Lecture 2 17.04.2018

Pre-Einstein spacetimes

(M, V) – affine space $\qquad \dim V = n \leftarrow$ dimension of (M,V)

$+ : M \to M$ s.t.

- $(p+u)+v = p+(u+v)$; $p+0 = p$
- $p+u = p \Rightarrow u = 0$
- $\forall p, q\ \exists u$ s.t. $q = p+u$

u is unique!

Line $\quad p, u \Rightarrow p(\lambda) = p + \lambda \cdot u$

Affine transformation

$(M,V) \Rightarrow (f, \alpha_f)$ s.t. $f(p+u) = f(p) + \alpha_f u$

f, α_f – bijections $\qquad \alpha_f : V \xrightarrow[\text{linear}]{} V$

Repère $\quad (o, e)$ s.t. $o \in M$, e – basis in V

$p = o + x^\mu(p) e_\mu$. This gives a map $p \to x^\mu(p)$.

$f(p) = o + x'^\mu e_\mu$

$\shortparallel$

$f(o) + \alpha_f (x^\mu e_\mu) = f(o) + x^\mu \alpha_f e_\mu = f(o) + x^\mu e_\nu \alpha^\nu{}_\mu =$

$= f(o) + (o - o) + x^\mu e_\nu \alpha^\nu{}_\mu = o + \underbrace{(f(o) - o)}_{\beta^\nu e_\nu} + \alpha^\nu{}_\mu x^\mu e_\nu$

$$\Rightarrow \boxed{x'^\nu = \alpha^\nu{}_\mu x^\mu + \beta^\nu}$$

$\alpha^\nu{}_\mu$: invertible matrix, "rotation"; β^ν: constant vector, "translation"

Model Spacetime is a 4-dimensional affine space. (M,V), $\dim V = 4$.

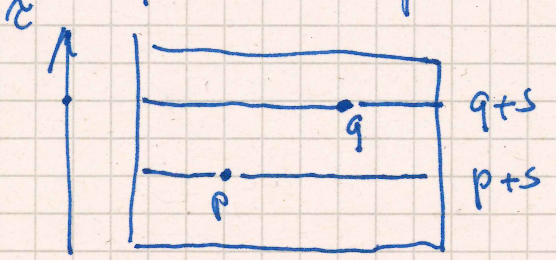
Absolute time

(M, V) equipped with $\tau : V \xrightarrow{\text{linear}} \mathbb{R}$, $\tau \neq 0$

Space like vectors $S = \{ v \in V \text{ s.t. } \tau(v) = 0 \}$

$p \in M \Rightarrow p + S = \{ q \in M \text{ s.t. } q - p \in S \}$

↑ hypersurface of events simultaneous with p.



p & q are simultaneous $\Leftrightarrow p - q \in S$.
$\Leftrightarrow \tau(p-q) = 0$.

Let $\theta \in M$. Then an absolute time measured from θ to p is $t = \tau(p - \theta)$

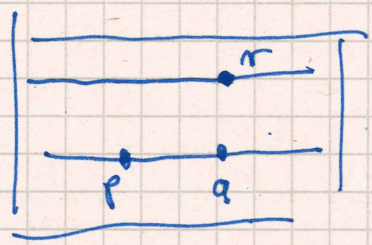
We equip (M, V, τ) with a metric structure in $S = \ker \tau$.

$h : S \times S \longrightarrow \mathbb{R}$, symmetric bilinear, positive definite.

Then, if p and q are simultaneous the distance between p and q is:

$$|p - q| = \sqrt{h(p-q, p-q)}$$

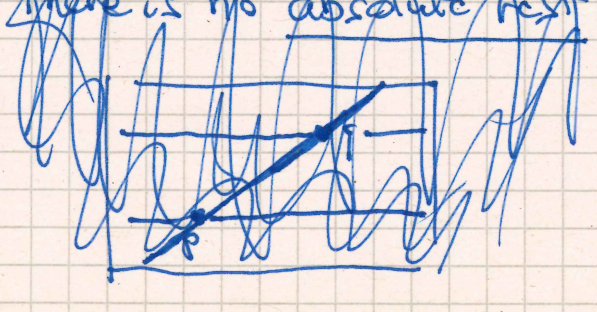
Note one can measure distances only between simultaneous events



$|p - q| = \sqrt{h(p-q, p-q)}$

but, dist(p, r) or dist(q, r) not defined.

~~There is no absolute rest~~



2) Arystotelian and Galilean spacetime.

