Lecture 16   29.06.2018

Cosmological models

In cosmology one assumes that gravitation is the main force that determines evolution of Universe.

1) Cosmological principles

- Perfect cosmological principle

"apart from local inhomogeneities the Universe is homogeneous at every place and at every time"

realization: Einstein Universe

$M = \mathbb{R} \times \mathbb{R}^3$
$$g = -dt^2 + R^2 \frac{dx^2+dy^2+dz^2}{\left(1+\frac{1}{4}(x^2+y^2+z^2)\right)^2}$$

satisfies Einstein's equations with $\Lambda = -\frac{1}{R^2}$, and dust of constant density $\rho = \frac{1}{4\pi R^2}$

- principle of spatial homogeneity: (Copernican principle)

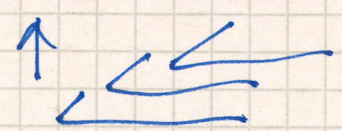
" apart ... the Universe is homogeneous at every place

- principle of spatial isotropy

" apart ... ... in every direction "

2) Mathematical formulation of spatial homogeneity:

Both gravitational field (metric) and matter fields admit a symmetry Group G, which when acting on (M, g) has ~~spatial orbits~~ SPACELIKE hypersurfaces as orbits.

t 

foliation of M by orbits of symmetry group t = const specify orbit

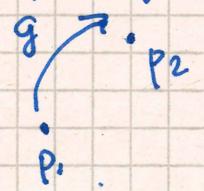
~~$g = -dt^2 + g_{ij}(x^k, t)\, dx^i dx^j$~~

$\cancel{\boxed{g = -dt^2 + R^2(t)\, g_{ij}(x^k)\, dx^i dx^j}}$ $\quad g = -dt^2 + g_{ij}(t, x^k)\, dx^i dx^j$

How $g_{ij}(x^k)$ looks like depends on the action of G on M

$G \supset G_0$ – which acts SIMPLY transitively on orbits – unique!

- exist – Bianchi models
- do not exist – Kantowski-Sachs.



Bianchi models

Choosing a point on an orbit one gets 1-1 correspondence between p and G_0 $\quad p \leftrightarrow g$ s.t. $p = g \cdot p_0$

One can identify $t = \text{const}$ with G_0.

θ^i – basis of Maurer-Cartan forms on G_0

$$\boxed{g = -dt^2 + g_{ij}(t)\, \theta^i \theta^j} \quad (*)$$

metric that satisfies Copernican principle.

Bianchi types of 3-dimensional Lie groups:

I $\quad \theta^1 = dx,\ \theta^2 = dy,\ \theta^3 = dz$

II $\quad \theta^1 = dx - z\,dy,\ \theta^2 = dy,\ \theta^3 = dz$

IV $\quad \theta^1 = dx,\ \theta^2 = e^x dy,\ \theta^3 = e^x (dz + x\,dy)$

V $\quad \theta^1 = dx,\ \theta^2 = e^x dy,\ \theta^3 = e^x dz$

VI $\quad \theta^1 = dx,\ \theta^2 = e^{Ax}(\cosh x\, dy - \sinh x\, dz);\ \theta^3 = e^{Ax}(-\sinh x\, dy + \cosh x\, dz)$

VII $\quad \theta^1 = dx,\ \theta^2 = e^{Ax}(\cos x\, dy - \sin x\, dz),\ \theta^3 = e^{Ax}(\sin x\, dy + \cos x\, dz)$

VIII $\begin{cases} \theta^1 = \cosh y \cos z \, dx - \sin z \, dy \\ \theta^2 = \cosh y \sin z \, dx + \cos z \, dy \\ \theta^3 = \sinh y \, dx + dz \end{cases}$ $\qquad G_0 = SO(1,2)$

IX $\begin{cases} \theta^1 = \cos y \cos z \, dx - \sin z \, dy \\ \theta^2 = \cos y \sin z \, dx + \cos z \, dy \\ \theta^3 = -\sin y \, dx + dz \end{cases}$ $\qquad G_0 = SO(3)$

substitute any of $\underline{I} - \underline{IX}$ to $(*)$ + Einstein equations $\Rightarrow$ Bianchi model

3) Mathematical formulation of spatial isotropy

Symmetry group G of matter and gravitational fields acts on spacelike hypersurfaces (which are its orbits) and at every point has $SO(3)$ as its isotropy subgroup.

$\underset{\Sigma}{\underset{\parallel}{\{t = \text{const}\}}} \Rightarrow g|_{t=\text{const}} = ?$

$$\underset{3}{\underset{\parallel}{\dim \Sigma}} = \dim G - \underset{3}{\underset{\parallel}{\dim \overset{SO(3)}{G_0}}}$$

$$\Downarrow$$

$$\boxed{\dim G = 6}$$

maximal symmetry ~~for~~ Σ

$\Rightarrow$ Metric on Σ must be metric of space of const. curvature.

$$g|_\Sigma = \frac{dx^2 + dy^2 + dz^2}{\left(1 + \frac{K}{4}(x^2 + y^2 + z^2)\right)^2}$$

$$g = -dt^2 + R^2(t)\,\frac{dx^2+dy^2+dz^2}{\left(1+\frac{k(t)}{4}(x^2+y^2+z^2)\right)^2}$$

$$T_p(M) = \mathbb{R}n \oplus T_p\Sigma \qquad n = \frac{\partial}{\partial t}$$

Σ ← orbit

vectors v^μ
- $\to v^0$ along n
- $\to \underline{\underline{v^i}}$ must be zero, otherwise it would distinguish direction and would brake isotropy.

