Lecture 3, 19.04.2018

Tensors

1) Tensors as multilinear maps.

V - vector space over $\mathbb{R}$, $\dim V = n < \infty$.

V^* - dual vector space to V

$$V^* := \{\lambda : V \xrightarrow[\text{linear}]{} \mathbb{R}\}$$

Evaluating $\lambda \in V^*$ on $v \in V$ defines a bilinear map

$$\langle \cdot , \cdot \rangle : V \times V^* \longrightarrow \mathbb{R}$$

$$\langle v, \lambda \rangle := \lambda(v)$$ (sometimes I will also denote $\lambda(v)$ as $v \lrcorner \lambda$).

obviously linear in the first argument, since λ is linear, but also linear in the second via:

$$(v, a\lambda + b\mu) \longmapsto \langle v, a\lambda + b\mu \rangle = a\,\lambda(v) + b\,\mu(v) = a\langle v, \lambda\rangle + b\langle v, \mu\rangle.$$

$$\forall a, b \in \mathbb{R} \quad \forall v \in V \quad \forall \lambda, \mu \in V^*.$$

Generalizing this we can consider ~~multilinear~~ maps

(1) $$T : \underbrace{V \times V \times \ldots \times V}_{s\text{-times}} \times \underbrace{V^* \times V^* \times \ldots \times V^*}_{r\text{-times}} \longrightarrow \mathbb{R}$$

which are <u>linear</u> in <u>each</u> of the arguments.

Definition

The ~~vector~~ space V^r_s of s-times <u>covariant</u> and r-times <u>contravariant</u> <u>tensors</u> is the ~~vector~~ space of all multilinear maps T as in (1)

Remarks

① V^r_s is denoted by $V^r_s = V\otimes \dots \otimes V \otimes V^* \otimes \dots V^* = \overset{r}{\otimes} V \otimes \overset{s}{\otimes} V^*$

It gets a structure of vector space by

$$(aT_1 + bT_2)(v_1, v_2, \dots v_s, \lambda^1, \lambda^2, \dots \lambda^r) =$$
$$= aT_1(v_1, v_2, \dots v_s, \lambda^1, \lambda^2, \dots \lambda^r) + bT_2(v_1, v_2, \dots v_s, \lambda^1, \dots \lambda^r),$$

② $\boxed{V^1_0 = V}$, indeed $v \in V$ can be considered as linear map

~~xxxxx~~ $v: V^* \to \mathbb{R}$ by

$$v(\lambda) := \langle v, \lambda \rangle$$

of course $\boxed{V^0_1 = V^*} \ni \lambda : V \to \mathbb{R}$.

③ $\dim V^r_s = n(r+s)$.

Let $e = \{e_\mu\}_{\mu=1}^n$ basis in V

$\omega = \{\omega^\mu\}_{\mu=1}^n$ dual basis in V^*, i.e. ω^μ are linear maps $V \to \mathbb{R}$

s.t. $\omega^\mu(e_\nu) = \delta^\mu_\nu$.

Then consider $e_{\mu_1} \otimes \dots \otimes e_{\mu_r} \otimes \omega^{\nu_1} \otimes \dots \omega^{\nu_s} \in V^r_s$ defined by

$$(e_{\mu_1} \otimes \dots e_{\mu_r} \otimes \omega^{\nu_1} \otimes \dots \otimes \omega^{\nu_s})(e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_s}, \omega^{\beta_1}, \dots, \omega^{\beta_r}) :=$$

$$= \omega^{\beta_1}(\mu_1) \dots \omega^{\beta_r}(e_{\mu_r})\, \omega^{\nu_1}(e_{\alpha_1}) \dots \omega^{\nu_s}(e_{\alpha_s})$$

Fact. $e_{\mu_1} \otimes \dots e_{\mu_r} \otimes \omega^{\nu_1} \otimes \dots \omega^{\nu_s}$ is a basis in V^r_s

$\Rightarrow \dim V^r_s = n \cdot r + n \cdot s = n(r+s)$.

Of course

$$V^r_s \ni T = \underbrace{T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}}_{\text{components of } T \text{ in basis } \{e\}} e_{\mu_1} \otimes e_{\mu_2} \otimes \dots \otimes e_{\mu_r} \otimes \omega^{\nu_1} \otimes \dots \otimes \omega^{\nu_s}$$

In particular:

$\boxed{\langle \cdot, \cdot \rangle = e_\alpha \otimes \omega^\alpha}$ Check!

$\langle v, \lambda \rangle = \langle v^\mu e_\nu, \lambda_\nu \omega^\nu \rangle = v^\mu \lambda_\mu \;\|\; (e_\alpha \otimes \omega^\alpha)(v, \lambda) = \omega^\alpha(v)\lambda(e_\alpha) = v^\alpha \lambda_\alpha$ ◁.

④ Change of basis

$v = v^\mu e_\mu$

$\lambda = \lambda_\nu \omega^\nu$

Take $a \in GL(n,\mathbb{R})$; Then $\omega' = \{\omega'^\mu\}$ defined by

$\omega'^\mu = a^\mu{}_\nu \omega^\nu$ is another basis in V^*

How the corresponding dual basis $e' = \{e'_\mu\}$ looks like?

