Lecture 13 5.06.2018

Basic physical properties of Schwarzschild metric

Spherically symmetric metric

$$g = -e^{2\mu} dt^2 + e^{2\nu} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$\mu = \mu(r,t)$, $\nu = \nu(r,t)$

$$\mathrm{Ric}(g) = 0$$

$$\Rightarrow \boxed{g = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)}$$

Schwarzschild 1916

Note that

- $\begin{pmatrix} \text{Spherical symmetry} \\ + \\ \mathrm{Ric}(g) \end{pmatrix} \Rightarrow \begin{pmatrix} \text{stationarity} \\ \text{staticity} \end{pmatrix}^{*}$ ✓

- m is integration constant

  weak field $g_{00} = -1 - \frac{2\varphi}{c^2} \Rightarrow \varphi = -\frac{mc^2}{r}$

  in Newton's theory $\varphi = -\frac{GM}{r}$

  $\Rightarrow m = \frac{GM}{c^2}$

Ad $*$ $X = \frac{\partial}{\partial t}$ is a Killing vector

$$g(X,X) = -\left(1 - \frac{2m}{r}\right) \begin{cases} \xrightarrow{r > 2m} \text{timelike} \\ \xrightarrow{r < 2m} \text{spacelike} \end{cases}$$

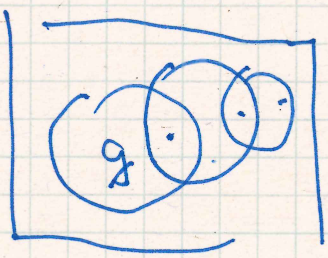
field is NOT stationary when $r < 2m$

hypersurface $r = 2m$?

- Maximal analytic extension of Schwarzschild.

Kruskal 1960!

$\mathbb{R}^4$



$$g = \underbrace{-\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}}}_{\text{will analyze this term.}} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$\boxed{r>2m}$

$$-\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} = -\left(1-\frac{2m}{r}\right)\left(dt^2 - \frac{dr^2}{\left(1-\frac{2m}{r}\right)^2}\right)$$

$$r^* = \int \frac{dr}{1-\frac{2m}{r}} = \int \frac{r-2m+2m}{r-2m}\,dr = r + 2m\int\frac{dr}{r-2m} =$$

$$= r + 2m\log\left(\frac{r}{2m}-1\right)$$

$u = t - r^*$

$v = t + r^*$

$2m < r < +\infty$

$-\infty < r^* < +\infty$

$$-\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} = -\left(1-\frac{2m}{r}\right)du\,dv$$

$\bar{u} = -4m\, e^{-\frac{u}{4m}}$

$\bar{v} = 4m\, e^{\frac{v}{4m}}$

$$d\bar{u}\,d\bar{v} = -16m^2\left(-\frac{1}{4m}\right)\left(\frac{1}{4m}\right)e^{-\frac{u}{4m}}e^{\frac{v}{4m}}\,du\,dv = e^{\frac{v-u}{4m}}\,du\,dv =$$

$$= e^{\frac{r^*}{2m}}\,du\,dv = e^{\frac{r}{2m}}\left(\frac{r}{2m}-1\right)du\,dv$$

$$= -e^{\frac{r}{2m}}\frac{r}{2m}\underbrace{\left(1-\frac{2m}{r}\right)du\,dv}$$

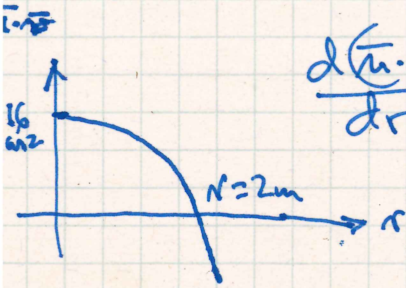
$$\boxed{g = \frac{2m}{r}e^{-\frac{r}{2m}}\,d\bar{u}\,d\bar{v} + r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)}$$

$$\underbrace{\begin{pmatrix} t \\ r \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}}_{\text{we made a coord transf in which}}$$

$$\bar{u}\bar{v} = -16m^2 e^{\frac{r^*}{2m}} = -16m^2\left(\frac{r}{2m}-1\right)e^{\frac{r}{2m}}$$

$$\bar{u}\bar{v} = -16m^2\left(\frac{r}{2m}-1\right)e^{\frac{r}{2m}}$$

When we can invert $r = r(\bar{u}\cdot\bar{v})$?



$$\frac{d(\bar{u}\cdot\bar{v})}{dr} = -16m^2 e^{\frac{r}{2m}}\left(\frac{1}{2m}+\frac{1}{2m}\left(\frac{r}{2m}-1\right)\right) = -4r e^{\frac{r}{2m}} \neq 0$$

$\cancel{\neq -16m^2 \neq}$ $\qquad r > 0$!

