Lecture 15 26.06.2018

Spherical stars

- Equations of motions from field equations.

$$E_{\mu\nu} = \Lambda g_{\mu\nu} + \varkappa T_{\mu\nu}$$

Bianchi id: $\underset{\overset{\shortparallel}{0}}{\nabla_\mu E^{\mu\nu}} = \varkappa \nabla_\mu T^{\mu\nu}$

Example perfect fluid

$$T^{\mu\nu} = (p+\varrho) u^\mu u^\nu + p g^{\mu\nu} \qquad u^\mu u_\mu = -1$$

$$0 = \nabla_\nu T^{\mu\nu} = u^\mu \nabla_\nu\big((p+\varrho)u^\nu\big) + (p+\varrho)u^\nu \nabla_\nu u^\mu + g^{\mu\nu}\nabla_\nu p \quad \Big|\cdot u_\mu$$

$$-\nabla_\nu\big((p+\varrho)u^\nu\big) + \underbrace{u^\nu \nabla_\nu p}_{\nabla_u p = \dot p} = 0. \quad \Rightarrow \quad \boxed{\dot p = \nabla_\nu\big((p+\varrho)u^\nu\big)} \quad (\mathrm{I})$$

$$(p+\varrho)u^\nu \nabla_\nu u^\mu = -u^\mu u^\nu \nabla_\nu p - g^{\mu\nu}\nabla_\nu p$$

$$\boxed{(p+\varrho)u^\nu \nabla_\nu u^\mu = -\underbrace{(u^\mu u^\nu + g^{\mu\nu})}_{\text{projection onto 3-plane} \perp \text{ to } u^\mu}\nabla_\nu p} \quad \mathrm{II}$$

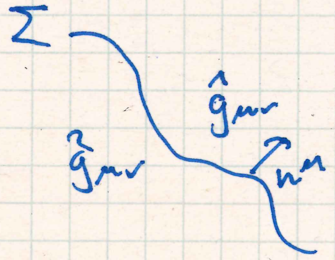
relativistic Euler's equation

e.g. DUST: $\boxed{p = 0, \ \varrho \neq 0} \Rightarrow \boxed{u^\nu \nabla_\nu u^\mu = 0}$

particles of dust move along geodesics.

$$(\mathrm{I}) \Rightarrow \boxed{0 = \nabla_\nu(\varrho u^\nu)} \leftarrow \text{continuity equation}$$

- Matching conditions Σ - spacelike



1) Ist fundamental form $\mathrm{I} = g|_\Sigma$

2) IInd fundamental form n^μ - unit normal

$$n^\mu n_\mu = -1, \quad h^\alpha{}_\beta = g^\alpha{}_\beta + n^\alpha n_\beta$$

$$\mathrm{II}_{\mu\nu} = \nabla_\alpha n_\beta h^\alpha{}_\mu h^\beta{}_\nu$$

Matching conditions: I and II must be CONTINUOUS along Σ.

2) Spherically symmetric static distribution of mass.

$$g = -e^{2\mu(r,t)}dt^2 + e^{2\nu(r,t)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Assume <u>static</u> $\quad \dot{\mu} = \dot{\nu} = 0$

$$E_{00} = \frac{1 + e^{-2\nu}(2r\nu' - 1)}{r^2}$$

$$E_{11} = \frac{-1 + e^{-2\nu}(2r\mu' + 1)}{r^2}$$

$$E_{22} = \frac{e^{-2\nu}}{r}\left[(1 + r\mu')(\mu' - \nu') + r\mu''\right]$$

Recall
This is in the orthonormal frame:
$\theta^0 = e^{\mu}dt$
$\theta^1 = e^{\nu}dr$
$\theta^2 = r\, d\theta$
$\theta^3 = r\sin\theta\, d\varphi$

