{\rho\mu\nu}-\cancel{A_{\rho\nu\mu}}=0\\ \cancel{A_{\nu\rho\mu}}-\cancel{A_{\nu\mu\rho}}=0\end{cases} \Leftarrow \text{ but; } A_{\mu\nu\rho}=-A_{\nu\mu\rho}$$

$$2A_{\rho\mu\nu}=0 \implies A_{\rho\mu\nu}\theta^\nu=A_{\rho\mu}=0. \quad \square.$$

How the Levi-Civita connection $\Gamma_{\mu\nu}$ of the induced metric $g=\delta_{\mu\nu}\theta^\mu\theta^\nu$ look like?

Calculate $d\vec{e}_\mu$:

$$\boxed{d\vec{e}_\mu=b_\mu\vec{n}+\gamma_{\nu\mu}\vec{e}_\nu}$$

This defines 1-forms b_μ and $\gamma_{\mu\nu}$ on Σ.

Obviously

$$\boxed{\begin{cases}b_\mu=d\vec{e}_\mu\cdot\vec{n}\\ \gamma_{\nu\mu}=d\vec{e}_\mu\cdot\vec{e}_\nu\end{cases}}$$

and are totally determined by specifying $\vec{r}=\vec{r}(x^\mu)$ defining Σ.

Fact

$\gamma_{\mu\nu}=-\gamma_{\nu\mu}$.

Proof

$$\delta_{\mu\nu}=\vec{e}_\mu\cdot\vec{e}_\nu \implies 0=d\vec{e}_\mu\cdot\vec{e}_\nu+\vec{e}_\mu\cdot d\vec{e}_\nu=\gamma_{\mu\nu}+\gamma_{\nu\mu} \quad \square.$$

Of course $d\theta^\mu$ is decomposable onto $\theta^\nu \wedge \theta^\sigma$ and one can look for $\Gamma_{\mu\nu}$ s.t $\quad d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu = 0$, $\Gamma_{\mu\nu} = \delta_{\mu\rho}\Gamma^\rho{}_\nu$, $\Gamma_{\mu\nu} + \Gamma_{\nu\mu} = 0$.

We have:

Proposition

1) $\Gamma_{\mu\nu} = \gamma_{\mu\nu} = d\vec{e}_\mu \cdot \vec{e}_\nu$

2) $b_{\mu\nu} = b_{\nu\mu}$ where $b_\mu = d\vec{e}_\mu \cdot \vec{n} = b_{\mu\nu}\theta^\nu$.

Proof

We use (2) on Σ:

$$0 = d^2\vec{r} = d(\vec{e}_\mu \theta^\mu) = d\vec{e}_\mu \wedge \theta^\mu + \vec{e}_\mu d\theta^\mu =$$

$$= d\vec{e}_\mu \wedge \theta^\mu - \vec{e}_\mu \Gamma^\mu{}_\nu \wedge \theta^\nu = (b_\mu \vec{n} + \gamma_{\nu\mu}\vec{e}_\nu)\wedge\theta^\mu - \vec{e}_\nu \Gamma^\nu{}_\mu \wedge \theta^\mu =$$

$$= b_\mu \wedge \theta^\mu \vec{n} + (\gamma_{\nu\mu} - \Gamma^\nu{}_\mu)\wedge\theta^\mu \vec{e}_\nu =$$

$$= b_\mu \wedge \theta^\mu \cdot \vec{n} + (\gamma_{\nu\mu} - \Gamma_{\nu\mu})\wedge\theta^\mu \vec{e}_\nu$$

$$\Rightarrow \begin{cases} b_\mu \wedge \theta^\mu = 0 \\ \underbrace{(\gamma_{\nu\mu} - \Gamma_{\nu\mu})}_{\text{antisymmetric + Cartan's lemma}} \wedge \theta^\mu = 0 \end{cases} \Rightarrow b_{\mu\nu}\theta^\nu \wedge \theta^\mu = 0 \Rightarrow b_{[\mu\nu]} = 0$$

$\Downarrow$

$b_{\mu\nu}$ is symetric.

$$\Rightarrow \Gamma_{\nu\mu} = \gamma_{\nu\mu}$$

$\square$.

<u>Def</u>

Form $b = b_{\mu\nu}\,\theta^\mu \cdot \theta^\nu$ where $b_\mu = d\vec{e}_\mu \cdot \vec{n} = b_{\mu\nu}\theta^\nu$ is called 2nd fundamental form for Σ.

We have

$$\boxed{d\vec{e}_\mu = b_\mu \vec{n} + \Gamma_{\nu\mu}\vec{e}_\nu} \quad (3)$$

where $\Gamma_{\nu\mu} = d\vec{e}_\mu \cdot \vec{e}_\nu$ are the Levi-Civita connection 1-forms for $g|_\Sigma = \delta_{\mu\nu}\,\theta^\mu\theta^\nu$.

<u>Proposition</u>

$$\boxed{d\vec{n} = -b_\mu \vec{e}_\mu} \quad (4)$$

<u>Proof</u>

$\vec{n}^2 = 1 \Rightarrow d\vec{n}\cdot\vec{n} = 0$ $\quad (*)$

$\vec{e}_\mu \cdot \vec{n} = 0 \Rightarrow \underbrace{d\vec{e}_\mu \cdot \vec{n}}_{= b_\mu} = -\vec{e}_\mu \cdot d\vec{n}$ $\quad (**)$

$\Rightarrow \quad d\vec{n} = \alpha\vec{n} + \beta_\mu \vec{e}_\mu$ and $\alpha = 0$

$(**) \longrightarrow \beta_\mu = -b_\mu$ $\quad \square.$

<u>Compatibility conditions for</u>

$$\begin{cases} d\vec{e}_\mu = b_\mu \vec{n} + \Gamma_{\nu\mu}\vec{e}_\nu \\ d\vec{n} = -b_\mu \vec{e}_\mu \end{cases}$$

$$d^2\vec{e}_\mu = 0 = db_\mu \vec{n} + b_\mu \wedge b_\varrho \vec{e}_\varrho + d\Gamma_{\nu\mu}\vec{e}_\nu - \Gamma_{\nu\mu}\wedge(b_\nu \vec{n} + \Gamma_{\varrho\nu}\vec{e}_\varrho)$$

$$= (db_\mu + b_\nu \wedge \Gamma_{\nu\mu})\vec{n} + (d\Gamma_{\nu\mu} - \Gamma_{\varrho\mu}\wedge\Gamma_{\nu\varrho} + b_\mu \wedge b_\nu)\vec{e}_\nu$$

$$\Rightarrow \quad \boxed{\begin{array}{l} db_\mu + b_\nu \wedge \Gamma_{\nu\mu} = 0 \\ \Omega_{\nu\mu} = b_\nu \wedge b_\mu \end{array}} \quad \begin{array}{l} \leftarrow \text{Codazzi} \\ \leftarrow \text{Gauss} \end{array}$$