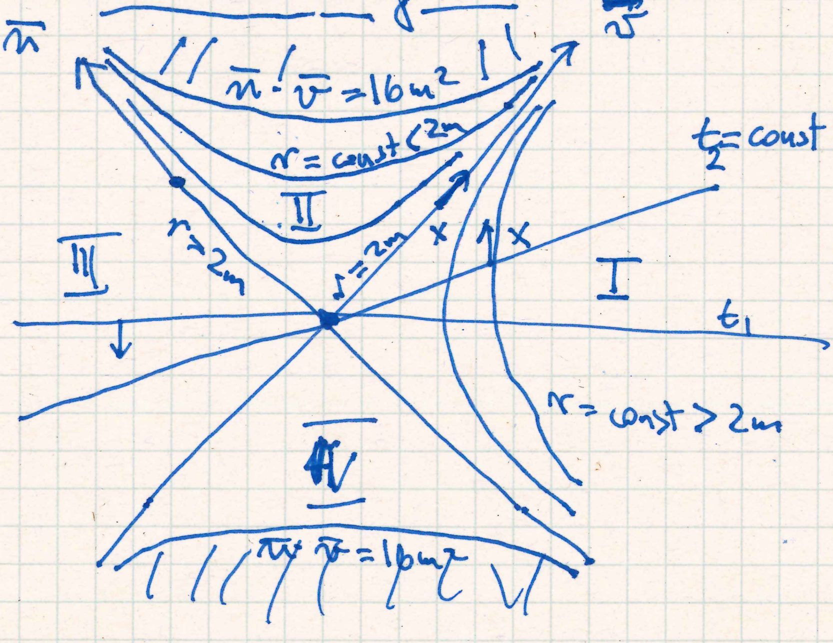
Thus implicit function theorem tells us that we can invert $r = r(\bar{u}\cdot\bar{v})$ for every r (including $r = 2m$)

$$g = \frac{2m}{r(\bar{u}\cdot\bar{v})} e^{-\frac{r(\bar{u}\cdot\bar{v})}{2m}} d\bar{u}\, d\bar{v} + r(\bar{u}\cdot\bar{v})^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right).$$

This covers $r > 0$ in Schwarzschild coordinates!

Note that if $r > 0$ then $\bar{u}\cdot\bar{v} < 16m^2$

Kruskal diagram



$$X = \partial_t = \frac{\partial\bar{u}}{\partial t}\partial_{\bar{u}} + \frac{\partial\bar{v}}{\partial t}\partial_{\bar{v}}$$

$$\frac{\bar{v}}{\bar{u}} = -e^{\frac{v+u}{4m}} = e^{\frac{t}{2m}}$$

$$\bar{u} = e^{-\frac{t}{2m}}\bar{v}$$

$g(X,X) > 0$	I, III
$g(X,X) < 0$	II, IV
$g(X,X) = 0$	$r = 2m$

Radial free fall

Equation of radial geodesic

$$g = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$\theta = \varphi = 0$$

$$\delta \int ds = 0$$

$$\cancel{ds = \sqrt{-\left(1-\frac{2m}{r}\right)dt^2}}$$

$$ds = \mathcal{L}(\lambda)\, d\lambda \qquad \lambda = r$$

$$ds^2 = -\left(1-\frac{2m}{r}\right)t'^2 + \frac{1}{1-\frac{2m}{r}} \quad \leftarrow \text{timelike} \Rightarrow < 0$$

$$ds = \sqrt{\left|\left(1-\frac{2m}{r}\right)t'^2 - \frac{1}{\underbrace{1-\frac{2m}{r}}_{X}}\right|}\, dr$$