<u>Einstein equations inside a star.</u>

$$\begin{cases} E_{\mu\nu} = \varkappa T_{\mu\nu} \\ T_{\mu\nu} = (\rho + p) u^{\mu} u^{\nu} + p g^{\mu\nu} \end{cases} \qquad u^{\mu} u_{\nu} = -1$$

$u_{\mu} = (u_t, u_r, u_{\theta}, u_{\varphi})$

Fluid is STATIC, should not move $\Rightarrow \boxed{u_r = u_{\theta} = u_{\varphi} = 0}$

$$-1 = u_t^2 g^{tt} \qquad g^{tt} = -e^{-2\mu}$$

$$\Rightarrow \boxed{u_t = e^{\mu}}$$

in the orthonormal frame

$$\Rightarrow \quad u = e^{\mu} dt \quad \Rightarrow \quad \boxed{u_0 = 1, \quad u_1 = u_2 = u_3 = 0}$$

Einstein's equations

$$\frac{1 + e^{-2\nu}(2r\nu' - 1)}{r^2} = \varkappa \rho \qquad (1)$$

$$\frac{-1 + e^{-2\nu}(2r\mu' + 1)}{r^2} = \varkappa p \qquad (2)$$

$$\frac{e^{-2\nu}}{r}\left[(1 + r\mu')(\mu' - \nu') + r\mu''\right] = \varkappa p \qquad (3)$$

Euler's equation $\quad \boxed{(\rho + p) u^{\nu} \nabla_{\nu} u^{\mu} = -h^{\mu\nu} \nabla_{\nu} p}$

$$h^{\mu\nu} = g^{\mu\nu} + u^{\mu} u^{\nu}$$

Spherical symmetry requires $\boxed{p = p(r),\ \varrho = \varrho(r)}$

$$u^\nu \nabla_\nu u^\mu = u^t \nabla_t u^\mu =$$
$$= u^t \left(\cancel{u^\mu{}_{,t}} + \Gamma^\mu{}_{\nu t} u^\nu\right) = u^t \Gamma^\mu{}_{tt} u^t = e^{-2\mu}\Gamma^\mu{}_{tt}$$

$$\Gamma^\mu{}_{tt} = \frac{1}{2} g^{\mu\varrho}\left(\cancel{g_{\varrho t,t}} + \cancel{g_{t\varrho,t}} - g_{tt,\varrho}\right) = -\frac{1}{2} g^{\mu r} g_{tt,r}$$

Only $\Gamma^r{}_{tt} \neq 0$, $\Gamma^r{}_{tt} = +\frac{1}{2} e^{-2\nu} e^{2\mu} 2\mu' = \mu' e^{2(\mu-\nu)}$

Euler's eqs for $\mu = r$

$$(\varrho + p)\mu' e^{-2\nu} = -e^{-2\nu} p' \implies \boxed{\mu' = -\frac{p'}{\varrho + p}}$$

other components of Euler's equation gives nothing.

$$(1) \implies e^{-2\nu}\left(\frac{1}{r^2} - \frac{2\nu'}{r}\right) - \frac{1}{r^2} = -\varkappa\varrho$$

$$X = e^{-2\nu},\ X' = -2\nu' e^{-2\nu}$$

$$\frac{X}{r^2} + \frac{X'}{r} - \frac{1}{r^2} = -\varkappa\varrho \quad \text{linear equation for } X$$

$$X = \boxed{e^{-2\nu} = 1 - \frac{\varkappa}{r}\int_0^r \varrho(r') r'^2 dr'}$$

$\varkappa = \frac{8\pi G}{c^4}$

$G = c^4 = 1$

Introduce

$$\boxed{m(r) := 4\pi \int_0^r r'^2 \varrho(r')\, dr'}$$

$$\boxed{e^{-2\nu} = 1 - \frac{2m(r)}{r}}$$

In particular if $r = R$ is (Schwarzschild) radius of a star

$$\boxed{4\pi \int_0^R r^2 \varrho(r)\, dr = m = \text{const}}$$

outside $Ric = 0$ + spher. symmetry

$\Downarrow$

outside we have Schwarzschild.

