Lecture 14 8.06.2018

- Energy functional for Schwarzschild.

$$E = \frac{1}{2}\int\left[\left(1-\frac{2m}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{2m}{r}} - r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)\right]ds$$

Timelike geodesic $\Rightarrow$ $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$ $c=1$

$$2\mathcal{L} = \boxed{\left(1-\frac{2m}{r}\right)\dot{t}^2 - \frac{\dot{r}^2}{1-\frac{2m}{r}} - r^2\left(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2\right) = 1} \quad (*)$$

- E.L. equation for θ:

$$\frac{d}{ds}\frac{\partial \mathcal{L}}{\partial\dot{\theta}} - \frac{\partial \mathcal{L}}{\partial\theta} = 0$$

$$\boxed{-(r^2\dot{\theta})^{\cdot} + r^2\sin\theta\cos\theta\,\dot{\varphi}^2 = 0}$$

this is automatically satisfied at $\theta = \frac{\pi}{2}$

From spherical symmetry, we can restrict to $\theta = \frac{\pi}{2}$ and forget about $\delta\theta$.

- $\mathcal{L}$ does not depend on t and φ $\Rightarrow$

$$\frac{d}{ds}\left[\left(1-\frac{2m}{r}\right)\dot{t}\right] = 0$$

$$\frac{d}{ds}\left[-r^2\dot{\varphi}\right] = 0$$

$$\Rightarrow \begin{cases} \left(1-\frac{2m}{r}\right)\dot{t} = \text{const} = E & (1) \\ -r^2\dot{\varphi} = \text{const} = -L & (2) \end{cases}$$

Instead of writing last E-L-equation for δr we use (*) inserting $\dot{t}$ and $\dot{\varphi}$:

$$\frac{E^2}{1-\frac{2m}{r}} - \frac{\dot{r}^2}{1-\frac{2m}{r}} - r^2\frac{L^2}{r^4} = 1$$

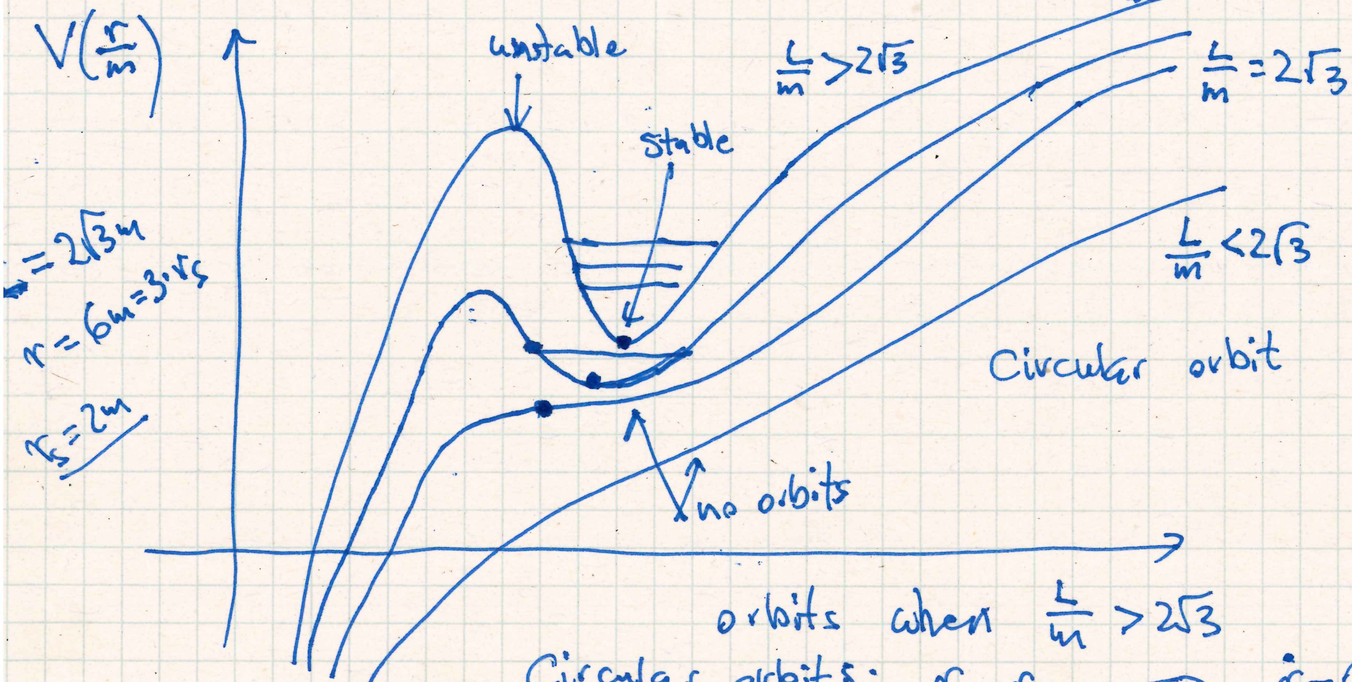
$$\boxed{E^2 = \underbrace{\dot{r}^2}_{\text{Kinetic energy}} + \underbrace{\left(1-\frac{2m}{r}\right)\left(1+\frac{L^2}{r^2}\right)}_{\text{Effective potential}}}$$