Spacetime is a 4 dimensional smooth manifold.
Points of this manifold = events in spacetime.
4-dimensions: because an event needs 4 coordinates to be identified (time, elevation, latitude, longitude) if on Earth.

No gravitational effects for today!

Galileo/Newton: First law of dynamics.

① There is a preferred class of motions, called free motions
② there exist reference frames relative to which the free motions have no acceleration. (inertial frames)

Remarks

Ad ① one adds that a body is in free motion when no external influences act on it.

Ad ② This is usually formulated that free motions are rectilinear (along straight lines) and uniform. relative to certain reference frames called inertial frames

This second formulation is equivalent to ② if one defines what rectilinearity and uniformity means.

⇝ This requires additional mathematical (geometric) structure on a manifold.

⟹ Notion of an AFFINE SPACE.

Reference frame: minimal set of physical objects (clocks, theodolites, rangefinders) using which one can associate (four) coordinates to events in a given region of spacetime.

Physical concept.

Mathematically, having a model (M, V, τ, h), we can find a mathematical object that realizes this consept.

A repère is good to define x^μ associated to p. everywhere in M.

It is convenient to restrict to repèrs in which 0^{th} component in x^μ, x^0 is an absolute time.

Or, better, to restrict to repers that maximally respects the geometry of (M, V, τ, h).

⇒ Inertial repère is a repere (θ, e) in (M, V, τ, h) such that:

1) $\tau(e_0) = 1$
2) $\tau(e_i) = 0$, $i = 1, 2, 3$
3) $h(e_i, e_j) = \delta_{ij}$ $i = 1, 2, 3$.

Having such a repère we have

$$p = \theta + x^0 e_0 + x^i e_i$$

In particular ~~...~~

$$\tau(p - \theta) = x^0 = t$$

↑ absolute time between p and θ.

Absolute time ≡ x^0 coordinate in an inertial ~~frame~~ repère related to θ.

Galilean transformation of (M, V, τ, h) is an affine transformation (f, γ_f), $f: M \xrightarrow[\text{invertible}]{} M$, $\gamma_f : V \xrightarrow[\text{linear}]{\text{invertible}} V$ preserving the structure i.e.

1) $\tau(\gamma_f(u)) = \tau(u)$

2) $h(v, \omega) = h(\gamma_f(v), \gamma_f(\omega)) \quad \forall\, v, \omega \in S = \ker \tau$.

Its form in inertial ~~frame~~ repère:

$$p = \theta + t e_0 + x^i e_i$$

and
$$f(p) = \theta + t' e_0 + x'^i e_i$$
$$\gamma_f(e_0) = P e_0 + V^i e_i$$
$$\gamma_f(e_i) = \cancel{Q_i e_0} + R^i{}_j e_i$$

$$\tau(\gamma_f(e_0)) = P \qquad \tau(\gamma_f(e_i)) = Q_i$$
$$\tau(\gamma_f(e_0)) = 1 \qquad \tau(\gamma_f(e_i)) = \tau(e_i) = 0$$

$\Rightarrow Q_i = 0, \; P = 1.$

$$h(\gamma_f(e_i), \gamma_f(e_j)) = h(R^k{}_i e_k, R^l{}_j e_l) = R^k{}_i R^l{}_j \delta_{kl}$$
$$h(\gamma_f(e_i), \gamma_f(e_j)) = h(e_i, e_j) = \delta_{ij}$$

$$\boxed{R^k{}_i R^l{}_j = \delta_{ij}} \Rightarrow R \text{ are orthogonal matrices w.r.t. } h_{ij} = \delta_{ij}$$

$$\Rightarrow f(p) = f(\theta + t e_0 + x^i e_i) = f(\theta) + t(e_0 + V^i e_i) + x^i R^k{}_i e_k = \theta + \underbrace{f(\theta) - \theta}_{\alpha^i e_i + t_0 e_0} + t e_0 + t V^i e_i + x^k R^i{}_k e_i$$

$$= \theta + (t + t_0) e_0 + (R^i{}_k x^k + t V^i + \alpha^i) e_i$$

$$\Rightarrow \boxed{\begin{aligned} t' &= t + t_0 \\ x'^i &= R^i{}_k x^k + t V^i + \alpha^i \end{aligned}}$$

Movements in (M, V, τ, h) are ~~curves~~ parametrized curves $\mathbb{R} \ni \lambda \longrightarrow p(\lambda) \in M$.