$$\frac{dr}{dt} < 0$$

$$\mathcal{L} = \sqrt{X t'^2 - \frac{1}{X}}$$

$$\frac{dt}{dr} < 0$$

$$\left.\begin{array}{l} \dfrac{d}{dr}\dfrac{\partial \mathcal{L}}{\partial t'} - \dfrac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \dfrac{\partial \mathcal{L}}{\partial t'} = -\sqrt{X_0} \\ \qquad\qquad\qquad\qquad\qquad\quad \parallel \\ \qquad\qquad\qquad\qquad\qquad \dfrac{X t'}{\sqrt{}} \end{array}\right\} (*)$$

$$X^2 t'^2 = X_0\left(+X t'^2 - \frac{1}{X}\right)$$

$$X(X - X_0)t'^2 = -\frac{X_0}{X} \Rightarrow t'^2 = \frac{X_0}{X^2(X_0 - X)}$$

$$\Rightarrow \underbrace{t' = -\frac{1}{X}\sqrt{\frac{X_0}{X_0 - X}}}_{(**)}$$

$$X = X_0 \Rightarrow t' \to -\infty$$

$$\Rightarrow \frac{dr}{dt} = 0$$

$$X_0 = X(r_0)$$

start.

Time of free fall for Schwarzschild radius:

$$T = -\int_{r_0}^{r=2m} \frac{1}{X}\sqrt{\frac{X_0}{X_0-X}} = \underbrace{\int_{r=2m}^{r_0} \frac{1}{X}\sqrt{\frac{X_0}{X_0-X}}}_{\text{DIVERGENT!}}$$

$$X = 1-\frac{2m}{r}$$

Schwarzschild time (as seen from the observer at r_0)

In proper time:

$$\frac{ds}{dr} = -\sqrt{Xt'^2-\frac{1}{X}} \underset{(*)}{=} \frac{Xt'}{\sqrt{X_0}} \underset{(**)}{=} -\frac{1}{\sqrt{X_0-X}}$$

$$S = -\int_{r_0}^{2m}\frac{dr}{\sqrt{X_0-X}} = \int_{2m}^{r_0}\frac{dr}{\sqrt{X_0-X}} = \int_{2m}^{r_0}\frac{dr}{\sqrt{\frac{2m}{r}-\frac{2m}{r_0}}} < \infty$$

$$\cancel{= \frac{1}{\sqrt{2m}}\int_{2m}^{r_0}}$$

FINITE!

Equations of motions of a test particle in Schwarzschild

$$g = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)$$

Remark

Geodesics — extremals of arc length

$$\delta \int \sqrt{|g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu|}\, d\lambda = 0.$$

This is indep. of parametrization, but not good for null geodesics.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \quad , \quad \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{1}{\mathcal{L} = 0}\,!$$

Energy functional

$$x^\mu = x^\mu(\lambda).$$

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \quad \leftarrow \text{not repar. inv. but}$$

E.L. exercise:

$$\ddot{x}^\mu + \left\{{\mu \atop \nu\rho}\right\}\dot{x}^\nu \dot{x}^\rho = 0.$$

geodesic eq. in affine parametrization!

What is this affine parameter?

$$ds = \sqrt{|g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu|}\, d\lambda$$

$$\left(\frac{ds}{d\lambda}\right)^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

$$\frac{d}{d\lambda}\left(\frac{ds}{d\lambda}\right)^2 = \frac{d}{d\lambda}\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right) = g_{\mu\nu,\rho}\dot{x}^\rho\dot{x}^\mu\dot{x}^\nu + \dots =$$

$$\underset{\text{E.L}}{\overset{\uparrow}{=}} \dots = 0.$$

$$\Rightarrow \boxed{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \text{ is a first integral for E.L. eqs.}}$$

$$2\frac{d^2s}{d\lambda^2}\frac{ds}{d\lambda} \Rightarrow s = a\lambda + b \rightsquigarrow \text{proper time}$$

unless $s \equiv 0$.

- Energy functional for Schwarzschild.

$$E = \frac{1}{2}\int\left[\left(1-\frac{2m}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{2m}{r}} - r^2\left(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2\right)\right]ds$$

Timelike geodesic $\Rightarrow$ $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$ $c=1$

$$2\mathcal{L} = \boxed{\left(1-\frac{2m}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{2m}{r}} - r^2\left(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2\right) = 1} \quad (*)$$

- E.L. equation for θ:

$$\frac{d}{ds}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = 0$$

$$\boxed{-\left(r^2\dot{\theta}\right)^{\cdot} + r^2\sin\theta\cos\theta\,\dot{\varphi}^2 = 0}$$

this is automatically satisfied at $\theta = \frac{\pi}{2}$

From spherical symmetry, we can restrict to $\theta = \frac{\pi}{2}$ and forget about $\delta\theta$.

