Lecture 12 1.06.2018

Spherical symmetry; Schwarzschild solution

1) stationary and static gravitational field

stationarity ≡ existence of timelike Killing

Def (M, g) is stationary iff there exist a vector field X on M s.t.

1) $\mathcal{L}_X g = 0$ (i.e. X is a Killing v.f.)
2) $g(X,X) < 0$ (i.e. X is timelike)

Locally take a coord. system in which $X = \frac{\partial}{\partial x^0}$, (x^0, x^i)

$$X^\mu = (1,0,0,0)$$

$$0 = \mathcal{L}_X g = X(g_{\mu\nu})\, dx^\mu \otimes dx^\nu + \cancel{g_{\mu\nu}\, dX^\mu \otimes dx^\nu} + \cancel{g_{\mu\nu}\, dx^\mu \otimes dX^\nu}$$

$$X(g_{\mu\nu}) = 0 \qquad g_{\mu\nu} = g_{\mu\nu}(x^i); \quad \partial_0 g_{\mu\nu} = 0$$

$$g(X,X) < 0 \implies g_{00} < 0.$$

$$\boxed{X \text{ stationary}} \overset{\text{locally}}{\Longleftrightarrow} \boxed{(x^0, x^i),\ X = \frac{\partial}{\partial x^0},\quad \frac{\partial g_{\mu\nu}}{\partial x^0} = 0,\quad g_{00} < 0}$$

Static is more than stationary

Def (M, g) is static iff

1) (M, g) is stationary
2) Field of hyperplanes $X^\perp = \{ Y \in TM : g(X,Y) = 0 \}$ is integrable.

Def

Field of k-planes spanned by $\{X_1, \dots X_k\}$ is integrable iff M is an union of k-dimensional submanifolds N such that

- none of any two submanifolds N intersect
- $\{X_1, \dots X_k\}$ are tangent to Ns.

Given such an integrable field of k-planes let

$X_1, \dots X_k, Y_{k+1}, \dots Y_n$ be a frame in M

Let

$\alpha_1, \dots \alpha_k, \beta_{k+1}, \dots \beta_n$ be a dual coframe.

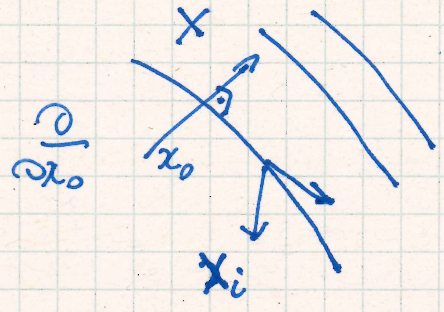
In particular

$$\mathrm{Span}_{FCM}(\beta_{k+1}, \dots \beta_n) = \{X_1, \dots X_k\}^{\perp}$$

Thm Frobenius

Field of k-planes spanned by $\{X_1, \dots X_k\}$ is integrable iff

$$d\beta_\alpha \wedge \beta_{k+1} \wedge \dots \wedge \beta_n = 0 \qquad \forall\ \alpha = k+1, \dots n$$



$$X = \frac{\partial}{\partial x^0}$$

$$0 = g(X, \underbrace{X_i}_{X^\perp}) = g_{0i}$$

$$g = g_{00}\, dt^2 + g_{ij}\, dx^i dx^j$$

$$g_{00} < 0, \quad g_{ij} = g_{ij}(x^k)$$

$$g_{00} = g_{00}(x^k)$$

$\Longrightarrow$ Spacetime (M, g) is static iff in the neighbourhood of any point there exists coordinates (x^0, x^i) such that the metric looks like

$$g = g_{00}\, dt^2 + g_{ij}\, dx^i dx^j$$

$$\partial_0 g_{\mu\nu} = 0, \quad g_{00} < 0$$

g_{ij} has signature $+++$

2) Spherical symmetry

Def (M, g) is spherically symmetric iff the isometry group (group preserving g) is $SO(3)$ with 2-dimensional orbits which are spacelike and have topology of S^2.

SO(3) is a symmetry group for $g \Rightarrow$

We must have 3 Killings, X_1, X_2, X_3 satisfying commutation relations of $\underline{so}(3)$.

We can choose X_1, X_2, X_3 to satisfy

$$[X_1, X_2] = X_3 \quad + \text{ cyclic permutation}$$

Locally we can introduce

$$(\underbrace{x^0, x^1}_{\text{labels orbits}}, \underbrace{x^2, x^3}_{\text{coordinates on orbits.}})$$

<u>Recall</u> (M, g), $\dim M = n \Rightarrow$ max. sym. group has dim $\frac{n(n+1)}{2}$ and such metrics $\Rightarrow$ metrics of constant curvature

$$h = \frac{\eta_{\mu\nu} dx^\mu dx^\nu}{\left(1 + \frac{k}{4} \eta_{\mu\nu} x^\mu x^\nu\right)^2}$$