The second compatibility conditions:

$$d^2\vec{n} = 0 = -db_\mu \vec{e}_\mu + b_\mu \wedge (b_\mu \vec{n} + \Gamma_{\nu\mu}\vec{e}_\nu) =$$

$$= (-db_\mu + b_\nu \wedge \Gamma_{\mu\nu})\vec{e}_\mu + b_\mu \wedge b_\mu \vec{n}$$ satisfied iff G-Codazzi satisfied.

<u>Thm</u>

Let (M, g) be an n-dimensional Riemannian manifold. Let b be a bilinear symmetric form on M and let (θ^μ) be an orthonormal coframe for g, $g = \delta_{\mu\nu}\theta^\mu\theta^\nu$.
Then b can be a second fundamental form for an isometric immersion of g in $\mathbb{R}^{n+1}$ with the standard euclidean metric <u>provided</u> that

$$(G\text{-}C)\quad \begin{cases} db_\mu + b_\nu \wedge \Gamma_{\nu\mu} = 0 \\ \Omega_{\nu\mu} = b_\nu \wedge b_\mu \end{cases}$$

where $\Gamma_{\mu\nu}$ and $\Omega_{\nu\mu}$ are respective L.C. connection 1-forms and curvature 2-forms in coframe (θ^μ) and b_ν is defined by $\quad b = b_{\mu\nu}\theta^\mu\theta^\nu, \quad b_\nu = b_{\nu\mu}\theta^\mu.$

<u>Rmk.</u> If $n = 2$ conditions (G-C) are also sufficient.

If the Gauss-Codazzi equations are satisfied the equations to be solved to get the isometric immersion are:

$$\begin{cases} d\vec{e}_\mu = b_\mu \vec{n} + \Gamma_{\nu\mu}\vec{e}_\nu \\ d\vec{n} = -b_\mu \vec{e}_\mu \end{cases}$$

<u>Example</u>

Take $\boxed{g = dx^2 + 2\cos\omega\, dx\, dy + dy^2}$, (M) $\qquad \omega = \omega(x,y)$

Curvature:

$$g = (dx + \cos\omega\, dy)^2 - \cos^2\omega\, dy^2 + dy^2 = (dx + \cos\omega\, dy)^2 + \sin^2\omega\, dy^2$$

$$\begin{cases} \theta^1 = dx + \cos\omega\, dy \\ \theta^2 = \sin\omega\, dy \end{cases} \qquad g = \theta^{1\,2} + \theta^{2\,2}$$

$$\boxed{\begin{aligned} d\theta^1 &= -\omega_x \sin\omega\, dx\, dy = -\omega_x\, dx\, \theta^2 = -\omega_x \theta^1\theta^2 \\ d\theta^2 &= \omega_x \cos\omega\, dx\, dy = \omega_x \cos\omega\, \theta^1 \frac{1}{\sin\omega}\theta^2 = \omega_x \cot\omega\, \theta^1\theta^2 \end{aligned}}$$

$$d\theta^1 = -\Gamma^1{}_2\theta^2 = -\Gamma_{12}\theta^2 = -\omega_x\theta^1\theta^2$$

$$\Rightarrow \Gamma_{12} = \omega_x\theta^1 + \alpha\theta^2$$

$$d\theta^2 = -\Gamma^2{}_1\theta^1 = \Gamma_{12}\theta^1 = \alpha\theta^2\theta^1 = \omega_x \cot\omega\, \theta^1\theta^2$$

$$\Rightarrow \boxed{\Gamma_{12} = \omega_x(\theta^1 - \cot\omega\, \theta^2)} = \omega_x\, dx$$

$$\Omega_{12} = d\Gamma_{12} + 0 = d(\omega_x dx) = \omega_{xy}\, dy\, dx = -\frac{\omega_{xy}}{\sin\omega}\theta^1 \wedge \theta^2$$

<u>Proposition</u>

Metric (M) has curvature K if

$$\boxed{\omega_{xy} = -K\sin\omega} \qquad K = K(x,y).$$

In particular: (M) has scalar curvature equal to

$\boxed{K=-1}$ ASSUMED FROM NOW, ON

iff function ω satisfies 'sine-Gordon' equation!

$$\boxed{\omega_{xy}=\sin\omega}$$

When g as in (M) can be isometrically immersed in $\mathbb{R}^3$ with $b = 2b_{xy}\,dx\,dy$ being the second fundamental form?

$$b = 2b_{xy}(\theta^1-\cot\omega\,\theta^2)\frac{1}{\sin\omega}\theta^2$$

$$b_1 = \frac{b_{xy}}{\sin\omega}\theta^2$$

$$b_2 = \frac{b_{xy}}{\sin\omega}\theta^1 - 2\frac{b_{xy}}{\sin\omega}\cot\omega\,\theta^2$$

Gauss equation:

$$-\theta^1\wedge\theta^2 = \Omega_{12} = b_1\wedge b_2 = -\frac{b_{xy}^2}{\sin^2\omega}\theta^1\wedge\theta^2$$

$$\Rightarrow \quad \boxed{b_{xy}=\sin\omega}$$

Thus $\boxed{b = 2\sin\omega\,dx\,dy}$

Codazzi equations: $b_1=\theta^2,\quad b_2=\theta^1-2\cot\omega\,\theta^2$

$$db_1 = \omega_x\cot\omega\,\theta^1\theta^2$$

C.E.1 $\quad \parallel$

$$-b_2\wedge\Gamma_{21} = b_2\Gamma_{12} = (\theta^1-2\cot\omega\,\theta^2)\,\omega_x(\theta^1-\cot\omega\,\theta^2) = -\omega_x\cot\omega\,\theta^2\theta^1 \quad ✓$$

C.E.$_2$ $$d\ell_2 = -\omega_x \theta^1\theta^2 + 2\frac{\omega_x}{\sin^2\omega}\theta^1\theta^2 - 2\cot^2\omega\, \omega_x \theta^1\theta^2 =$$

$$= \omega_x \frac{-\sin^2\omega + 2 - 2\cos^2\omega}{\sin^2\omega}\theta^1\theta^2 = \omega_x\theta^1\theta^2$$

$$\parallel$$

$$-b_1\Gamma_{12} = -\theta^2\omega_x\theta^1$$

✓

Thus if
$$\boxed{\begin{array}{ll} g = dx^2 + 2\cos\omega\, dx\, dy + dy^2 & \\ b = 2\sin\omega\, dx\, dy & \omega_{xy} = \sin\omega \end{array}}$$

then g can be isometrically immersed in $\mathbb{R}^3$ with b being the II^{nd} fundamental form.

