Lecture 10 25.05.2018

Predictions of linearized Einstein's gravity

- Comment on conventions:

	WE	OTHERS
Signature of a metric	$-+++$	$+---$
Ricci	$R^{\mu}{}_{\nu\mu\rho}$	$R^{\mu}{}_{\nu\rho\mu}$
Cosmological constant	$E_{\mu\nu}+\Lambda g_{\mu\nu}=\varkappa T_{\mu\nu}$	$E_{\mu\nu}=\Lambda g_{\mu\nu}+\varkappa T_{\mu\nu}$

As we have seen in ~~the~~ an example of Einstein's Universe

$$g=-c^2dt^2+R^2 g_{S^3}$$

with our conventions $\Lambda=+\frac{1}{R^2}$ positive!

This is NOT the standard convention!
Even in the news they say that one of the most striking discovery of MODERN cosmology is that observed cosmological constant is POSITIVE because the Universe is EXPANDING quicker than it was assumed before

Quicker expansion $\Longrightarrow$ positive cosmological constant acts as REPULSIVE gravitational force

Einstein introduced Λ to have STATIC Universe so Einstein's cosmological constant acts as ATTRACTIVE gravitational force, so it should have OPPOSITE sign to the one from modern cosmology. So ~~...~~ to be in accordance with newspapers' convention ~~...~~ attractive Λ must be NEGATIVE

So instead of

$$E_{\mu\nu} + \Lambda g_{\mu\nu} = \varkappa T_{\mu\nu}$$

we should adopt a convention that Einstein's equations with cosmological constant are

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \Lambda g_{\mu\nu} + \varkappa T_{\mu\nu}}$$

Then Λ in Einstein's Universe will be NEGATIVE; and Λ in the current model of the Universe will be positive, as in newspapers.

- If <u>no matter</u> $\Rightarrow T_{\mu\nu} \equiv 0 \Rightarrow$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \Lambda g_{\mu\nu} \quad \Big| \text{ take trace}$$

$$R - \frac{1}{2} \cdot 4 \cdot R = \Lambda \cdot 4$$

$$-R = 4\Lambda$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (-4\Lambda) = \Lambda g_{\mu\nu}$$

$$R_{\mu\nu} + 2\Lambda g_{\mu\nu} = \Lambda g_{\mu\nu}$$

$$\boxed{R_{\mu\nu} = -\Lambda g_{\mu\nu}}$$

If in addition $\Lambda \equiv 0$ i.e.
if $T_{\mu\nu} \equiv \Lambda \equiv 0$ then Einstein's equations

$$\boxed{R_{\mu\nu} = 0}$$

Einstein's equations in vacuum

- Summary of linearized gravity

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \Lambda g_{\mu\nu} + \frac{8\pi G}{c^4} T_{\mu\nu}}$$

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}} \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}$$

$$|h_{\mu\nu}| << 1$$

gauge freedom $\quad h_{\mu\nu} \longrightarrow h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \quad , \quad h_{\mu\nu} \eta^{\mu\nu} =: h .$$

De Donder gauge condition

$$\partial^\mu \bar{h}_{\mu\nu} = 0$$

still enables gauge with ξ^μ satisfying $\quad \Box \xi^\mu = 0$

$$\Box = \eta_{\mu\nu} \partial^\mu \partial^\nu$$

Linearized Einstein's equations in deDonder gauge:

$$\boxed{-\frac{1}{2} \Box \bar{h}_{\mu\nu} = \Lambda (\eta_{\mu\nu} + h_{\mu\nu}) + \frac{8\pi G}{c^4} T_{\mu\nu}}$$

Without cosmological constant:

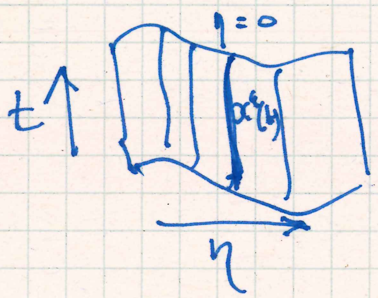
$$\boxed{\Box \bar{h}_{\mu\nu} = -16 \frac{\pi G}{c^4} T_{\mu\nu}}$$

- **Tidal forces** – geodesic deviation equation – Jacobi equation

a) Newton's theory:

$x^a(t)$ – a trajectory of a particle in space

$x^a(t,\eta)$ – 1-parameter family of such trajectories



$$x^a(t,0) = x^a(t)$$

$$\frac{\partial}{\partial\eta} = \frac{\partial x^a}{\partial\eta}\partial_a$$

$\frac{\partial}{\partial\eta}\Big|_{\eta=0}$ with components $\frac{\partial x^a}{\partial\eta}\Big|_{\eta=0}$ is a "separation" vector separating trajectory $x^a(t)$ from the infinitesimally close one.