<u>metric on an orbit</u>

$$\left.\begin{array}{l} \dim M = 2, \quad \frac{n(n+1)}{2} = \frac{2\cdot 3}{2} = 3 \\ \dim SO(3) = 3 \end{array}\right] \Rightarrow \text{metric on orbits}$$

$$h = \frac{dx^2 + dy^2}{\left(1 + \frac{k}{4}(x^2 + y^2)\right)^2}$$

metric on S^2 i.e. in other coordinates

$$h = d\theta^2 + \sin^2\theta \, d\varphi^2.$$

$$(x^0, x^1, \theta, \varphi)$$

$$g\Big|_{x^0, x^1 = \text{const}} = (\cdot)(d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$$\mathcal{L}_{X_i} g = 0 \qquad \begin{cases} X_1 = \partial/\partial\varphi \\ X_2 = \sin\varphi \, \partial_\theta - \cos\varphi \, \mathrm{ctg}\,\theta \, \partial_\varphi \\ X_3 = \cos\varphi \, \partial_\theta - \sin\varphi \, \mathrm{ctg}\,\theta \, \partial_\varphi \end{cases}$$

$\Rightarrow$

$$\boxed{\begin{gathered} g = g_{AB}(x^C)\,dx^A dx^B + r^2(x^C)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \\ x^A = (x^0, x^1) \quad , \quad A, B, C = 0, 1 \end{gathered}}$$

Three cases depending on causality of $X_\mu = \partial_\mu r$:

1) $X_\mu = \partial_\mu r$ is spacelike $\qquad \partial_\mu r \cdot \partial^\mu r > 0$
2) $X_\mu = \partial_\mu r$ is null $\qquad < 0$
3) $X_\mu = \partial_\mu r$ is timelike. $\qquad = 0.$

Ad1)

$$\partial_\mu r\, \partial_\nu r\, g^{\mu\nu} > 0.$$

$\Rightarrow \partial_\mu r \neq 0$ and r is indep. of θ and φ

$\Rightarrow$ ~~...~~ we can take it as a coordinate $x^1 = r$.

$\Rightarrow$

$$g = g_{00}\, dx^{0\,2} + g_{01}\, dx^0 dr + g_{11}\, dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$\begin{pmatrix} x^1 = r \\ x^0 \end{pmatrix} \rightarrow \begin{pmatrix} r \\ x^0(r, t) \end{pmatrix}$$

$$g = g_{00}\left(x^0_r\, dr + x^0_t\, dt\right)^2 + g_{01}\left(x^0_r\, dr + x^0_t\, dt\right) dr + g_{11}\, dr^2 + \ldots$$

$$\Rightarrow \ldots + \left(2 g_{00}\, x^0_r\, x^0_t + g_{01}\, x^0_t\right) dr\, dt + \ldots$$

$$2 g_{00}\, x^0_r + g_{01} = 0$$

$$x^0_r = -\frac{g_{01}}{2 g_{00}} \quad \text{one can solve for } x^0_r.$$

$2 g_{00} < 0$

$$\Rightarrow \quad g = g_{00} {x^0_t}^2 dt^2 + x^0_r (g_{00} x^0_r + g_{01}) dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$$= g_{00} {x^0_t}^2 dt^2 - \frac{g_{01}}{2 g_{00}} \left(-\frac{g_{01}}{2} + g_{01} \right) dr^2 + \ldots =$$

$$= \underbrace{g_{00} (x^0_t)^2}_{<0} dt^2 - \underbrace{\frac{g_{01}^2}{4 g_{00}}}_{>0} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$$\boxed{g = -e^{2\mu(r,t)} dt^2 + e^{2\nu(r,t)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2).}$$

$$\mu = \mu(r,t) \qquad \nu = \nu(r,t) \qquad \frac{\partial}{\partial r} \text{ spacelike}$$

Not stationary in general

Ex Analyze the other two possibilities in the same way.

Schwarzschild metric

Impose $Ric(g) = 0$ conditions.

Result

$$\boxed{g = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)}$$

Schwarzschild 1916.

Remark

he assumed $\mu = \mu(r)$, $\nu = \nu(r)$
stationarity

Birkhoff 1923 has shown that this assumption is not needed $\Rightarrow$ follows from the equations.