__Examples__ of solutions to
$$\begin{cases} d\vec{e}_\mu = b_\mu \vec{n} + \Gamma_{\nu\mu}\vec{e}_\nu \\ d\vec{n} = -b_\mu \vec{e}_\mu \end{cases}$$

__Homework__

Show that the surface in $\mathbb{R}^3$
$$z = -\sqrt{1-x^2-y^2} + \log\frac{1+\sqrt{1-x^2-y^2}}{\sqrt{x^2+y^2}}$$
correspond to a solution of the sine-gordon equation
$$\omega_{xy} = \sin\omega,$$
written in coordinates $x = \frac{\tau+\xi}{\sqrt{2}}$, $y = \frac{\tau-\xi}{\sqrt{2}}$, not depending on ξ, and vanishing when $\tau\to\infty$.

Immersing n-dimensional manifolds in $\mathbb{R}^{n+k}$.

$$g = dy^{1\,2} + \dots + dy^{n+k\,2} = (d\vec{r})^2$$

$$\Sigma_n = \{ \vec{r} = (y^1, \dots, y^{n+k}) \in \mathbb{R}^{n+k}, \quad \vec{r} = \vec{r}(x^1, \dots x^n)$$
$$dx^1 \wedge \dots \wedge dx^n \neq 0 \ \}.$$

$$g|_\Sigma = |d\vec{r}|^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = \delta_{\mu\nu}\theta^\mu\theta^\nu = \theta^{1\,2} + \dots + \theta^{n\,2}$$

$\{\theta^\mu\}$ orthonormal coframe, $\mu = 1, \dots n$

$\{e_\mu\}$ dual frame

$$d\vec{r} = e_\mu(\vec{r})\theta^\mu \qquad e_\mu(\vec{r}) = \vec{e}_\mu$$

$\vec{e}_\mu \cdot \vec{e}_\nu = \delta_{\mu\nu}$ - orthonormal vectors in $\mathbb{R}^{n+k}$

Let $(\vec{n}_a)$ $a = 1, \dots k$ be vectors in $\mathbb{R}^{n+k}$ which are

- normal to Σ at each point of Σ $\quad \vec{n}_a \cdot \vec{e}_\mu = 0$
- orthogonal to each other $\quad \vec{n}_a \cdot \vec{n}_b = 0$ $\quad a \neq b$
- unital $\quad \vec{n}_a^2 = 1 \quad \forall a = 1, \dots k$

$\Big\}\ \vec{n}_a \cdot \vec{n}_b = \delta_{ab}$

$(\vec{e}_\mu, \vec{n}_a)$ is orthonormal basis in $\mathbb{R}^{n+1}$ attached to each point of Σ!

$$\begin{cases} d\vec{e}_\mu = b_{\mu a}\vec{n}_a + \gamma_{\nu\mu}\vec{e}_\nu \\ d\vec{n}_a = \alpha_{ab}\vec{n}_b + \beta_{\mu a}\vec{e}_\mu \end{cases}$$

$$\Rightarrow \begin{cases} \gamma_{\nu\mu} = d\vec{e}_\mu \cdot \vec{e}_\nu \\ b_{\mu a} = d\vec{e}_\mu \cdot \vec{n}_a \\ \alpha_{ab} = d\vec{n}_a \cdot \vec{n}_b \\ \beta_{\mu a} = d\vec{n}_a \cdot \vec{e}_\mu \end{cases}$$

$$0 = d(\vec{e}_\mu \cdot \vec{e}_\nu) = \gamma_{\nu\mu} + \gamma_{\mu\nu} \quad \Rightarrow \quad \gamma_{\mu\nu} = -\gamma_{\nu\mu}$$

$$0 = d^2\vec{r} = d\vec{e}_\mu \theta^\mu + \vec{e}_\mu \, d\theta^\mu =$$

$$= (b_{\mu a}\vec{n}_a + \gamma_{\nu\mu}\vec{e}_\nu)\theta^\mu + \vec{e}_\mu \, (-\Gamma_{\mu\rho}\theta^\rho) =$$

$$= b_{\mu a}\theta^\mu \vec{n}_a + (\gamma_{\nu\mu} - \Gamma_{\nu\mu}) \wedge \theta^\mu \vec{e}_\nu$$

cartan's lemma.

$$\Rightarrow \quad b_{\mu a}\theta^\mu = 0 \quad \text{and} \quad \boxed{\gamma_{\nu\mu} = \Gamma_{\nu\mu}}$$

$$b_{\mu\rho a}\,\theta^\rho \wedge \theta^\mu = 0 \qquad\qquad b_{\mu a} = b_{\mu\rho a}\theta^\rho$$

$$\Rightarrow \quad \boxed{b_{\mu\rho a} = b_{\rho\mu a}}$$

$\boxed{b_a = b_{\mu\rho a}\theta^\mu\theta^\rho}$ ← second fundamental form along $\vec{n}_a$.

$$\boxed{d\vec{e}_\mu = b_{\mu a}\vec{n}_a + \Gamma_{\nu\mu}\vec{e}_\nu}$$

$b_{\mu a} \rightsquigarrow b_a$

↑ second fundamental form.