Eff. potential:

$$V_{eff}(r) = \left(1-\frac{2m}{r}\right)\left(1+\frac{L^2}{r^2}\right)$$

$$V'(r) = 0 \iff r_\pm = \frac{L^2 \pm L\sqrt{L^2-12m^2}}{2m}$$

$$V''(r) = 0 \iff r = \frac{3L^2 \pm \sqrt{3}\sqrt{3L^4-32L^2m^2}}{2m}$$



orbits when $\frac{L}{m} > 2\sqrt{3}$

Circular orbits: $r_+, r_- \implies \dot{r} = 0$

$$E^2 = V_{eff}(r_+) = \left(1-\frac{2m}{r_+}\right)\left(1+\frac{L^2}{r_+^2}\right)$$

Dependence on φ (orbits)

$$r' = \frac{dr}{d\varphi} = \frac{\dot r}{\dot\varphi} \Rightarrow \dot r = r'\dot\varphi \underset{(2)}{=} r'\frac{L}{r^2}$$

$\Rightarrow$ radial equation:

$$E^2 = r'^2\frac{L^2}{r^4} + \left(1-\frac{2m}{r}\right)\left(1+\frac{L^2}{r^2}\right)$$

Change of variable $\quad r \to u = \frac{1}{r}$

$$r' = -\frac{u'}{u^2}$$

$$E^2 = \frac{u'^2}{\cancel{u^4}} L^2 \cancel{u^4} + (1-2mu)(1+L^2u^2)$$

$$(u'^2+u^2)L^2 = E^2 - 1 + 2mu + 2mL^2u^3$$

$$u'^2 + u^2 = \frac{E^2-1}{L^2} + \frac{2m}{L^2}u + 2mu^3 \qquad \Big| \frac{d}{d\varphi}$$

$$2u'u'' + 2uu' = \frac{2m}{L^2}u' + 6mu^2u'$$

Assume that $u' \neq 0$ beyond circular orbits regime.

$$\boxed{u'' + u = \frac{m}{L^2} + 3mu^2}$$

Comparison with Newton theory:

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right] - \tilde V(r)$$

φ - cyclic $\Rightarrow r^2\dot\varphi = L = \text{const}$

radial equation

$$\frac{d^2r}{dt^2} - r\dot\varphi^2 + \frac{\partial\tilde V}{\partial r} = 0$$

$$\dot{r} = r'\dot{\varphi} = r'\frac{L}{r^2} = -\frac{u'}{u^2}Lu^2 = -Lu'$$

$$\ddot{r} = -Lu''\dot{\varphi} = -Lu''Lu^2 = -L^2u''u^2$$

$$-L^2u''u^2 - \frac{1}{u}L^2u^4 + \frac{\partial\tilde{V}}{\partial r} = 0$$

$$\boxed{u'' + u = \frac{1}{L^2}\frac{\partial\tilde{V}}{\partial r}\frac{1}{u^2}}$$

If $\tilde{V}$ is a Newtonian potential: $\tilde{V} = -\frac{M}{r}$

$$\boxed{u'' + u = \frac{M}{L^2}} \quad \text{Newton}$$

$$\boxed{u'' + u = \frac{m}{L^2} + 3mu^2} \quad \text{Einstein}$$

How big is $3mu^2$ when compared to $\frac{m}{L^2}$

$$\frac{3mu^2}{\frac{m}{L^2}} = 3u^2L^2 = 3\frac{1}{r^2}(r^2\dot{\varphi})^2 = 3(r\dot{\varphi})^2 = \frac{3v_\perp^2}{c^2}$$

in Solar system $\frac{3v_\perp^2}{c^2}$ is biggest at Merkury.