- $\mathcal{L}$ does not depend on t and φ $\Rightarrow$

$$\frac{d}{ds}\left[\left(1-\frac{2m}{r}\right)\dot{t}\right] = 0$$

$$\frac{d}{ds}\left[-r^2\dot{\varphi}\right] = 0$$

$$\Rightarrow \begin{cases} \left(1-\frac{2m}{r}\right)\dot{t} = \text{const} = E & (1) \\ -r^2\dot{\varphi} = \text{const} = -L & (2) \end{cases}$$

Instead of writing last E-L-equation for δr we use (*) inserting $\dot{t}$ and $\dot{\varphi}$:

$$\frac{E^2}{1-\frac{2m}{r}} - \frac{\dot{r}^2}{1-\frac{2m}{r}} - r^2\frac{L^2}{r^4} = 1$$

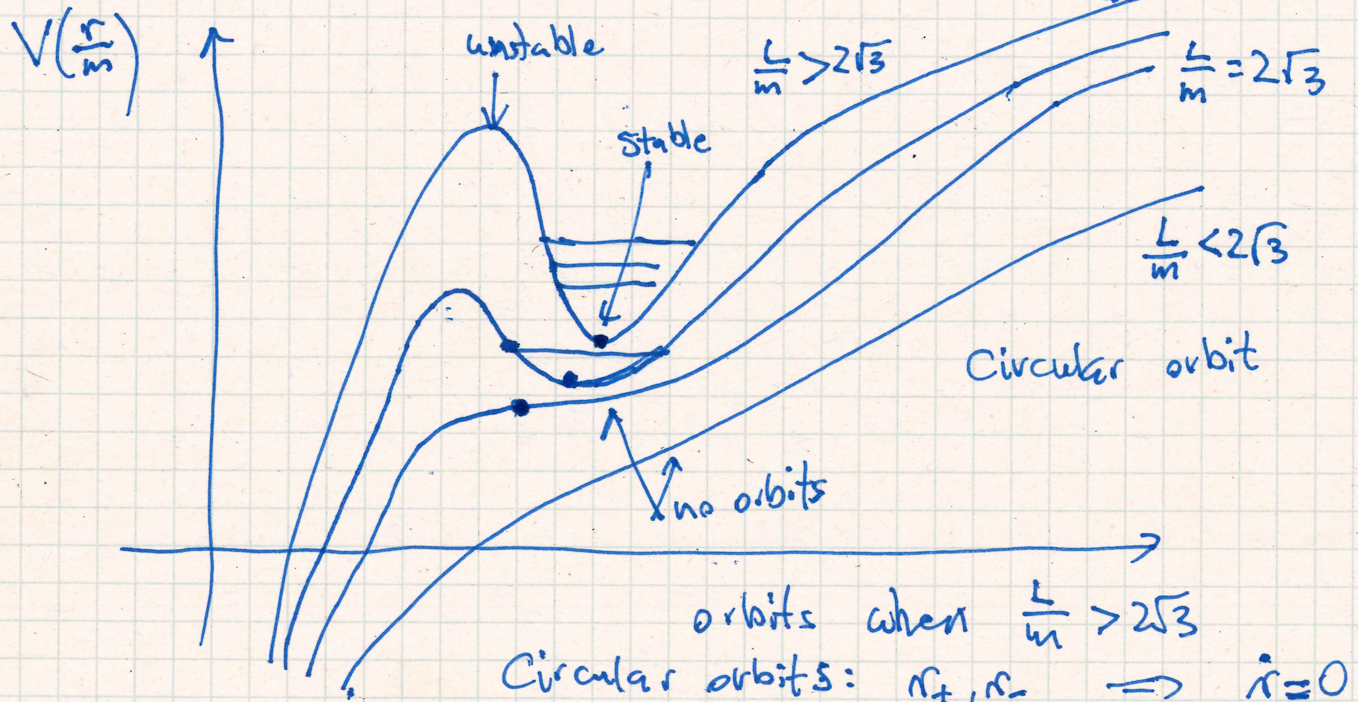
$$\boxed{E^2 = \underbrace{\dot{r}^2}_{\text{Kinetic energy}} + \underbrace{\left(1-\frac{2m}{r}\right)\left(1+\frac{L^2}{r^2}\right)}_{\text{Effective potential}}}$$