$$\vec{n}_a \cdot \vec{n}_b = \delta_{ab} \Rightarrow d\vec{n}_a \cdot \vec{n}_b + \vec{n}_a \cdot d\vec{n}_b = 0$$

$$\boxed{\alpha_{ab} = -\alpha_{ba}}$$

$$0 = d(\vec n_a \cdot \vec e_\mu) = d\vec n_a \vec e_\mu + \vec n_a d\vec e_\mu = b_{\mu a} + d\vec n_a \vec e_\mu = b_{\mu a} + \beta_{\mu a}$$

$$\beta_{\mu a} = -b_{\mu a}$$

$\Rightarrow$

$$\begin{cases} d\vec e_\mu = b_{\mu a}\vec n_a + \Gamma_{\nu\mu}\vec e_\nu \\ d\vec n_a = \alpha_{ab}\vec n_b - b_{\mu a}\vec e_\mu \end{cases} \qquad \begin{cases} b_{\mu a} = d\vec e_\mu \cdot \vec n_a \\ \alpha_{ab} = d\vec n_a \cdot \vec n_b \end{cases}$$

$$\alpha_{ab} = -\alpha_{ba}$$

$b_{\mu\nu a} = b_{\nu\mu a}$ where $b_{\mu a} = b_{\mu\nu a}\theta^\nu$

Compatibility:

$$0 = \underline{db_{\mu a}\vec n_a} - b_{\mu a}(\underline{\alpha_{ab}\vec n_b} - b_{\mu a}\vec e_\mu) +$$
$$+ d\Gamma_{\nu\mu}\vec e_\nu - \Gamma_{\nu\mu}(b_{\underline{\nu a}}\vec n_a + \Gamma_{\rho\nu}\vec e_\rho) =$$
$$= (db_{\mu a} - b_{\mu b}\alpha_{ba} + b_{\nu a}\Gamma_{\nu\mu})\vec n_a +$$
$$+ (d\Gamma_{\nu\mu} - \Gamma_{\rho\mu}\Gamma_{\nu\rho} + b_{\mu a}b_{\nu a})\vec e_\nu$$

$$\Rightarrow \boxed{\begin{aligned} db_{\mu a} + b_{\nu a}\Gamma_{\nu\mu} - b_{\mu b}\alpha_{ba} &= 0 \\ \Omega_{\nu\mu} &= b_{\nu a}\wedge b_{\mu a} \end{aligned}} \quad \begin{aligned} &\leftarrow \text{Codazzi} \\ &\leftarrow \text{Gauss} \end{aligned}$$

$$0 = \underline{d\alpha_{ab}\vec n_b} - \alpha_{ab}d\vec n_b - db_{\mu a}\vec e_\mu + b_{\mu a}d\vec e_\mu =$$
$$= d\alpha_{ab}\vec n_b - \alpha_{ab}(\alpha_{bc}\vec n_c - b_{\mu b}\vec e_\mu) +$$
$$+ (b_{\nu a}\Gamma_{\nu\mu} - b_{\mu b}\alpha_{ba})\vec e_\mu - b_{\mu a}(b_{\mu b}\vec n_b + \Gamma_{\nu\mu}\vec e_\nu)$$

$$d\alpha_{ab} - \alpha_{ac}\wedge\alpha_{cb} - b_{\mu a}\wedge b_{\mu b} = 0$$

$$\cancel{\alpha_{ab} b_{\mu b}} + \cancel{b_{\nu a}\Gamma_{\nu\mu}} - \cancel{b_{\mu b}\,\alpha_{ba}} - \cancel{b_{\nu a}\Gamma_{\mu\nu}} = 0$$

$$\Rightarrow \boxed{\begin{aligned} &\Omega_{\nu\mu} = b_{\nu a}\wedge b_{\mu a} \\ &d b_{\mu a} + b_{\nu a}\wedge\Gamma_{\nu\mu} - b_{\mu b}\wedge\alpha_{ba} = 0 \\ &d\alpha_{ab} - \alpha_{ac}\wedge\alpha_{cb} = b_{\mu a}\wedge b_{\mu b} \end{aligned}} \quad \begin{aligned} &\longleftarrow \text{Gauss} \\ &\longleftarrow \text{Codazzi} \\ &\longleftarrow \text{Ricci} \end{aligned}$$

α_{ab} are (sometimes) called torsions

<u>Thm</u> Schlaffli (Cartan).

Any analytic Riemannian manifold of dimension n can be locally isometrically embedded in a Riemannian ~~man~~ <u>flat</u> manifold of dimension $N \le \frac{n(n+1)}{2}$

If <u>flat</u> is replaced by Ricci flat then $N \le n+1$

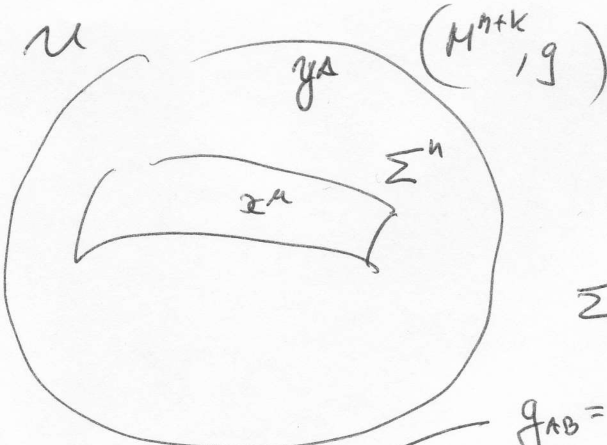
(Campbell 1926
A course of differential geometry
Clarendon Press Oxford)

1

L. 23

SB 12.02.2018

Local isometric embedding in (M^{n+k}, g).



y^A - local coordinates in $U \subset M^{n+k}$

$A = 1, 2, \ldots, n, \ldots, n+k$

$$g = g_{AB}\, dy^A dy^B \qquad g_{AB} = g_{AB}(y^C)$$

$$\Sigma^n = \{\, y^A = y^A(x^\mu),\ dx^1 \wedge \ldots \wedge dx^n \neq 0 \,\}$$

$g_{AB} = \ddot{g}_{AB}(x^\alpha)$

$$g|_\Sigma = \underbrace{g_{AB}\, y^A{}_{,\mu}\, y^B{}_{,\nu}}_{\tilde{g}_{\mu\nu}}\, dx^\mu dx^\nu = \tilde{g}_{\mu\nu}\, dx^\mu dx^\nu = \delta_{\mu\nu}\, \theta^\mu \theta^\nu$$

orthonormal frame.