$(2) \implies$ ~~$\frac{1}{r^2}(-1 + \frac{1-2m}{r})$~~

$$\frac{1}{r^2}\left(-1 + \left(1 - \frac{2m}{r}\right)\left(1 - 2r\frac{p'}{\varrho + p}\right)\right) = \varkappa p$$

$$+\frac{2}{r}\left(1 - \frac{2m}{r}\right)\frac{p'}{\varrho + p} + \frac{2m}{r^3} = \varkappa p$$

$$\frac{p'}{\varrho+p} = -\frac{m+4\pi p r^3}{r^2\left(1-\frac{2m}{r}\right)} \Rightarrow \boxed{p' = -(\varrho+p)\frac{m+4\pi p r^2}{r^2\left(1-\frac{2m}{r}\right)}}$$

Oppenheimer-Volkoff equation.

Summary – Spherically symmetic star:

for $r \geq R$ $\quad g = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$

for $r < R$ $\quad g = -e^{2\mu(r)}dt^2 + \frac{dr^2}{1-\frac{2m(r)}{r}} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$

$$(1)\left\{\begin{aligned} & m(r) = \int_0^r 4\pi\,\varrho(r')\,r'^2\,dr' \\ & m(R) = m \\ & p' = -(\varrho+p)\frac{m(r)+4\pi p r^3}{r^2\left(1-\frac{2m}{r}\right)} \\ & \mu' = \frac{m(r)+4\pi p r^3}{r^2\left(1-\frac{2m}{r}\right)} \\ & e^{2\mu(r)} = 1-\frac{2m}{r} \end{aligned}\right.$$

these AUTOMATICALLY satisfy (3).

We need equation of state $\boxed{p = p(\varrho)}$ (2) which should be added.

and initial/boundary values

$$\boxed{\begin{aligned} & p(R) = 0, \\ & m_c = m(0) = 0. \end{aligned}}$$

Example

$\rho = \rho_0 = \text{const.}$ STIFF star.

$$m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & r \leq R \\ M = \frac{4}{3}\pi R^3 \rho_0 & r = R \end{cases}$$

$$\frac{m(r)}{M} = \frac{r^3}{R^3} \qquad m(r) = M\frac{r^3}{R^3}$$

$$\boxed{p' = (-\rho_0 + p)\, r \, \frac{\frac{M}{R^3} + 4\pi p}{1 - \frac{2Mr^2}{R^3}}}$$ linear equation for p

Integrating:

$$p = \rho_0 \frac{\sqrt{1-\frac{2Mr^2}{R^3}} - \sqrt{1-\frac{2M}{R}}}{3\sqrt{1-\frac{2M}{R}} - \sqrt{1-\frac{2Mr^2}{R^3}}}$$

$$\Rightarrow \mu' = \frac{M\frac{r^3}{R^3} + 4\pi p r^3}{r^2\left(1 - \frac{2Mr^2}{R^3}\right)}$$

$$e^{\mu} = \frac{3}{2}\left(1-\frac{2M}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2Mr^2}{R^3}\right)^{1/2}$$

Discussion

$$p_c = \rho_0 \frac{1 - \sqrt{1-\frac{2M}{R}}}{3\sqrt{1-\frac{2M}{R}} - 1}$$

1. p_c monotonically increases with ~~[illegible]~~ M and decreases with R.

2. if $9\left(1-\frac{2M}{R}\right) - 1 = 0 \Leftrightarrow 9\cdot\frac{2M}{R} = 8 \Leftrightarrow \frac{2M}{R} = \frac{2^3}{3^2}$

$\Rightarrow$ pressure in the center is infinite

$\Rightarrow \frac{2M}{R} \leq \frac{2^3}{3^2}$ for a star with constant density

3. if $t = \text{const} \Rightarrow g_{t=\text{const}} = +\frac{dr^2}{1-\frac{2Mr^2}{R^3}} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$

metric on a 3-dim sphere of radius $a = \sqrt{\frac{R^3}{2m}}$

$r = a\sin\chi$
$dr = a\cos\chi\, d\chi$

- <u>Kerr metric</u> – stationary and axially symmetric.

