Lecture 11 29.05.2018

Symmetries, de Sitter and Schwarzschild solutions

1. Lie derivative of tensor fields.

- one parameter groups of transformations

$t \in \mathbb{R}$ $\quad \varphi_t : M \xrightarrow[\text{diff.}]{} M$ $\quad$ s.t.

$$(\varphi_{s+t} =) \varphi_{t+s} = \varphi_t \circ \varphi_s$$

$$\Rightarrow \begin{cases} \varphi_0 = id_M \\ \varphi_{-t} = (\varphi_t)^{-1} \end{cases}$$

define a vector field

$$X(f) = \frac{d}{dt} f \circ \varphi_t \Big|_{t=0}$$

- Lie derivative w.r.t. vector field X:

$$\underset{X}{\mathcal{L}} : \text{tensor fields} \xrightarrow[\mathbb{R}\text{-linear}]{} \text{tensor fields}$$

such that

a) $\underset{X}{\mathcal{L}}$ preserves tensor type $\binom{r}{s} \to \binom{r}{s}$

b) $\underset{X}{\mathcal{L}}$ is $\mathbb{R}$-linear

c) $\underset{X}{\mathcal{L}}(K \otimes L) = \underset{X}{\mathcal{L}} K \otimes L + K \otimes \underset{X}{\mathcal{L}} L$

d) $\underset{X}{\mathcal{L}}$ commutes with contractions

e) on forms : $\underset{X}{\mathcal{L}} \omega = X \lrcorner d\omega + d(X \lrcorner \omega)$

$\Rightarrow$ in particular : $\underset{X}{\mathcal{L}} \circ d = d \circ \underset{X}{\mathcal{L}}$, $\underset{X}{\mathcal{L}} f = X(f)$.

f) on vector fields:

$$\underset{X}{\mathcal{L}} Y = [X, Y].$$

g) $[\underset{X}{\mathcal{L}}, \underset{Y}{\mathcal{L}}] = \underset{[X,Y]}{\mathcal{L}}$.

Example

Lie derivative of a metric on a Riemannian manifold.

e^μ a frame in which $g = g_{\mu\nu}\, e^\mu \otimes e^\nu$, $g_{\mu\nu} = \text{const}$

$$\mathcal{L}_X e^\mu = X \lrcorner\, de^\mu + d(X \lrcorner\, e^\mu) =$$
$$= X \lrcorner\, (-\omega^\mu{}_\nu \wedge e^\nu) + dX^\mu$$
$$= dX^\mu + \omega^\mu{}_\nu X^\nu - \Gamma^\mu{}_{\nu\rho} X^\rho e^\nu$$
$$= DX^\mu - \Gamma^\mu{}_{\nu\rho} X^\rho e^\nu$$

$$\mathcal{L}_X g = \cancel{X(g_{\mu\nu})\, e^\mu \otimes e^\nu} + g_{\mu\nu}\, \mathcal{L}_X e^\mu \otimes e^\nu + g_{\mu\nu}\, e^\mu \otimes \mathcal{L}_X e^\nu$$
$$= g_{\mu\nu}\, DX^\mu \otimes e^\nu + g_{\mu\nu}\, e^\mu \otimes DX^\nu$$
$$- g_{\mu\nu} \Gamma^\mu{}_{\alpha\beta} X^\beta e^\alpha \otimes e^\nu - g_{\mu\nu}\, e^\mu \otimes \Gamma^\nu{}_{\alpha\beta} X^\beta e^\alpha$$
$$= g_{\mu\nu} \nabla_\alpha X^\mu e^\alpha \otimes e^\nu + g_{\mu\nu} \nabla_\alpha X^\nu e^\mu \otimes e^\alpha$$
$$- \left(\Gamma_{\nu\alpha\beta} X^\beta e^\alpha \otimes e^\nu + \Gamma_{\mu\alpha\beta} X^\beta e^\mu \otimes e^\alpha\right) =$$
$$= (\nabla_\alpha X_\beta + \nabla_\beta X_\alpha)\, e^\alpha \otimes e^\beta$$

$$\mathcal{L}_X g = (\nabla_\alpha X_\beta + \nabla_\beta X_\alpha)\, e^\alpha \otimes e^\beta$$

$$\boxed{\mathcal{L}_X g = 0} \iff \boxed{\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0}$$

Killing equation.