$(\theta^1, \ldots \theta^n)$ orthonormal frame on Σ

$(e_1, \ldots e_n)$ dual frame.

$$dy^A = e_\mu(y^A)\theta^\mu = e_\mu{}^A \theta^\mu$$

$e_\mu(y^A) = e_\mu^A$ n-vectors at each point $x \in \Sigma^n$

These vectors have values at $T_x M^{n+k}$

$$\left.\begin{aligned} g_{AB}\, dy^A dy^B|_\Sigma &= e_\mu^A e_\nu^B \theta^\mu \theta^\nu g_{AB} \\ &\,\| \\ \delta_{\mu\nu}\theta^\mu \theta^\nu & \end{aligned}\right\} \Rightarrow \boxed{g_{AB}\, e_\mu^A e_\nu^B = \delta_{\mu\nu}}$$

or:

$$g(e_\mu, e_\nu) = \delta_{\mu\nu}$$

We supplement e_μ^A by n_a^A ~~a~~ so that (e_μ^A, n_a^B) is an orthonormal basis in $T_x M^{n+k}$ at each point $x \in \Sigma^n$.

We again have:

$$\boxed{\begin{cases} de_\mu{}^A = b_{\mu a} n_a{}^A + \gamma_{\nu\mu} e_\nu{}^A \\ dn_b{}^A = \alpha_{ab} n_a{}^A + \beta_{b\mu} e_\mu{}^A \end{cases} \quad \text{and} \quad dy^A = e_\mu{}^A \theta^\mu.}$$

We have to supplement this by the compatibility conditions $d^2 = 0$.

We in addition have:

1) $\boxed{d\theta^\mu + \Gamma_{\mu\nu} \wedge \theta^\nu = 0}$ and $\Gamma_{\mu\nu} = -\Gamma_{\nu\mu}$ are Levi-Civita connection 1-forms for $g|_\Sigma$ in frame θ^μ.

2) $$\left.\begin{array}{l} Dg_{AB} = dg_{AB} \mp \Gamma_{AB} \mp \Gamma_{BA} \\ \text{and} \\ d(dy^A) + \Gamma^A{}_B \wedge dy^B = 0 \end{array}\right\} \text{ in } \underline{M}$$

$$\Rightarrow \left.\begin{array}{l} dg_{AB} = \Gamma_{AB} + \Gamma_{BA} \\ \Gamma^A{}_B \wedge dy^B = 0 \end{array}\right\} \text{ in } M$$

restricting to Σ we have:

$$dg_{AB} = (\Gamma_{AB\rho} + \Gamma_{BA\rho}) \wedge \theta^\rho$$

$$e_\mu^B \Gamma^A{}_{B\rho} \theta^\rho \wedge \theta^\mu = 0$$

$$\Rightarrow \boxed{\Gamma^A{}_{B\rho} e^B_\mu = \Gamma^A{}_{B\mu} e^B_\rho} \quad \boxed{dg_{AB} = (\Gamma_{AB\rho} + \Gamma_{BA\rho}) \wedge \theta^\rho}$$

3) $$\begin{cases} g(e_\mu, e_\nu) = \delta_{\mu\nu} = g_{AB}\, e_\mu{}^A e_\nu{}^B \\ g(e_\mu, n_a) = 0 = g_{AB}\, e_\mu{}^A n_a{}^B \\ g(n_a, n_b) = 0 = g_{AB}\, n_a{}^A n_b{}^B \end{cases}$$

$$\begin{cases} b_{\mu a} = de_\mu^A\, n_a^B\, g_{AB} = g(de_\mu, n_a) \\ \gamma_{\nu\mu} = de_\mu^A\, e_\nu^B\, g_{AB} = g(de_\mu, e_\nu) \\ \alpha_{ab} = dn_b^A\, n_a^B\, g_{AB} = g(dn_b, n_a) \\ \beta_{a\nu} = dn_a^A\, e_\nu^B\, g_{AB} = g(dn_a, e_\nu) \end{cases}$$

<u>Closing the system:</u>

$$0 = d^2 y^A = \left(b_{\mu a} n^A{}_a + \gamma_{\nu\mu} e_\nu{}^A\right) \wedge \theta^\mu + e_\mu{}^A \left(-\Gamma_{\mu\nu} \wedge \theta^\nu\right)$$

$$\Rightarrow \boxed{\begin{array}{l} b_{\mu a} \wedge \theta^\mu = 0 \\ (\gamma_{\nu\mu} - \Gamma_{\nu\mu}) \wedge \theta^\nu = 0 \end{array}}$$

Since $b_{\mu a} = b_{\mu\nu a}\theta^\nu \Rightarrow b_{\mu\nu a}\theta^\nu \wedge \theta^\mu = 0 \Rightarrow b_{\mu\nu a} = b_{\nu\mu a}$

$\Rightarrow \boxed{b_a = b_{\mu\nu a}\theta^\mu \theta^\nu}$ second fundamental form.

<u>What we know about $\gamma_{\nu\mu}$?</u>

$$0 = d\delta_{\mu\nu} = dg_{AB}\, e_\mu{}^A e_\nu{}^B + g_{AB}\, de_\mu{}^A e_\nu{}^B + g_{AB}\, e_\mu{}^A de_\nu{}^B =$$

$$= dg_{AB}\, e_\mu{}^A e_\nu{}^B + \gamma_{\nu\mu} + \gamma_{\mu\nu}$$

$$dg_{AB}\, e_\mu{}^A e_\nu{}^B = \left(\Gamma_{AB\rho} + \Gamma_{BA\rho}\right) e^A{}_\mu\, e^B{}_\nu\, \theta^\rho$$

$$dg_{AB}\, e^A{}_\nu\, e^B{}_\mu = \left(\Gamma_{AB\rho} + \Gamma_{BA\rho}\right) e^A{}_\nu\, e^B{}_\mu\, \theta^\rho$$

REMARKS on terminology

6.24

12.09.2008

$$\Sigma^n \subset (M^{n+k}, g)$$

$(e_\mu{}^A, n_a{}^B)$ - orthonormal frame on Σ^n

$$de_\mu^A = b_{\mu a} n^A{}_a + \gamma_{\nu\mu} e_\nu{}^A.$$

$$b_{\mu a} = b_{\mu\nu a}\theta^\nu \qquad b_{\mu\nu a} = b_{\nu\mu a}$$

$$\Rightarrow \boxed{B^A = b_{\mu\nu a}\theta^\mu\theta^\nu \cdot n_a^A} \quad \text{second fundamental form.}$$