Among them, there are parametrized lines

$$p(\lambda) = p + \lambda u.$$

A movement is rectilinear and uniform iff it is a line

$$p(t) = p + tu$$

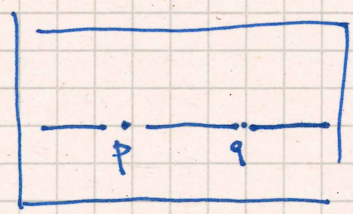
parameterized by an absolute time t.

The movement in an inertial ~~frame~~ repere is described in terms of a curve

$$x^\mu(t) = x^\mu_0 + t u^\mu \quad \text{in } \mathbb{R}^4$$

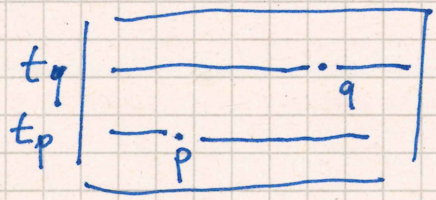
$$\Rightarrow \quad \frac{d^2 x^\mu}{dt^2} = 0 \Longrightarrow \text{movement without acceleration, UNIFORM.}$$

In Galilean physics **no absolute rest!**



$$|p-q| = \sqrt{h(p-q, p-q)}$$

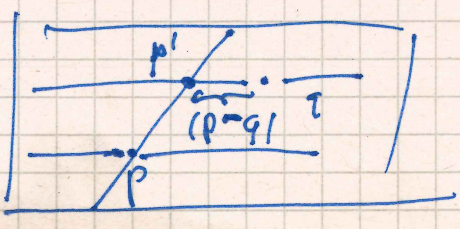
But



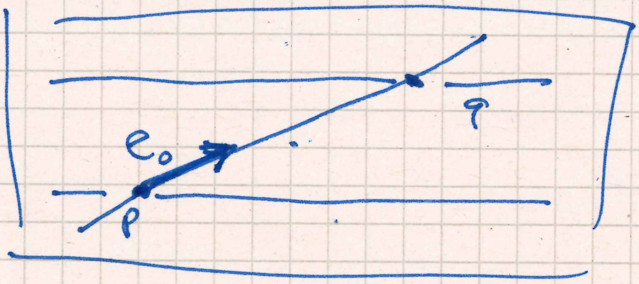
What is the distance between p and q? One can find infinitely many answers:

One can travel with constant velocity on straight line (which is not distinguishable from the rest!) from hypersurface t_p to t_q and say that

$$|p-q| = \sqrt{h(p'-q, p'-q)}$$



But what if we take another velocity? E.g. as on the picture:



If we take an inertial repere in which

$$e_0 = \frac{q-p}{\tau(q-p)}$$

Then $x^\mu(p) = (t_p, \underline{\underline{x^i}})$, $x^\mu(q) = (t_q, \underline{\underline{x^i}})$

This shows that in this reference frame point p is <u>in the same</u> place <u>that point</u> q !

<u>Aristotelian vs Galilean models of space-time</u>

In Galileo model we have (M, V, τ, h)

h is in $S = \ker \tau$.

<u>Aristotelian</u> model <u>has more structure</u>!

For Aristotle $V = \mathbb{R} \oplus S$

We have: $V \ni u = (u^0, \vec{u})$ and TWO PROJECTIONS

$$\left.\begin{aligned} \tau &= pr_{\mathbb{R}} \\ \sigma &= pr_S \end{aligned}\right\} \quad \begin{aligned} \tau(u) &= u^0 \\ \sigma(u) &= \vec{u}. \end{aligned}$$