Miracolo

take $\tilde{V} = -\frac{m}{r} - m\frac{L^2}{r^3}$

then

$$\frac{1}{L^2}\frac{\partial\tilde{V}}{\partial r}\frac{1}{u^2} = \frac{1}{L^2}\left(mu^2 + 3mL^2u^4\right)\frac{1}{u^2} = \frac{m}{L^2} + 3mu^2$$

Relativistic effects in Schwarzschild ~~...~~

modify ~~N...~~ Newtonian potential from $V = -\frac{m}{r}$ to $V = -\frac{m}{r} - \frac{mL^2}{r^3}$

Perihelion advance

$3mu^2$ – small perturbation

$$u(\varphi) = \frac{1}{p}(1 + e\cos\varphi) \qquad \text{Keplerian motion}$$

$$p = a(1-e^2) = \frac{L^2}{m}$$

$$\left(\text{because} \quad u'' + u = \frac{1}{p}\right)$$

Method of succesive aproximations.

$$u^0 = \frac{1}{p}(1 + e\cos\varphi)$$

$$\boxed{u^{(1)''} + u^{(1)}} = \frac{m}{L^2} + 3mu^{0\,2} =$$

$$= \boxed{\frac{m}{L^2} + 3\frac{m^3}{L^4}(1 + e\cos\varphi)^2} \qquad (*)$$

consider three equations:

$$v'' + v = \begin{cases} A \\ A\cos\varphi \\ A\cos^2\varphi \end{cases}$$

particular solutions

$$v = \begin{cases} A \\ \frac{1}{2}A\varphi\sin\varphi \\ \frac{1}{2}A - \frac{1}{6}A\cos 2\varphi \end{cases}$$

=> general solution to *

$$\boxed{u^{(1)} = \frac{m}{L^2}(1 + e\cos\varphi) + 3\frac{m^3}{L^4}\left(1 + e\varphi\sin\varphi + \frac{e^2}{2} - \frac{e^2}{6}\cos 2\varphi\right)}$$

Perihelion – point in an orbital motion ~~orbi~~ of a body around the Sun when its distance to the Sun is ~~the~~ smallest.

Let us put perihelion when $\varphi = 0$.

next perihelion will be when $\varphi = 2\pi + \Delta\varphi$

This happens when $u' = 0$.

$$0 = u' = -\frac{me}{L^2}\sin\varphi + \frac{3m^3}{L^4}\left(e\sin\varphi + e\varphi\cos\varphi + \frac{e^2}{3}\sin 2\varphi\right)\Big|_{2\pi+\Delta\varphi}$$

$$0 = -\frac{me}{L^2}\sin\Delta\varphi + \frac{3m^3}{L^4}e\,\underbrace{(2\pi+\Delta\varphi)\cos\Delta\varphi}_{\text{small}} + \underbrace{\frac{3m^3}{L^4}e\left(\sin\Delta\varphi + \frac{e}{3}\sin 2\Delta\varphi\right)}_{\text{Small}}$$

$$\tan\Delta\varphi = \frac{6\pi m^2}{L^2}$$

$$\tan\Delta\varphi \approx \Delta\varphi$$

$$\boxed{\Delta\varphi = \frac{6\pi m^2}{L^2} = 3\pi\,\frac{\frac{2GM}{c^2}}{a(1-e^2)}}$$

43 seconds / 100 years for Mercury

This result was presented by Einstein in a lecture to the Prussian Academy of Sciences in Berlin on 18 November 1915. (Before the correct Einstein's equations!)

He said to Fokker that when he saw the result with so good agreement with observations he was so exited that he got heart palpitations.

This 43 seconds were lacking since 1859 when Le Verrier incorporated corrections to the perihelion from other planets in solar system.