$\eta \in (T\Sigma)^\perp$, $X, Y \in T\Sigma$ $\qquad X = X^\mu e_\mu \quad Y = Y^\nu e_\nu$
$\eta = N_a n_a$

$$H_\eta(X,Y) = g(B(X,Y),\eta) = g_{AB} B^A(X,Y)\eta^B =$$

$$= g_{AB} b_{\mu\nu a} X^\mu Y^\nu n_a^A N_b n_b^B =$$

$$= b_{\mu\nu a} X^\mu Y^\nu N_a =$$

$$= (b_{\mu\nu a} Y^\nu N_a) X^\mu$$

$g(\cancel{S_\eta(X), Y})$

~~Observe that S_η defined on T_pM by~~

~~$S_\eta(X)$~~

Define

$$S_\eta : T_pM \longrightarrow TM \quad \text{by:}$$

$$[S_\eta(X)]_\mu = b_{\mu\nu a} X^\nu N_a e_\mu$$

$$g(S_\eta(X), Y) = H_\eta(X,Y) = g(S_\eta(Y), X)$$

S_η is a symetric operator on $T_p\Sigma$ for each $p \in \Sigma$

There exists an orthonormal basis in $T_p\Sigma$ s.t.

$$S_\eta(e_\mu) = \lambda_\mu e_\mu \qquad \lambda_\mu \text{ - real eigenvalues.}$$

no summation

(e_μ, n_a) and e_μ diagonalize S_η.

If $|\eta| = 1$ and Σ^n is hypersurface $(k=1)$ then we can take (e_μ, η) and η is unique if we want (e_μ, η) to agree with the orientation of $T_p M$

e_μ - are called principal directions

λ_μ - are called principal curvatures

invariants of the immersion

$\det S_\eta = \lambda_1 \dots \lambda_n$ Gauss-Kronecker curvature

$\frac{1}{n} \mathrm{Tr}\, S_n = \frac{\lambda_1 + \dots + \lambda_n}{n}$ mean curvature

~~Easy to see that if all λ_μ distinct then e_μ are unique.~~

Isoparametric hypersurfaces

$\Sigma^n \subset (M^{n+1}, g)$, where (M^{n+1}, g) is a space of constant curvature, is called ISOPARAMETRIC iff all its principal curvatures are constant.

Results:

1) $\boxed{M^{n+1} = \mathbb{R}^{n+1}} \Rightarrow$ isoparametric Σ has at most $\boxed{\text{two}}$ distinct principal curvatures

and must be an open subset of

a) hyperplane
or
b) hypersphere
or
c) spherical cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k}$

Levi-Civita for $n+1=3$ 1937
Segre for arbitrary n. 1938

2) $M^{n+1} = \mathbb{H}^{n+1} \Rightarrow$ isoparametric Σ has at most 2 distinct principal curvatures

~~and~~ must be either $\left\{ \begin{array}{l} \text{open subset of } \mathbb{S}^k \times \mathbb{H}^{n-k} \\ \text{or be totally umbilic.} \end{array} \right\}$

Cartan 1938

3) $M^{n+1} = \mathbb{S}^{n+1}$ more interesting situation!

Cartan 1938 found isoparametric $\Sigma^n \subset \mathbb{S}^{n+1}$

with 1, 2, 3 and 4 distinct principal curvatures.

Münzner : number g of distinct principal curvatures of an isoparametric hypersurface $\Sigma^n \subset \mathbb{S}^{n+1}$ can be 1, 2, 3, 4 or 6.

$g \leq 3$ Cartan:

$g=1 \Rightarrow \Sigma$ is a great or small sphere in S^{n+1}

$g=2 \Rightarrow \Sigma$ is a standard product of two spheres

$S^k(r) \times S^{n-k}(s) \subset S^{n+1}$

$g=3 \Rightarrow$ all the principal curvatures have to have the same multiplicity 1, 2, 4, or 8.

$g=6 \Rightarrow$ all have the same multiplicity $m=1$ or 2,

Cartan / Münzner:

Isoparametric hypersurface in S^{n+1} is given in terms of ~~as a level surface of a~~ a ~~function~~ polynomial

$$F : \mathbb{R}^{n+2} \longrightarrow \mathbb{R}$$

of degree g satisfying

$$|\nabla F|^2 = g^2 (x^{1\,2} + \dots + x^{n+2\,2})^{g-1}$$

$$\Delta F = \frac{m_2 - m_1}{2} g^2 (x^{1\,2} + \dots + x^{n+2\,2})^{\frac{g}{2}-1}$$

$g=4$
Cartan examples
with $m=1$ in S^5
$m=2$ in S^9

where m_1 and m_2 are multiplicities of principal curvatures, which either are all equal or there are only two different multiplicites.

Then $\Sigma^n = \left\{ x^i \in \mathbb{R}^{n+2} \text{ s.t. } \begin{matrix} F = \text{const} \\ x^{1\,2} + \dots + x^{n+2\,2} = 1 \end{matrix} \right\}$

Cartan

$$g=3 \Rightarrow \begin{cases} 1) & |\nabla F|^2 = 9\left(x^{1\,2}+\dots+x^{n+2\,2}\right)^2 \\ 2) & \Delta F = 0 \end{cases}, \qquad m_1 = m_2$$

$$F = F_{\mu\nu\rho}\, x^\mu x^\nu x^\rho$$

$$\nabla_\mu F = 3 F_{\mu\nu\rho}\, x^\nu x^\rho$$

$$|\nabla F|^2 = 9\, F_{\mu\nu\rho}\, x^\nu x^\rho\, F_{\mu\alpha\beta}\, x^\alpha x^\beta$$

$$\parallel$$

$$9 \quad g_{\nu\rho}\, x^\nu x^\rho\, g_{\alpha\beta}\, x^\alpha x^\beta$$

$$\Rightarrow \boxed{g^{\mu\sigma} F_{\mu(\nu\rho} F_{\alpha\beta)\sigma} = g_{(\nu\rho}\, g_{\alpha\beta)}} \quad 1)$$

$$\boxed{g^{\mu\nu} F_{\mu\nu\rho} = 0} \quad 2)$$

What are the dimensions $n+2$ in which a symmetric tensor with properties 1) and 2) exist?