Exercise

$$g = -e^{2\mu(r,t)}dt^2 + e^{\nu(r,t)}dt^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$\begin{cases} e^0 = e^{\mu}dt \\ e^1 = e^{\nu}dr \\ e^2 = r\,d\theta \\ e^3 = r\sin\theta\, d\varphi \end{cases}$$

$$de^{\mu} + \omega^{\mu}{}_{\nu} \wedge e^{\nu} = 0 \quad , \quad \omega_{\mu\nu} + \omega_{\nu\mu} = 0$$

$\Rightarrow$ ~~[illegible]~~

Connection

$$\boxed{\begin{aligned} &\omega_{01} = -e^{-\mu}\dot{\nu}\, e^1 - e^{-\nu}\mu' e^0 \\ &\omega_{02} = \omega_{03} = 0 \\ &\omega_{12} = -\frac{e^{-\nu}}{r}e^2 \\ &\omega_{13} = -\frac{e^{-\nu}}{r}e^3 \qquad \omega_{23} = -\frac{\cot\theta}{r}e^3 \end{aligned}}$$

Ricci

$$R_{11} = e^{-2\mu}(-\dot{\mu}\dot{\nu} + \dot{\nu}^2 + \ddot{\nu}) + \frac{e^{-2\nu}}{r}\left(2\nu' - r(\mu'^2 - \mu'\nu' + \mu'')\right)$$

$$R_{00} = e^{-2\mu}(\dot{\mu}\dot{\nu} - \dot{\nu}^2 - \ddot{\nu}) + \frac{e^{-2\nu}}{r}\left(2\mu' + r(\mu'^2 - \mu'\nu' + \mu'')\right)$$

$$R_{01} = \frac{2e^{-\mu-\nu}}{r}\dot{\nu}$$

$$R_{02} = 0 = R_{03}$$

$$\|$$

$$R_{12} = R_{13} = R_{23}$$

$$R_{22} = \frac{1}{r^2} + \frac{e^{-2\mu}}{r^2}\left(-1 + r(\nu' - \mu')\right) = R_{33}$$

$$R_{ij} = \begin{pmatrix} \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ 0 & 0 & \bullet & 0 \\ 0 & 0 & 0 & \bullet \end{pmatrix}$$

Einstein

$$
\begin{cases}
E_{00} = \dfrac{1 + e^{-2\nu}(2r\nu' - 1)}{r^2} \\[2mm]
E_{01} = \dfrac{2 e^{-\mu-\nu}\dot{\nu}}{r} \qquad E_{02} = E_{03} = 0 \\[2mm]
E_{11} = \dfrac{-1 + e^{-2\nu}(2r\mu' + 1)}{r^2} \qquad E_{12} = E_{13} = 0 \\[2mm]
E_{22} = e^{-2\mu}\left[\dot{\mu}\dot{\nu} - \dot{\nu}^2 - \ddot{\nu}\right] + \dfrac{e^{-2\nu}}{r}\left[(1 + r\mu')(\mu' - \nu') + r\mu''\right] = E_{22}, \\[2mm]
E_{23} = 0
\end{cases}
$$

3) Coupled Einstein-Maxwell system (spherical symmetry)

$$
\begin{cases}
R_{\mu\nu} = \Lambda g_{\mu\nu} + T_{\mu\nu} \\
T_{\mu\nu} = F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
dF = d{*}F = 0
\end{cases}
$$

$\mathcal{L}_{X_i} F = 0.$ $\rightsquigarrow$ Radial electric and magnetic field.

$$F = E\, e^0 \wedge e^1 + B\, e^2 \wedge e^3 \qquad E = E(r,t), \quad B = B(r,t).$$

$$dF = 0 \Rightarrow d(B\, e^2 \wedge e^3) = 0$$

$$\dot{B} e^{-\mu}\, e^0 \wedge e^2 \wedge e^3 + \frac{2B + rB'}{r} e^{-\nu}\, e^1 \wedge e^2 \wedge e^3 = 0$$

$$\dot{B} = B, \quad B = \frac{b}{r^2}$$

$$*(e^0 \wedge e^1) = e^2 \wedge e^3$$
$$*(e^2 \wedge e^3) = -e^0 \wedge e^1$$

$d{*}F = 0$

$$\Rightarrow d(E\, e^2 \wedge e^3) = 0 \Rightarrow \boxed{E = \frac{e}{r^2}}$$

$$T_{\mu\nu} = \begin{pmatrix} \frac{b^2+e^2}{2r^4} & & & \\ & -\frac{b^2+e^2}{2r^4} & & \\ & & \frac{b^2+e^2}{2r^4} & \\ & & & \frac{b^2+e^2}{2r^4} \end{pmatrix}$$

$$E_{\mu\nu} = \varkappa T_{\mu\nu} + \Lambda g_{\mu\nu}$$

$$e^{2\mu} = \left(1 - \frac{2m}{r} + \frac{\varkappa(e^2+b^2)}{2r^2} + \frac{\Lambda}{3} r^2\right) = e^{-2\nu}$$

$$\boxed{g = -\left(1 - \frac{2m}{r} + \frac{\varkappa(e^2+b^2)}{2r^2} + \frac{\Lambda}{3} r^2\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r} + \frac{\varkappa(e^2+b^2)}{2r^2} + \frac{\Lambda}{3} r^2} dr^2 + r^2\left(d\theta^2 + \sin^2\vartheta \, d\varphi^2\right)}$$

m – mass
e, – electric, b – magnetic charge
Λ – cosmological constan

$\Lambda = 0 \implies$ Reisner - Nordström
$m = e = b = 0$ – deSitter / anti deSitter