$$A_{\alpha(\beta\gamma)} = \frac{1}{2}\left(A_{\alpha\beta\gamma} + A_{\alpha\gamma\beta}\right)$$

$$A_{[\alpha\beta]\gamma} = \frac{1}{2}\left(A_{\alpha\beta\gamma} - A_{\beta\alpha\gamma}\right)$$

$$A_{\alpha(\beta\gamma)} = \frac{1}{2}\left(A_{\alpha\beta\gamma} + A_{\alpha\gamma\beta}\right)$$

$$A_{[\gamma\alpha]\beta} = \frac{1}{2}\left(A_{\gamma\alpha\beta} - A_{\alpha\gamma\beta}\right]$$

$$\cancel{A_{\beta(\gamma\alpha)} = \frac{1}{2}\left(A_{\beta\gamma\alpha} + A_{\beta\alpha\gamma}\right)}$$

$$A_{[\beta\gamma]\alpha} = \frac{1}{2}\left(A_{\beta\gamma\alpha} - A_{\gamma\beta\alpha}\right)$$

$$+\ A_{\alpha(\beta\gamma)} + A_{[\gamma\alpha]\beta} = \frac{1}{2}\left(A_{\alpha\beta\gamma} + A_{\gamma\alpha\beta}\right)$$

$$+\ A_{\gamma(\alpha\beta)} + A_{[\beta\gamma]\alpha} = \frac{1}{2}\left(A_{\gamma\alpha\beta} + A_{\beta\gamma\alpha}\right)$$

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$$A_{\alpha(\beta\gamma)} + A_{\gamma(\alpha\beta)} + A_{\beta(\gamma\alpha)}$$

$$+ A_{[\gamma\alpha]\beta} + A_{[\beta\gamma]\alpha} + A_{[\alpha\beta]\alpha} = \cancel{A_{\alpha\beta\gamma}}$$

$$= A_{\gamma\alpha\beta}$$

2. Transformation groups

M - manifold

G - Lie group $\Leftarrow$ manifold

$$\left\{\begin{array}{l} \cdot\ G \times G \to G \\ (a\cdot b)\cdot c = a\cdot(b\cdot c) \\ \exists e \text{ s.t. } e\cdot a = a\cdot e = a \\ \forall a\ \exists a^{-1} \text{ s.t. } a\cdot a^{-1} = a^{-1}\cdot a = e \\ \text{map} : (a,b) \longmapsto a\cdot b^{-1} \quad \text{smooth analytic} \end{array}\right.$$

(right) action of G in M

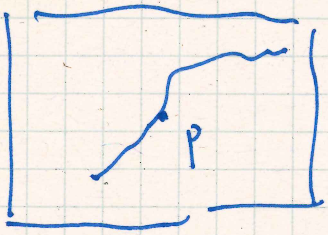
$$M \times G \to M$$

$$(p, a) \xmapsto{\text{smooth}} p\cdot a = R_a(p)$$

$$\left\{\begin{array}{l} (p\cdot a)b = p\cdot(a\cdot b) \\ p\cdot e = p \end{array}\right.$$

G acts effectively on M iff $\forall e \neq a \in G\ \exists p \in M$ s.t. $pa \neq p$.

Orbit : $\mathcal{O}_p = \{ pa \text{ s.t. } a \in G \}$



If $\mathcal{O}_p = M$ then G acts transitively on M

then M is ~~the~~ homogeneous w.r.t. action of G.

Isotropy group of p

$$H_p = \{ a \in G \text{ s.t. } pa = p \}$$

$$\mathcal{O}_p = G/H_p$$

$$\boxed{\dim \mathcal{O}_p = \dim G - \dim H_p}$$

3) G-invariant tensors

Assume that we have a smooth right action of G in M

Let $a(t)$ be a curve in G s.t.

$$a(0) = e$$
$$a(t+t') = a(t)\cdot a(t')$$

Such a curve is called 1-parameter subgroup of G

$$\Big(\underset{\substack{\| \\ a(0)=e}}{a(t-t)} = a(t)\cdot a(-t) \Rightarrow a(t)^{-1} = a(-t) \Big)$$