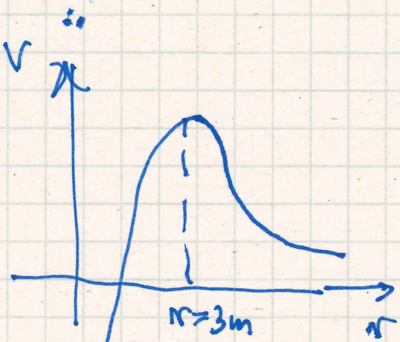
- **Deflection of light.**

Changing the last equation

$$\frac{E^2}{1-\frac{2m}{r}} - \frac{\dot{r}^2}{1-\frac{2m}{r}} - \frac{L^2}{r^2} = 1 \quad \text{from } 1 \text{ to } 0$$

we will get equations for null geodesics.

$$E^2 = \dot{r}^2 + \left(1-\frac{2m}{r}\right)\frac{L^2}{r^2}$$



now $V(r) = \left(1-\frac{2m}{r}\right)\frac{L^2}{r^2}$

$V'(r) = 0 \iff r = 3m$

unstable circular orbit.

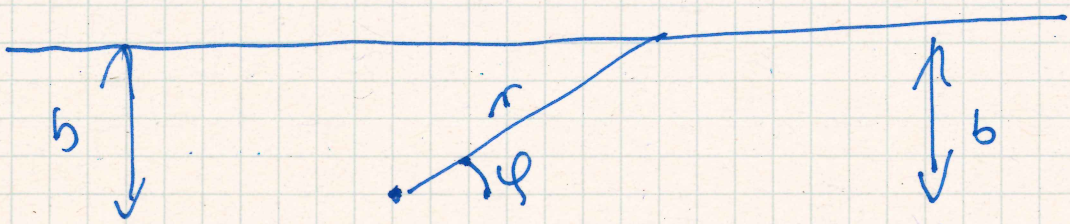
- Equation of trajectories parametrized by φ:

$$E^2 = r'^2\frac{L^2}{r^4} + \left(1-\frac{2m}{r}\right)\frac{L^2}{r^2} \qquad u = \frac{1}{r}$$

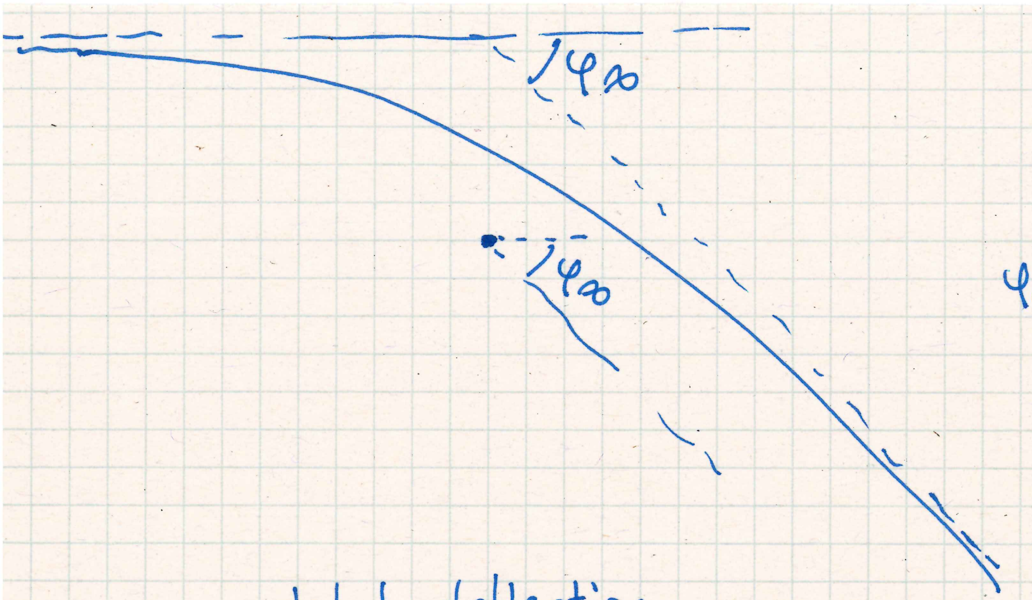
$$u'^2 + u^2 = \frac{E^2}{L^2} + 2mu^3 \qquad \frac{d}{d\varphi}$$

$$\boxed{u'' + u = 3mu^2}$$

$m = 0 \implies u = \frac{1}{b}\sin\varphi$ (equation of a straight line)