If $x^a(t,\eta)$ are trajectories of particles in free fall we have

$$\frac{d^2x^a}{dt^2} + \frac{\partial\varphi}{\partial x^a} = 0 \qquad \frac{\partial}{\partial\eta}\Big|_{\eta=0}$$

$$0 = \frac{d^2n^a}{dt^2} + \frac{\partial^2\varphi}{\partial\eta\,\partial x^a}\Big|_{\eta=0} = \frac{d^2n^a}{dt^2} + \frac{\partial^2\varphi}{\partial x^b\partial x^a}\frac{\partial x^a}{\partial\eta}\Big|_{\eta=0} =$$

$$= \frac{d^2n^a}{dt^2} + \frac{\partial^2\varphi}{\partial x^a\partial x^b}n^a$$

Separation vector satisfies

$$\boxed{\frac{d^2n^a}{dt^2} + \frac{\partial^2\varphi}{\partial x^a\partial x^b}n^b = 0}$$

$\frac{\partial^2\varphi}{\partial x^a\partial x^b}$ is responsible for ~~between~~ "tidal force" between close trajectories.

Recall: $\frac{\partial^2\varphi}{\partial x^a\partial x^b} = R^a{}_{0b0}$

$$\boxed{\frac{d^2n^a}{dt^2} + R^a{}_{0b0}\,n^b = 0}$$

e.g. Earth–Moon in free fall of Sun grav

Obviously

$$[n, u] = 0$$

$x^a(s, \eta)$ trajectories in free fall $\Rightarrow$

$$\nabla_u u = 0. \quad \Big| \; \nabla_n$$

$\Downarrow$

$$\nabla_n \nabla_u u = 0$$

$$R(n, u)u = [\nabla_n, \nabla_u]u - \cancel{\nabla_{[n,u]}} u$$

$$= \cancel{\nabla_n(\nabla_u u)} - \nabla_u(\nabla_n u)$$

$$\Rightarrow \quad \nabla_u[\nabla_n u] + R(n, u)u = 0.$$

$$0 = Q(n, u) = \nabla_n u - \nabla_u n - \cancel{[n, u]}$$

$$\Rightarrow \quad \nabla_u \nabla_u n + R(n, u)u = 0$$

$$\boxed{\nabla_u^2 n + R(n, u)u = 0}$$

$$\nabla_u^2 n^\mu + R^\mu{}_{\nu\rho\sigma} n^\nu u^\rho u^\sigma = 0$$

$$\boxed{\nabla_u^2 n^\mu = R^\mu{}_{\sigma\rho\nu} u^\rho u^\sigma n^\nu}$$

eq. Linear in $\underline{\underline{n^\mu}}$

"tidal forces"

Riemann.

Gravitational waves in linearized theory

$$\Box \bar{h}_{\mu\nu} = -16\pi G/c^4 \, T_{\mu\nu}$$

Vacuum : $T_{\mu\nu} = 0$

$$\boxed{\Box \bar{h}_{\mu\nu} = 0}$$

solutions: monochromatic plane wave:

$$\bar{h}_{\mu\nu} = \mathrm{Re}\left(a_{\mu\nu} e^{-ik_\varsigma x^\varsigma}\right)$$

$$0 = \Box \bar{h}_{\mu\nu} = \eta^{\alpha\beta}\partial_\alpha \mathrm{Re}\left(-ik_\beta a_{\mu\nu} e^{-ik_\varsigma x^\varsigma}\right) =$$

$$= \eta^{\alpha\beta} \mathrm{Re}\left(-k_\beta k_\alpha a_{\mu\nu} e^{-ik_\varsigma x^\varsigma}\right) =$$

$$= \mathrm{Re}\left(-(k_\alpha k^\alpha) a_{\mu\nu} e^{-ik_\varsigma x^\varsigma}\right)$$

$$\boxed{k_\alpha k^\alpha = 0} \iff k = k^\alpha e_\alpha \ \text{ s.t. } g(k,k) = 0$$

k - propagation vector of the wave.

De Donder condition

$$0 = \partial^\mu \bar{h}_{\mu\nu} \iff \boxed{k^\mu a_{\mu\nu} = 0}$$

Freedom in choosing $a_{\mu\nu}$

$$a_{\mu\nu} = a_{\nu\mu} \implies 10 \text{ components}$$

$$a_{\mu\nu} k^\nu = 0 \implies 4 \text{ conditions}$$

Gauge freedom $\Box \xi_\nu = 0 \iff \xi_\nu = \mathrm{Re}\left(i C_\nu e^{ik_\alpha x^\alpha}\right)$

$$a_{\mu\nu} \mapsto a_{\mu\nu} + k_\mu C_\nu + k_\nu C_\mu - \eta_{\mu\nu} k_\varsigma C^\varsigma$$

$$\implies 4 \text{ parameter freedom}$$

$10 - 4 - 4 = 2$ degrees of freedom!

$a_{\mu\nu}$ - has only 2 physically relevant components.

How do choose them?