Now, an absolute time t between p and q is

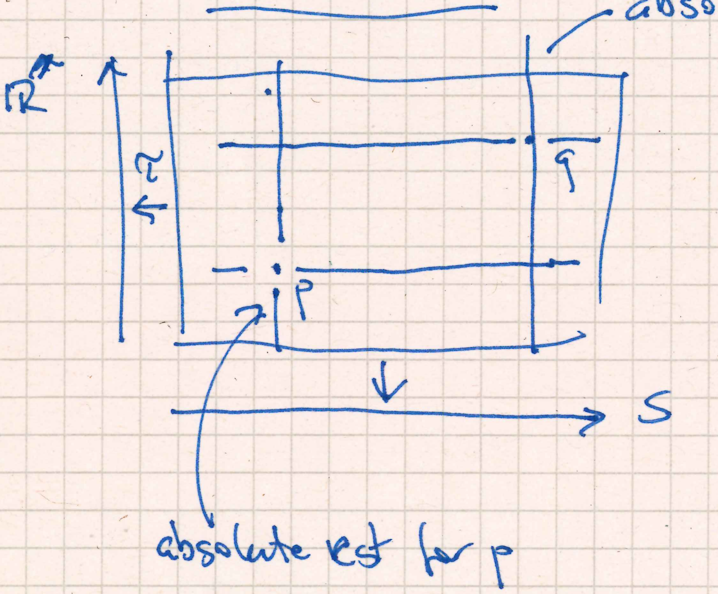
$$t_{pq} = \tau(p-q)$$

and absolute distance between p and q is

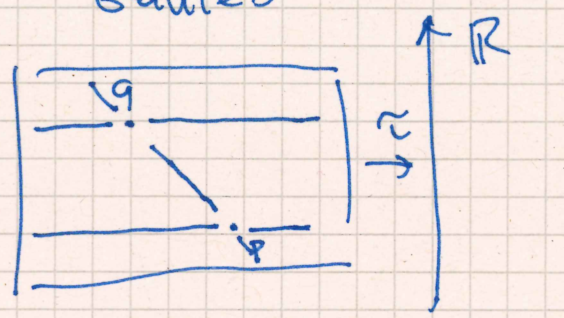
$$|p-q| = \sqrt{h(\sigma(p-q), \sigma(p-q))}.$$

In particular q is in a world line of p staing in <u>absolute rest</u> iff $\tau(q-p)=0$.

<u>Aristotle</u>



Galileo



$(M, \mathbb{R}\oplus S, h \text{ in } S)$

$(M, V, \tau: V\to\mathbb{R}, h \text{ in } \ker\tau = S)$

Product $V=\mathbb{R}\oplus S$

$u=(u^0, 0)$ timelike
$u=(0, \vec{u})$ spacelike vectors

bundle $S \to V \xrightarrow{\tau} \mathbb{R}$

<u>Minkowski</u>

Aristotle $(M, V=\mathbb{R}\oplus S, h \text{ in } S)$

Galileo $(M, V, \tau: V\to\mathbb{R}, h \text{ in } S=\ker\tau)$

<u>Minkowski</u> ABANDON τ, absolute time.

<u>But</u>, have a structure that enables to distinguish what are timelike and what are spacelike vectors.

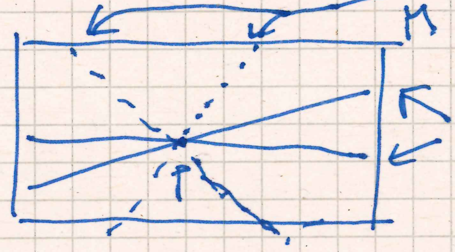
Minkowski spacetime: (M, V) affine 4-dim space with $g : V \times V \to \mathbb{R}$, symmetric, bilinear form of signature $(-,+,+,+)$.

Classification of vectors $v \in V$:

if $g(v,v) < 0$ — v is called timelike
if $g(v,v) > 0$ — v is called spacelike
if $g(v,v) = 0$ — v is called null (optical)

‖ no absolute time!
‖ no absolute rest!

null cone passing through p



two different spacelike hypersurfaces ~~[crossed out]~~ passing through $p \in M$

Inertial repère (o, e) s.t.

$$g(e_0, e_0) = -1, \quad g(e_0, e_i) = 0, \quad g(e_i, e_j) = \delta_{ij}.$$

e_0 – unital timelike
e_i – orthonormal spacelike

Poincaré transformation is an affine transformation (f, λ_f), $f : M \to M$, $\lambda_f : V \to V$ in Minkowski spacetime (M, V) s.t.

$$g(\lambda_f(v), \lambda_f(w)) = g(v, w).$$

λ_f is called **Lorentz transformation** for V

Example 2-dimensional Minkowski spacetime

(M, V, g) $g: V \times V \to \mathbb{R}$ signature $(-,+)$

$g(e_0,e_0) = -1, \quad g(e_1,e_1) = 1, \quad g(e_0,e_1) = 0.$

$p = 0 + x^0 e_0 + x^1 e_1$

Poincaré transformation: (f, λ_f) s.t.