This defines a 1-parameter transformation group on M

$$\varphi_t : M \to M$$
$$\varphi_t(p) = p\cdot a(t).$$

$$\left(\begin{array}{l} \varphi_0(p) = p\cdot a(0) = p\cdot e = p \\ \varphi_{t+t'}(p) = \dots = \varphi_t \circ \varphi_{t'}(p) \end{array} \right)$$

$\Rightarrow$ this defines a vector field X on M

$$X(f) = \frac{d}{dt} f\circ\varphi_t \Big|_{t=0}.$$

Given a vector field X we search for φ_t s.t.

$$X(f) = \frac{d}{dt} f\circ\varphi_t \Big|_{t=0}$$

$$X(f)\circ\varphi_s = \frac{d}{dt} f\circ\varphi_t\circ\varphi_s \Big|_{t=0}$$

$$= \frac{d}{dt} f\circ\varphi_{t+s} \Big|_{t=0}$$

~~Tensor field K is G-invariant iff~~

$$\cancel{\mathcal{L}_X K = 0}$$

~~for all X coming from all 1-par. subgroups of G.~~

~~Tensor field K is G-invariant iff~~

$$\cancel{\mathcal{L}_X K = 0.}$$

$t' = t+s$

$dt = dt'$

$$= \frac{d}{dt'} f\circ\varphi_{t'}$$

$$= \frac{d}{ds} f\circ\varphi_s \Big|_{t'=s}$$

$f = x^\mu$

$$\boxed{X^\mu(x(s)) = \frac{d}{ds} x^\mu(s)}$$

4) Killing vector field for a tensor field K

is X s.t. $\mathcal{L}_X K = 0$.

~~[illegible]~~ If X, Y two Killings then

$$\mathcal{L}_{[X,Y]} K = \mathcal{L}_X \circ \mathcal{L}_Y K - \mathcal{L}_Y \circ \mathcal{L}_X K = 0.$$

Killing fields form a Lie algebra.

Lie algebra of symmetries of K.

5) Maximal number of symmetries on a Riemannian mfd.

Obviously, when (M, g) is flat.

$$R^\mu{}_{\nu\rho\sigma} \equiv 0 \Rightarrow \Gamma^\mu{}_{\nu\rho} = 0$$

Killing equation

$$\mathcal{L}_X g = 0 \Leftrightarrow \nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$$

$$\boxed{\partial_\mu X_\nu + \partial_\nu X_\mu = 0.} \quad *$$

$$\Downarrow \quad \partial_{(\mu} X_{\nu)} = 0$$

$$\boxed{\partial_\rho \partial_{(\mu} X_{\nu)} = 0.}$$

but

$$\boxed{\partial_{[\rho} \partial_{\mu]} X_\nu = 0}$$

$$\xrightarrow{Ex} \partial_\rho \partial_\mu X_\nu = 0$$

$$\Rightarrow \partial_\mu X_\nu = a_{\mu\nu} = \text{const}$$

$$\stackrel{*}{\Rightarrow} a_{\mu\nu} = -a_{\nu\mu} \qquad \Rightarrow X_\nu = a_{\mu\rho} x^\rho + b_\nu$$

$a_{\mu\nu}$ — $\frac{n(n-1)}{2}$
parameters
b_ν — n

$$\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2} \quad \text{parameters.}$$

$a_{\mu\nu}$ — Lorentz group
b_ν — translations
} Poincare group.

6) Decomposition of Riemann tensor:

$$R^{\mu\nu}{}_{\rho\sigma} = C^{\mu\nu}{}_{\rho\sigma} + a\, \check{R}^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]} + b R\, \delta^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]}$$

$$\boxed{\check{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R}$$

~~t~~raceless Ricci

$$R^{\mu\nu}{}_{\rho\sigma} = C^{\mu\nu}{}_{\rho\sigma} + \frac{a}{4}\left(\check{R}^{\mu}{}_{\rho}\delta^{\nu}_{\sigma} - \check{R}^{\nu}{}_{\rho}\delta^{\mu}_{\sigma} - \check{R}^{\mu}{}_{\sigma}\delta^{\nu}_{\rho} + \check{R}^{\nu}{}_{\sigma}\delta^{\mu}_{\rho}\right)$$

$$+ \frac{b}{4} R\left(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\nu}_{\rho}\delta^{\mu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho} + \delta^{\nu}_{\sigma}\delta^{\mu}_{\rho}\right)$$