Choose of 4-velocity of an observer u^μ

$$\begin{cases} a_{\mu\nu}k^\nu = 0 & \text{4 eqs} \\ a_{\mu\nu}u^\nu = 0 & \text{3 eqs} \quad \text{since } u_\mu u^\mu = -1 \text{ (3 components)} \\ a^\mu{}_\mu = 0 & \text{1-eq} \end{cases}$$

TT gauge - (transverse - traceless gauge)

$$u^\mu = (1,0,0,0)$$

$$k^\mu = (\omega,0,0,\omega) \quad \text{wave travels in } z\text{-direction}$$

$$a_{\mu\nu} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ & \left(\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & a_+ & a_\times & 0 \\ 0 & a_\times & -a_+ & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

$\{e_\mu\}$ - frame of vector fields

$$e_+ = e_1 \otimes e_1 - e_2 \otimes e_2$$

$$e_\times = e_1 \otimes e_2 + e_2 \otimes e_1$$

$c=1$

$$\bar{h}_{+\mu\nu} = \mathrm{Re}\left[a_+ e_{+\mu\nu} e^{-i\omega(t-z)}\right]$$

$$\bar{h}_{\times\mu\nu} = \mathrm{Re}\left[a_\times e_{\times\mu\nu} e^{-i\omega(t-z)}\right]$$

2 different polarizations.

Evolution of a ball of dust in a field of monochromatic wave.

h^{TT} ~ $h_{\mu\nu}$ of a grav. wave in TT-gauge

Riemann tensor in linearized theory

$$R^{\mu}{}_{\nu\sigma\rho} = \frac{1}{2}\left(h^{\mu}{}_{\rho,\nu\sigma} - h^{\mu}{}_{\sigma,\nu\rho} - \partial_\sigma \partial^\mu h_{\nu\rho} + \partial_\rho \partial^\mu h_{\nu\sigma} \right)$$

$$u^\mu = (1,0,0,0) \qquad h_{0\mu} \equiv 0 .$$

$$R^{\mu}{}_{0\sigma 0} = -\frac{1}{2} h^{\mu}{}_{\sigma,00} = -\frac{1}{2}\ddot{h}^{\mu}{}_{\sigma}$$

Jacobi equation for 2 particles close to each other in the dust.

$$\nabla^2_u n^\mu + R^{\mu}{}_{\nu\rho\sigma} u^\nu n^\rho u^\sigma = 0$$

$$\nabla^2_u n^\mu + R^{\mu}{}_{0\rho 0} n^\rho = 0$$

$$\boxed{\ddot{n}^\mu - \frac{1}{2}\ddot{h}^{\mu}{}_{\sigma} n^\sigma = 0}$$

$$h^{TT\,0}{}_{\mu} = 0 = h^{TT\,3}{}_{\mu} = 0$$

$$\left.\begin{aligned} \ddot{n}^0 &= 0 \\ \ddot{n}^3 &= 0 \end{aligned}\right\} \text{ no longitudinal acceleration.}$$

relative acceleration only in x,y plane.

Solving by the method of successive approximations:

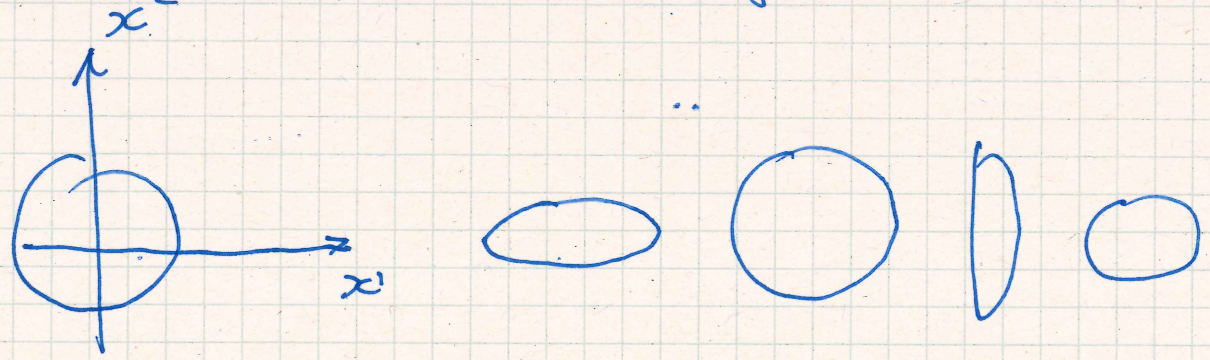
Start $n^\mu_{(0)} = \text{cont}$

$$\ddot{n}^\mu_{(1)} = \frac{1}{2}\ddot{h}^\mu{}_\sigma n^\sigma_{(0)}$$

$$\Rightarrow \quad n^\mu_{(1)} = \frac{1}{2}\underbrace{h^{TT\mu}{}_\sigma}_{\text{time dependence.}} n^\sigma_{(0)} + (\text{const})^\mu$$

time $\longrightarrow$

$+$



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