Cartan

$n+2 = 5, 8, 14, 26.$

dim 5: $\quad A \in M_{3\times 3}(\mathbb{R})$ s.t. $\quad A^T = A, \quad \mathrm{Tr}\, A = 0$

Space of such matrices is a 5-dim. vector space $\simeq \mathbb{R}^5$

$$A = \begin{pmatrix} x^5 - \sqrt{3}\, x^4 & \sqrt{3}\, x^3 & \sqrt{3}\, x^2 \\ & x^5 + \sqrt{3}\, x^4 & \sqrt{3}\, x^1 \\ & & -2x^5 \end{pmatrix} \Rightarrow F = \tfrac{1}{2} \det A$$

Satisfies 1), 2) with $g=3$.

Why 5, 8, 14, 26 ?

Because

$\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

Take $A \in M_{3\times 3}(\mathbb{K})$ s.t. $A^{+} = A$, $\operatorname{Tr} A = 0$

$$n = 2 + 3 \cdot \begin{cases} 1 \\ 2 \\ 4 \\ 8 \end{cases}$$

$$F = \tfrac{1}{2} \det A$$

<u>Problem</u> define det for $A \in M_{3\times 3}(\mathbb{H})$
$M_{3\times 3}(\mathbb{O})$.

$n =$ as above

<u>Define</u>: $G \subset SO(n)$ by

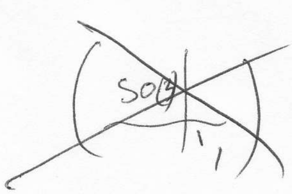
$$G \ni a \iff F(ax, ax, ax) = F(x, x, x).$$

Check that

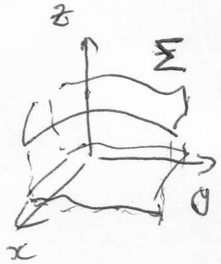
$$\left.\begin{array}{lcccc} G = & SO(3) & SU(3) & Sp(3) & F_4 \\ n = & 5 & 8 & 14 & 26 \end{array}\right\}$$ each group being in a n dimensional irreducible representation

this in particular means that

$SO(3)$ seats in a nonstandard way in $SO(5)$



Minimal surfaces in $\mathbb{R}^3$: Enneper-Weierstress formula



$z = u(x,y)$

$$g_{|\Sigma} = dx^2 + dy^2 + dz^2 =$$

$$= (1+u_x^2)dx^2 + 2u_x u_y\, dx\, dy + (1+u_y^2)dy^2$$

metric tensor
$$g_{\mu\nu} = \begin{pmatrix} 1+u_x^2 & u_x u_y \\ u_x u_y & 1+u_y^2 \end{pmatrix}$$

$$\sqrt{\det g_{\mu\nu}} = \sqrt{1+u_x^2+u_y^2}$$

$$S[u] = \int_u \sqrt{1+u_x^2+u_y^2}\; dx \wedge dy$$

$$\delta S = 0 \Rightarrow \Sigma \text{ is minimal.}$$

Euler-Lagrange equations: $\mathcal{L} = \sqrt{1+u_x^2+u_y^2}$

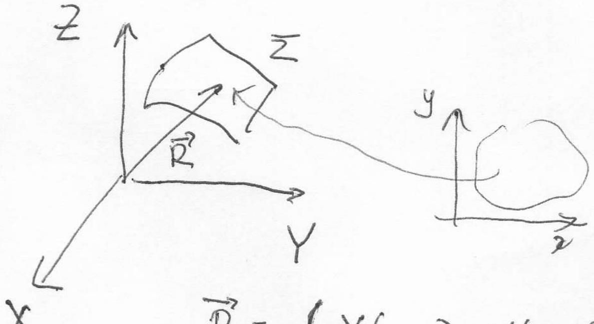
$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} + \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_y} - \frac{\partial \mathcal{L}}{\partial u} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\frac{u_x}{\sqrt{}} + \frac{\partial}{\partial y}\frac{u_y}{\sqrt{}} = 0$$

$$\rightsquigarrow \boxed{(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0} \quad *$$

Easy to see that $u = ax+by+c$ satisfies $*$, but ~~it~~ is difficult to find other solutions.

Another approach Weierstrass ~ 1865



$$\vec{R} = (X(x,y), Y(x,y), Z(x,y))$$

e.g. $X = x$, $Y = y$, $Z = u(x,y)$.

$$g = d\vec{R}^2 \quad ; \quad g|_\Sigma = (\vec{R}_x dx + \vec{R}_y dy)^2 =$$
$$= \vec{R}_x^2 dx^2 + 2\vec{R}_x \vec{R}_y dx\, dy + \vec{R}_y^2 dy^2.$$

$$\det g = \vec{R}_x^2 \cdot \vec{R}_y^2 - (\vec{R}_x \cdot \vec{R}_y)^2 = (\vec{R}_x \times \vec{R}_y)^2 = L^2$$

minimal condition:

$$\delta \int L \, dx \wedge dy = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\frac{\partial L}{\partial \vec{R}_x} + \frac{\partial}{\partial y}\frac{\partial L}{\partial \vec{R}_y} = 0$$

$$2\mathcal{L}\,\delta\mathcal{L} = \delta\mathcal{L}^2 \overset{\text{w.r.t. } \delta R_x}{=} \delta\left(\vec{R}_x \times \vec{R}_y\right)^2 =$$

$$= 2\left(\delta\vec{R}_x \times \vec{R}_y\right)\cdot\left(\vec{R}_x \times \vec{R}_y\right)$$

$$= 2\left(\vec{R}_y \times \left(\vec{R}_x \times \vec{R}_y\right)\right)\delta\vec{R}_x$$

$$(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a}$$

$$\Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\vec{R}_x} = \frac{1}{\mathcal{L}}\left(\vec{R}_y \times \left(\vec{R}_x \times \vec{R}_y\right)\right)$$

$$\Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\vec{R}_y} = \frac{1}{\mathcal{L}}\left(\vec{R}_x \times \left(\vec{R}_y \times \vec{R}_x\right)\right)$$