$$R^{\nu}{}_{\sigma} = \frac{a}{4}\left(-\check{R}^{\nu}{}_{\sigma} - \check{R}^{\nu}{}_{\sigma} + n\check{R}^{\nu}{}_{\sigma}\right)$$

$$+ \frac{b}{4} R\left(n\delta^{\nu}_{\sigma} - \delta^{\nu}_{\sigma} - \delta^{\nu}_{\sigma} + n\delta^{\nu}_{\sigma}\right)$$

$$\check{R}^{\nu}{}_{\sigma} + \frac{1}{n}\delta^{\nu}_{\sigma} R + \frac{a(n-2)}{4}\check{R}^{\nu}{}_{\sigma} + \frac{n-1}{2} b\, \delta^{\nu}_{\sigma} R$$

$$\frac{a(n-2)}{4} = 1 \qquad a = \frac{4}{n-2}$$

$$\frac{b(n-1)}{2} = \frac{1}{n} \qquad b = \frac{2}{n(n-1)}$$

$$\boxed{\underset{\text{Riemann}}{R^{\mu\nu}{}_{\rho\sigma}} = \underset{\text{Weyl}}{C^{\mu\nu}{}_{\rho\sigma}} + \frac{4}{n-2}\check{R}^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]} + \frac{2}{n(n-1)}\delta^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]}}$$

Maximal symmetry:

group has to preserve C, $\check{R}$, R

One has to put $C = \check{R} = 0$.

$$\boxed{R^{\mu\nu}{}_{\rho\sigma} = \frac{2}{n(n-1)} \delta^{[\mu}_{[\rho} \delta^{\nu]}_{\sigma]}}$$

$$\boxed{\Omega^\mu{}_\nu = \frac{1}{2} R^{\mu}{}_{\nu\rho\sigma}\, \theta^\rho \wedge \theta^\sigma = \frac{R}{n(n-1)} \theta^\mu \wedge \theta^\nu}$$

II$^{\text{n}}$ B.I.

$$0 = D\Omega^\mu{}_\nu = \frac{DR}{n(n-1)} \theta^\mu \wedge \theta^\nu = \frac{dR}{n(n-1)} \theta^\mu \wedge \theta^\nu$$

$n = 2 \Rightarrow$ no conditions.

$n \geq 3 \Rightarrow R = \text{const}$.

Spaces of constant curvature.

Thm

Every (M, g) having maximal number of Killing symmetries can be locally represented in coordinates x^μ in which the metric g reads

$$g = \frac{\eta_{\mu\nu}\, dx^\mu dx^\nu}{\left(1 + \frac{K}{4} \eta_{\mu\nu} x^\mu x^\nu\right)^2} \qquad K = \text{const}$$

Exercise: the curvature of this metric

$$\Omega^{\mu\nu} = -K\, \theta^\mu \wedge \theta^\nu \qquad (-R = n(n-1)K)$$

$$\theta^\mu = \frac{dx^\mu}{1 + \frac{K}{4} \eta_{\mu\nu} x^\mu x^\nu}$$

Moreover:

Every (M, g) as in the thm can be obtained as part of a hypersurface

$$M = \{ y \in \mathbb{R}^{4+1} \text{ s.t. } K {y^0}^2 + g_{\mu\nu} y^\mu y^\nu = k r^2 \}$$

$$K = \frac{1}{k r^2}$$

and $g = K\, {dy^0}^2 + g_{\mu\nu}\, dx^\mu dx^\nu \big|_M$.

~~4~~) Anti deSitter space.

$$g = \frac{-dt^2 + dx^2 + dy^2 + dz^2}{\left(1 + \frac{K}{4}\left(-t^2 - x^2 - y^2 - z^2\right)\right)^2}$$

$$0 = \check{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \Rightarrow$$

$$\Rightarrow R_{\mu\nu} = \frac{1}{4} g_{\mu\nu} R = \frac{-1}{4} g_{\mu\nu}\, 12K = -3K g_{\mu\nu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -3K g_{\mu\nu} + \frac{1}{2} g_{\mu\nu}\, 12K =$$

$$= \underbrace{+3K}_{= \Lambda} g_{\mu\nu}$$

$\Lambda > 0$ i.e. $K > 0$ De Sitter

$\Lambda < 0$ i.e. $K < 0$ anti De Sitter.

After suitable change of variables one can $g = -(1 - Kr^2) dt^2 + \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2$.