<u>Def</u>

Gravitational field is stationary and axially symmetric iff it admits two Killing field X, Y s.t

a) $X^2 < 0$

b) $Y^2 > 0$ except the axis where $Y = 0$

c) orbits of Y are closed

d) $[X, Y] = 0$.

$\Rightarrow$ one can choose coordinates t, φ s.t. $X = \frac{\partial}{\partial t}$, $Y = \frac{\partial}{\partial \varphi}$

period 2π.

$$\boxed{g = e^{2\mu}(dt + \omega\, d\varphi)^2 - e^{-2\mu}\left(\gamma_{AB}\, dx^A dx^B + W^2 d\varphi^2\right) \quad A,B = 1,2 \qquad \mu = \mu(x^A),\ \gamma_{AB} = \gamma_{AB}(x^C),\ W = W(x^A),\ \omega = \omega(X^A)}$$

$$Ric(g) = 0 \Rightarrow \underbrace{\nabla_A \nabla^A}_{\text{cov. derivatives w.r.t. } \gamma_{AB}} W = 0$$

$W = \operatorname{Re} f(x^1 + i x^2)$ $\quad x^1, x^2$ – coords in which metric diagonalizes.

Taking $f(x^1 + i x^2) = \rho + i z$

$$\boxed{g = e^{2\mu}(dt + \omega d\varphi)^2 - e^{-2\mu}\left(e^{2k}(d\rho + dz^2) + \rho^2 d\varphi^2\right)}$$

Weyl's coordinates.

Boyer-Lindquist coordinates.

$$g = -\frac{\Delta}{\varrho^2}\left(dt - a\sin^2\vartheta\, d\varphi\right)^2 + \frac{\sin^2\vartheta}{\varrho^2}\left[(r^2+a^2)\,d\varphi - a\,dt\right]^2 + \varrho^2\left(d\theta^2 + \frac{dr^2}{\Delta}\right)$$

Kerr's solution.

$$\varrho^2 = r^2 + a^2\cos^2\vartheta$$

charged

$$\Delta = r^2 - 2mr + a^2 \quad + (Ge^2)$$

$\frac{m}{G}$ – mass $\cdot\, c^{-2}$

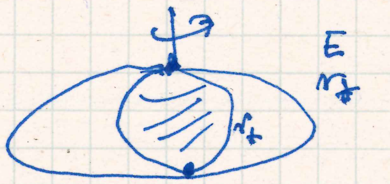
$\frac{ma}{G}$ – angular momentum

$$m = \frac{GM}{c^2}, \quad a = \frac{J}{Mc}, \quad e = \frac{Q}{c^2}$$

$\Delta = 0 \qquad r_\pm = m \pm \sqrt{m^2 - a^2 - Ge^2}$

horizon
$g_{rr} = \infty$

$$r_+ = m + \sqrt{m^2 - a^2}$$



$$-\frac{\Delta}{\varrho^2} + \frac{a^2\sin^2\vartheta}{\varrho^2} \sim$$

$$\sim (r^2 - 2mr + a^2) + a^2\sin^2\vartheta$$

$$-r^2 + 2mr - a^2\cos^2\vartheta = 0$$

$$\cancel{\Delta = 4m^2 - 4a^2\cos^4\vartheta}$$

$$\Delta = 4m^2 - 4a^2\cos^2\vartheta$$

$$r^E_\pm = \frac{2m \pm 2\sqrt{m^2 - a^2\cos^2\vartheta}}{2}$$

$$r^E_\pm = m \pm \sqrt{m^2 - a^2\cos^2\vartheta}$$

$$g(X,X) = 0$$