From the very begining I could have restricted my attention to coordinates (x,y) in which

$$g|_\Sigma = e^{2\varphi}\left(dx^2 + dy^2\right)$$ i.e. to the

coordinates in which $\vec{R}_x^{\,2} = \vec{R}_y^{\,2}$ and $\vec{R}_x \cdot \vec{R}_y = 0$.

$$\Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\vec{R}_x} = \vec{R}_x \quad , \quad \frac{\partial\mathcal{L}}{\partial\vec{R}_y} = \vec{R}_y$$

$$\Rightarrow \quad \boxed{\vec{R}_{xx} + \vec{R}_{yy} = 0}$$

$$+ \begin{cases} \vec{R}_x^{\,2} = \vec{R}_y^{\,2} \\ \vec{R}_x \cdot \vec{R}_y = 0 \end{cases}$$

$$\Rightarrow \boxed{\vec{R} = \mathrm{Re}\,\vec{F}(z)}$$

holomorphic in $z = x+iy$

+ conditions $\begin{cases} \vec{R}_x^{\,2} = \vec{R}_y^{\,2} \\ \vec{R}_x \cdot \vec{R}_y = 0 \end{cases}$

$$\vec{F} = \vec{R} + i\vec{S}$$

$$\vec{F}'(z) = \frac{d\vec{F}}{dz} = \frac{1}{2}(\partial_x - i\partial_y)\vec{F} =$$

$$= \frac{1}{2}(\vec{F}_x - i\vec{F}_y) = \frac{1}{2}(\vec{R}_x + i\vec{S}_x - i\vec{R}_y + \vec{S}_y)$$

$$\underset{\uparrow \text{ Cauchy-Riemann}}{=\!=\!=\!=} \frac{1}{2}(\vec{R}_x - i\vec{R}_y - i\vec{R}_y + \vec{R}_x) =$$

$\begin{cases} \vec{R}_x = \vec{S}_y \\ \vec{R}_y = -\vec{S}_x \end{cases}$

$$= \vec{R}_x - i\vec{R}_y$$

$$\vec{F}' \cdot \vec{F}' = (\vec{R}_x - i\vec{R}_y)^2 = \vec{R}_x^{\,2} - \vec{R}_y^{\,2} - 2i\vec{R}_x\vec{R}_y = 0$$

$\vec{F}'$ must be a holomorphic complex NULL vector in $\mathbb{C}^3$

$$\vec{F}' = (X, Y, Z)$$

$$A = \begin{pmatrix} Z & X - iY \\ X + iY & -Z \end{pmatrix} \Rightarrow \det A = -Z^2 - X^2 - Y^2 = -\vec{F}'^2 = 0$$

$$\Rightarrow A = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} (\psi_1, \psi_2) = \begin{pmatrix} \varphi_1\psi_1 & \varphi_1\psi_2 \\ \varphi_2\psi_1 & \varphi_2\psi_2 \end{pmatrix}$$

$$\begin{cases} Z = \varphi_1\psi_1 = -\varphi_2\psi_2 \\ X - iY = \varphi_1\psi_2 \\ X + iY = \varphi_2\psi_1 \end{cases} \Rightarrow \psi_2 = -\frac{\varphi_1}{\varphi_2}\psi_1$$

$$X - iY = -\frac{\varphi_1^2}{\varphi_2}\psi_1$$

$$X + iY = \varphi_2\psi_1$$

$$\frac{\psi_1}{\varphi_2} = \alpha$$

$$\left.\begin{array}{l} \begin{cases} X - iY = -\varphi_1^2\alpha \\ X + iY = \varphi_2^2\alpha \end{cases} \\ Z = \varphi_1\varphi_2\cdot\alpha \\ \varphi = \frac{\varphi_1}{\sqrt{\alpha}}, \quad \psi = \frac{\varphi_2}{\sqrt{\alpha}} \end{array}\right\} \Rightarrow \begin{cases} Z = \varphi\psi \\ X - iY = -\varphi^2 \\ X + iY = \psi^2 \end{cases}$$

$$\boxed{\begin{cases} X = \dfrac{\psi^2 - \varphi^2}{2} \\ Y = \dfrac{-\varphi^2 - \psi^2}{2i} \\ Z = \varphi\psi \end{cases}}$$

$\Rightarrow$

$$\cancel{\begin{cases} Z = \varphi\psi \\ X = -\varphi^2 \\ Y = \psi^2 \end{cases}}$$

$$\frac{\psi^4 - 2\psi^2\varphi^2 + \varphi^4}{4} - \frac{\varphi^4 + 2\psi^2\varphi^2 + \psi^4}{4} + \psi^2\varphi^2$$

$$\shortparallel$$

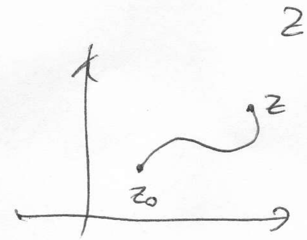
$$0$$

ok

$$\Rightarrow \quad \vec{F}'(z) = \begin{pmatrix} \dfrac{\varphi^2 - \psi^2}{2} \\ \dfrac{\varphi^2 + \psi^2}{2i} \\ \varphi\psi \end{pmatrix}$$

where

$\psi = \psi(z)$
$\varphi = \varphi(z)$ holomorphic



$$\Rightarrow \quad \vec{F}(z) = \int \begin{pmatrix} \dfrac{\varphi^2 - \psi^2}{2} \\ \dfrac{\varphi^2 + \psi^2}{2i} \\ \varphi\psi \end{pmatrix} dz'$$

integration along any curve starting at $z' = z_0$ to $z' = z$

variable.

$$\vec{R} = \operatorname{Re} \vec{F}.$$

— . —

minimal surfaces are special case of harmonic maps.

$$(M, h) \xrightarrow{\varphi} (N, g)$$

$$(x^\mu, h_{\mu\nu}) \qquad (y^A, g_{AB})$$

$$\cancel{\Delta_h \varphi^A + \Gamma^A_{BC} \varphi^B \varphi^C = 0}$$

~~$\cancel{g}$~~

~~$M \subset \mathbb{R}^1$~~

$M \subset \mathbb{R}^1$

$\Downarrow$

$\varphi(M)$ is a geodesic.

$$\boxed{\Delta_h \varphi^A + \Gamma^A_{BC} \varphi^B_{,\mu} \varphi^C_{,\nu} h^{\mu\nu} = 0}$$