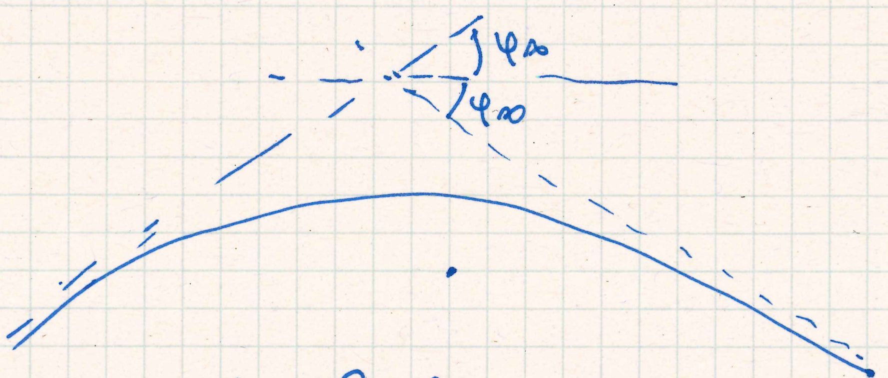


$$\frac{b}{r} = \sin\varphi \implies u = \frac{1}{b}\sin\varphi.$$



φ_∞ when $r \to \infty$

total deflection



$$\Delta\varphi = 2 \cdot \varphi_\infty$$

$\Delta\varphi$ small $\Rightarrow$ φ_∞ small.

$$u^0 = \frac{1}{b}\sin\varphi$$

$$u^{1\,''} + u^1 = 3m\, u^{0\,2}$$

$$u'' + u = \frac{3m}{b^2}\left(1 - \cos^2\varphi\right)$$

particular solution:

$$u_1 = \frac{3m}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right)$$

$$\Rightarrow \quad u = \frac{1}{b}\sin\varphi + \frac{3m}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi\right)$$

$\varphi_\infty \Leftrightarrow r \to \infty \Leftrightarrow u \to 0$

$$0 = \frac{1}{b}\sin\varphi_\infty + \frac{3m}{b^2}\left(1 + \frac{1}{3}\cos 2\varphi_\infty\right) \qquad \varphi_\infty \ll 1$$

$$\boxed{\varphi_\infty = -\frac{2m}{b}} \Rightarrow \Delta\varphi = \frac{4m}{b} = \frac{4GM}{c^2 b}$$

For sun $\Delta\varphi = 1.75'' \dfrac{R_\odot}{b}$

- How field equations determine movement of particles?

$$E_{\mu\nu} = \Lambda g_{\mu\nu} + \varkappa T_{\mu\nu}$$

$$\underbrace{\nabla_\mu E^{\mu\nu}}_{=0} = \varkappa \nabla_\mu T^{\mu\nu}$$

$$\Rightarrow \quad \boxed{\nabla_\mu T^{\mu\nu} = 0}$$

Example Perfect fluid

$$\cancel{T^{\mu\nu} = (p+\varrho) g^{\mu\nu}}$$

$$T^{\mu\nu} = (p+\varrho) u^\mu u^\nu + p g^{\mu\nu} \qquad u^\mu u_\mu = -1$$

$$0 = \nabla_\mu T^{\mu\nu} = u^\mu \nabla_\nu\left[(p+\varrho)u^\nu\right] + (p+\varrho) u^\nu \nabla_\nu u^\mu + g^{\mu\nu}\nabla_\nu p = 0$$

$u_\mu |\cdot$

$$-\nabla_\nu\left[(p+\varrho)u^\nu\right] + \underbrace{u^\nu \nabla_\nu p}_{\nabla_u p = \dot{p}} = 0 \quad \Rightarrow \quad \boxed{\dot{p} = \nabla_\nu\left[(p+\varrho)u^\nu\right]} \; (\mathrm{I})$$

$$(p+\varrho) u^\nu \nabla_\nu u^\mu = -u^\mu u^\nu \nabla_\nu p - g^{\mu\nu}\nabla_\nu p$$

$$\boxed{\underbrace{(p+\varrho) u^\nu \nabla_\nu u^\mu}_{\text{relativistic Euler's equation.}} = -\underbrace{(g^{\mu\nu} + u^\mu u^\nu)\nabla_\nu p}_{\text{projection on the space } \perp \text{ to } u_\mu}} \; (\mathrm{II})$$

e.g. DUST: $p = 0$, $\varrho \neq 0$

$\boxed{u^\nu \nabla_\nu u^\mu = 0}$ ← particles in the dust move along geodesics.

$\Rightarrow (\mathrm{I}) \Rightarrow \boxed{\nabla_\nu(\varrho u^\nu) = 0}$ ← continuity equation!