v^0 must be constant on the orbits, otherwise spatial homogeneity would be broken.

$$v^0 = v^0(t)$$

$T^{\mu\nu}$
- T^{00}
- $T^{0i} = 0$
- $T^{i0} = 0$
- $T^{ij} \sim g^{ij}$

$$T^{00} = \rho(t)$$
$$T^{ij} = p(t)\,g^{ij}$$

$$E^{\mu\nu} = \Lambda g^{\mu\nu} + 8\pi G T^{\mu\nu}$$

$\Rightarrow \quad \dot{k} = 0 \quad$ $\underline{\underline{k = \text{const}}}$

$$\boxed{g = -dt^2 + R^2(t)\,\frac{dx^2+dy^2+dz^2}{\left(1+\frac{k}{4}(x^2+y^2+z^2)\right)^2}}$$

Robertson–Walker metric.

Only two equations

$$\begin{cases} \frac{8\pi}{3} G\rho = \frac{\dot{R}^2 + K}{R^2} + \frac{\Lambda}{3} \\ G\frac{8\pi}{3}(\rho + 3p) = -\frac{2\ddot{R}}{R} - \frac{2}{3}\Lambda \end{cases}$$

+ equation of state.

From now on DUST $p = 0$

$$\begin{cases} \frac{8}{3} G\pi\rho = \frac{\dot{R}^2 + K}{R^2} + \frac{\Lambda}{3} \qquad (1) \\ \frac{2\ddot{R}}{R} = -\frac{\dot{R}^2 + K}{R^2} - \Lambda \end{cases}$$

$$G\frac{8}{3}\pi\dot{\rho} = \frac{2\dot{R}\ddot{R}}{R^2} - 2\dot{R}\frac{\dot{R}^2 + K}{R^3} = \frac{\dot{R}}{R}\left(-\frac{\dot{R}^2 + K}{R^2} - \Lambda - 2\frac{\dot{R}^2 + K}{R^2}\right) =$$

$$= \frac{\dot{R}}{R}\left(-3\frac{\dot{R}^2 + K}{R^2} - \Lambda\right) = -8\pi\rho\frac{\dot{R}}{R}$$

$$\dot{\rho} + 3\rho\frac{\dot{R}}{R} = 0 \Rightarrow \dot{\rho}R + 3\rho\dot{R} = 0$$

$$\Rightarrow (\rho R^3)^{\cdot} = 0$$

$$\Rightarrow \rho R^3 = \text{const}$$

$$\Rightarrow \frac{4}{3}\pi\rho R^3 = M$$

inserting in (1)

$$\boxed{\frac{2MG}{R} = \dot{R}^2 + k + \frac{\Lambda}{3}R^2}$$

$$\boxed{\frac{1}{2}\dot{R}^2 - \frac{GM}{R} + \frac{\Lambda}{6}R^2 = -\frac{k}{2}}$$

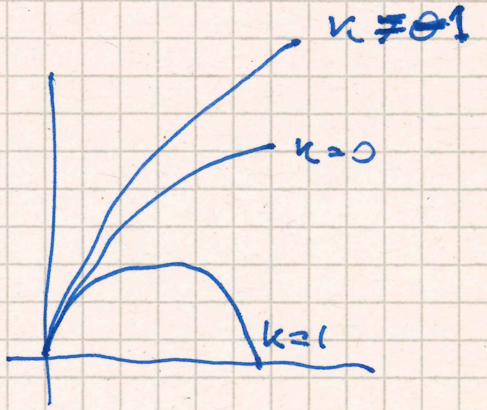
Friedman $\Lambda = 0$

Le Maitre $\Lambda \neq 0$.

$\Lambda = 0.$

(A) $k = 0 \implies \frac{dR}{dt} = \sqrt{\frac{2MG}{R}}$

$\implies R = \sqrt[3]{\frac{9MG}{2}}\, t^{2/3}$



(B) $k = 1 \implies t = MG(\eta - \sin\eta)$

$R = MG(1 - \cos\eta)$

(C) $k = -1 \quad t = MG(\operatorname{sh}\eta - \eta)$

$R = MG(\operatorname{ch}\eta - \eta)$

4) Measurable cosmological parameters $\quad \Lambda = 0.$

$$\frac{8\pi G}{3}\rho = \frac{\dot R^2 + k}{R^2} = -\frac{2\ddot R}{R}$$

(1) H - Hubble "constant" $\quad \boxed{H := \frac{\dot R}{R}}$

$$\frac{8\pi G}{3}\rho = 2q H^2 = 2q\frac{\dot R^2}{R^2} = -\frac{2\ddot R}{R}$$

$$q = -\frac{\ddot R}{\dot R^2} R \qquad qH^2 = -\frac{\ddot R}{R}$$

(2) deccelaration parameter.

(3) Critical density.

$$\boxed{\frac{8\pi G}{3}\rho_c := H^2} \implies \frac{8\pi G}{3}\rho = 2q\frac{8\pi G}{3}\rho_c$$

$$\frac{k}{R^2} = \frac{8\pi G}{3}\rho - H^2 = \frac{8\pi G}{3}(\rho - \rho_c) = \frac{8\pi G}{3}\rho_c(2q - 1)$$

$\operatorname{sgn}(k) = \operatorname{sgn}(2q-1) \quad$ Density parameter $\Omega = \frac{\rho}{\rho_c} = 2q$