$$f(p+u) = f(p) + \lambda_f(u)$$

$$g(\lambda_f u, \lambda_f v) = g(u,v)$$

$$\lambda_f(e_0) = A e_0 + B e_1$$

$$\lambda_f(e_1) = C e_0 + D e_1$$

$$-1 = g(e_0,e_0) = g(\lambda_f(e_0), \lambda_f(e_0)) = g(Ae_0 + Be_1, Ae_0 + Be_1) = -A^2 + B^2$$

$$\Rightarrow \boxed{A^2 = 1 + B^2} \qquad |A| \geq 1$$

$$0 = g(e_0,e_1) = g(\lambda_f e_0, \lambda_f e_1) = g(Ae_0 + Be_1, Ce_0 + De_1) = -AC + BD$$

$$C = \frac{B \cdot D}{A}$$

$$1 = g(e_1,e_1) = g(\lambda_f e_1, \lambda_f e_1) = g(Ce_0 + De_1, Ce_0 + De_1) = -C^2 + D^2$$

$$1 = -\frac{B^2 D^2}{1+B^2} + D^2 = D^2\left(\frac{1+B^2-B^2}{1+B^2}\right) = \frac{D^2}{1+B^2}$$

$$\boxed{D^2 = 1 + B^2 = A^2} \Rightarrow \boxed{D = \varepsilon A,\ C = B\varepsilon} \qquad \varepsilon = \pm 1 \qquad \boxed{\varepsilon = 1}$$

$$\lambda_f(e_0) = \sqrt{1+B^2}\, e_0 + B e_1$$

$$\lambda_f(e_1) = \varepsilon B e_0 + \varepsilon \sqrt{1+B^2}\, e_1$$

$$B = \frac{-v/c}{\sqrt{1 - v^2/c^2}} \qquad 1 + B^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$\lambda_f(e_0) = \frac{1}{\sqrt{1 - v^2/c^2}} \left(e_0 - \frac{v}{c} e_1\right)$$

$$\lambda_f(e_1) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(-\frac{v}{c} e_0 + e_1\right)$$

$$f(p) = \theta + x'^0 e_0 + x'^1 e_1$$

$$\parallel$$

$$f(0) + x^0 \lambda_f(e_0) + x^1 \lambda_f(e_1) =$$

$$= \theta + \underbrace{(f(0) - \theta)}_{x_0^0 e_0 + x_0^1 e_1} + x^0 \frac{1}{\sqrt{\ }} \left(e_0 - \frac{v}{c} e_1\right) + x^1 \frac{1}{\sqrt{\ }} \left(-\frac{v}{c} e_0 + e_1\right)$$

$$= \theta + \left[\frac{1}{\sqrt{\ }} \left(x^0 - \frac{v}{c} x^1\right) + x_0^0\right] e_0 +$$

$$+ \left[\frac{1}{\sqrt{\ }} \left(-\frac{v}{c} x^0 + x^1\right) + x_0^1\right] e_1$$

$x^0 = c \cdot t$

$$\begin{cases} t' = \dfrac{1}{\sqrt{1 - v^2/c^2}} \left(t - \dfrac{v}{c^2} x^1\right) + t_0 \\ x'^1 = \dfrac{1}{\sqrt{1 - v^2/c^2}} \left(x^1 - v t\right) + x_0^1 \end{cases}$$

Poincaré transformation.

$(t_0, x_0) = (0, 0) \Rightarrow$ Lorentz transformation

Note

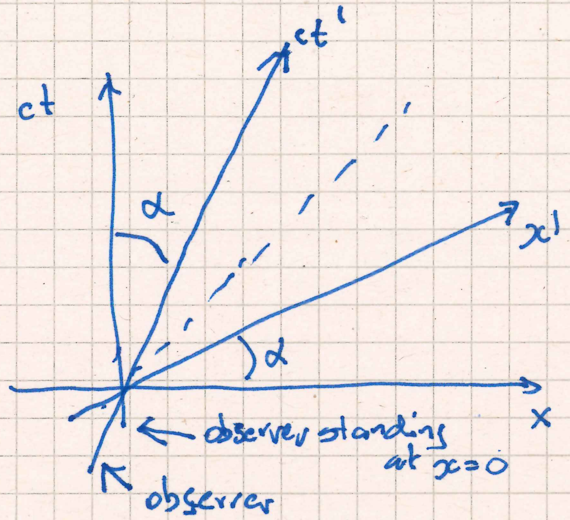
$c \to \infty$ then

$$\begin{cases} t' = t + t_0 \\ x' = x - vt + x_0 \end{cases}$$

Galileo transf.

c interpreted as speed of light.

In Galilean physics was assumed ∞.



moving with velocity v w.r.t. observer standing at $x=0$.

$$t_0 = 0$$
$$x_0 = 0$$

$$ct' = 0$$

$$ct = \frac{v}{c} x$$

$$\frac{v}{c} \leq 1$$

$$\mathrm{tg}\,\alpha = \frac{v}{c}$$

$$x' = 0 \qquad x = \frac{v}{c} ct$$