Eff. potential:

$$V_{eff}(r) = \left(1-\frac{2m}{r}\right)\left(1+\frac{L^2}{r^2}\right)$$

$$V'(r) = 0 \iff r_\pm = \frac{L^2 \pm L\sqrt{L^2-12m^2}}{2m}$$

$$V''(r) = 0 \iff r = \frac{3L^2 \pm \sqrt{3}\sqrt{3L^4-32L^2m^2}}{2m}$$



orbits when $\frac{L}{m} > 2\sqrt{3}$

Circular orbits: $r_+, r_- \implies \dot{r} = 0$

$$E^2 = V_{eff}(r_+) = \left(1-\frac{2m}{r_+}\right)\left(1+\frac{L^2}{r_+^2}\right)$$

Dependence on φ (orbits)

$$r' = \frac{dr}{d\varphi} = \frac{\dot r}{\dot\varphi} \Rightarrow \dot r = r'\dot\varphi \underset{(2)}{=} r'\frac{L}{r^2}$$

$\Rightarrow$ radial equation:

$$E^2 = r'^2\frac{L^2}{r^4} + \left(1-\frac{2m}{r}\right)\left(1+\frac{L^2}{r^2}\right)$$

Change of variable $\quad r \to u = \frac{1}{r}$

$$r' = -\frac{u'}{u^2}$$

$$E^2 = \frac{u'^2}{\cancel{u^4}}L^2\cancel{u^4} + (1-2mu)(1+L^2u^2)$$

$$(u'^2+u^2)L^2 = E^2 - 1 + 2mu + 2mL^2u^3$$

$$u'^2+u^2 = \frac{E^2-1}{L^2} + \frac{2m}{L^2}u + 2mu^3 \qquad \Big| \frac{d}{d\varphi}$$

$$2u'u''+2uu' = \frac{2m}{L^2}u' + 6mu^2u' \qquad \Big|$$ Assume that $u'\neq 0$ beyond circular orbits regime.

$$\boxed{u''+u = \frac{m}{L^2} + 3mu^2}$$

Comparison with Newton theory:

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right] - \tilde V(r)$$

φ - cyclic $\Rightarrow r^2\dot\varphi = L = \text{const}$

radial equation

$$\frac{d^2r}{dt^2} - r\dot\varphi^2 + \frac{\partial\tilde V}{\partial r} = 0$$

$$\dot{r} = r'\dot{\varphi} = r'\frac{L}{r^2} = -\frac{u'}{u^2}Lu^2 = -Lu'$$

$$\ddot{r} = -Lu''\dot{\varphi} = -Lu''Lu^2 = -L^2u''u^2$$

$$-L^2u''u^2 - \frac{1}{u}L^2u^4 + \frac{\partial\tilde{V}}{\partial r} = 0$$

$$\boxed{u'' + u = \frac{1}{L^2}\frac{\partial\tilde{V}}{\partial r}\frac{1}{u^2}}$$

If $\tilde{V}$ is a Newtonian potential: $\tilde{V} = -\frac{M}{r}$

$$\boxed{u'' + u = \frac{M}{L^2}} \quad \text{Newton}$$

$$\updownarrow$$

$$\boxed{u'' + u = \frac{m}{L^2} + 3mu^2} \quad \text{Einstein}$$

How big is $3mu^2$ when compared to $\frac{m}{L^2}$

$$\frac{3mu^2}{\frac{m}{L^2}} = 3u^2L^2 = 3\frac{1}{r^2}(r^2\dot{\varphi})^2 = 3(r\dot{\varphi})^2 = \frac{3v_\perp^2}{c^2}$$

in Solar system $\frac{3v_\perp^2}{c^2}$ is biggest at Merkury.

Miracolo

take $\tilde{V} = -\frac{m}{r} - m\frac{L^2}{r^3}$

then

$$\frac{1}{L^2}\frac{\partial\tilde{V}}{\partial r}\frac{1}{u^2} = \frac{1}{L^2}\left(mu^2 + 3mL^2u^4\right)\frac{1}{u^2} = \frac{m}{L^2} + 3mu^2$$

Relativistic effects in Schwarzschild $\rightsquigarrow$

modify Newtonian potential from $V = -\frac{m}{r}$ to $V = -\frac{m}{r} - \frac{mL^2}{r^3}$