We have: $e'_\mu = e_\nu b^\nu{}_\mu$, but

$$\delta_\mu{}^\nu = \omega'^\nu(e'_\mu) = a^\nu{}_\rho \omega^\rho (e_\sigma b^\sigma{}_\mu) = a^\nu{}_\rho b^\sigma{}_\mu \delta^\rho{}_\sigma = $$
$$= a^\nu{}_\rho b^\rho{}_\mu$$

$\Rightarrow \quad a \cdot b = \mathbb{1} \Rightarrow b = a^{-1}$ $\boxed{\omega^\mu = a^{-1\mu}{}_\rho \omega'^\rho \qquad e_\nu = e'_\alpha a^\alpha{}_\nu}$

$\Updownarrow \qquad \Updownarrow$

$$\left.\begin{aligned} \omega &\to \omega' = a \cdot \omega \\ e &\to e' = e a^{-1} \end{aligned}\right\} \qquad \boxed{\omega'^\mu = a^\mu{}_\nu \omega^\nu \Leftrightarrow e'_\mu = e_\nu a^{-1\nu}{}_\mu}$$

How the components change:

$$\left.\begin{aligned} v = v'^\mu e'_\mu &= v'^\rho e'_\rho \\ \parallel \\ v^\nu e_\nu = v^\nu e'_\rho a^\rho{}_\nu \end{aligned}\right\} \Rightarrow \boxed{v'^\rho = a^\rho{}_\nu v^\nu}$$

$$\left.\begin{aligned} \lambda = \lambda'_\mu \omega'^\mu &= \lambda'_\rho \omega'^\rho \\ \parallel \\ \lambda_\mu \omega^\mu = \lambda_\mu a^{-1\mu}{}_\rho \omega'^\rho \end{aligned}\right\} \Rightarrow \boxed{\lambda'_\rho = \lambda_\mu a^{-1\mu}{}_\rho}$$

Generally:

$$T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} \longmapsto \boxed{T'^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} = a^{\mu_1}{}_{\alpha_1} \ldots a^{\mu_r}{}_{\alpha_r} T^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s} \cdot a^{-1\beta_1}{}_{\nu_1} \ldots a^{-1\beta_s}{}_{\nu_s}}$$

This is a physicist's traditional definition of a tensor.

More precisely, for physicists, an r-times contravariant and s-times covariant tensor T is an equivalence class of pairs (e, T) in which e is a basis in V, $e = \{e_\mu\}_{\mu=1}^{n}$, and T is a collection of $n^{(r+s)}$ numbers, $T = \{T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}\}$, called components of T, where the equivalence relation is given by

Physicists' definition of a tensor

$$\boxed{\begin{aligned}&(e, T) \sim (e', T') \text{ iff } \exists\, a \in GL(n, \mathbb{R}) \text{ s.t.}\\ &e'_\mu = e_\nu a^{-1\nu}{}_\mu \quad \text{and} \quad T'^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} =\\ &\qquad = a^{\mu_1}{}_{\alpha_1} \dots a^{\mu_r}{}_{\alpha_r} T^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s} a^{-1\beta_1}{}_{\nu_1} \dots a^{-1\beta_s}{}_{\nu_s}\end{aligned}}$$

— * —

We generalize this concept a bit, but for this we need some preparations.

2) Pseudotensors, tensor densities.

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① G – group i.e. set G with a ~~map~~ map $\cdot : G \times G \to G$ s.t.

$$* \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$** \quad \exists\, \mathbb{1} \in G \text{ s.t. } \forall a \in G \ \ \mathbb{1} \cdot a = a \cdot \mathbb{1} = a$$

$$*** \quad \forall a \in G\ \exists\, a^{-1} \in G \text{ s.t. } a \cdot a^{-1} = a^{-1} \cdot a = \mathbb{1}.$$

② homomorphism between groups

$$G_1 \xrightarrow{\rho} G_2$$

$$\rho(a \cdot b) = \rho(a) \cdot \rho(b). \qquad \forall a, b \in G_1$$

③ action of a group G on a set S

$$G \times S \xrightarrow{\varphi} S$$

$$(a, s) \longmapsto \varphi_a(s) \quad \text{s.t.}$$

- $\varphi_{\mathbb{1}}(s) = s \quad \forall s \quad \Leftrightarrow \quad \varphi_{\mathbb{1}} = id_S$
- $\varphi_{a \cdot b}(s) = \varphi_a(\varphi_b(s)) \quad \forall s \Leftrightarrow \quad \varphi_{ab} = \varphi_a \circ \varphi_b.$

④ Effectiveness, freedom, transitivity

- An action φ of G in S is *effective* iff
  $$\forall \underset{\substack{\neq \\ 1\!\!1}}{a} \in G \quad \exists s \in S \quad \text{s.t.} \quad \varphi_a(s) \neq s$$
- An action φ of G in S is *free* iff
  $$\forall \underset{\substack{\neq \\ 1}}{a} \in G \quad \forall s \in S \quad \varphi_a(s) \neq s.$$
- An action is *transitive* iff
  $$\forall s_1, s_2 \in S \quad \exists a \in G \quad \text{s.t.} \quad s_2 = \varphi_a(s_1).$$
  (*Simply* transitive if a is unique for each pair s_1, s_2).

⑤ W - vector space,

$$GL(W) := \{ c : W \xrightarrow[\text{bijections}]{\text{linear}} W \}$$

this is a group with multiplication

$$c_1 \cdot c_2 = \underbrace{c_1 \circ c_2}_{\text{composition of linear maps.}}$$

Example

$W = \mathbb{R}^n$, $\quad GL(W) := GL(n, \mathbb{R})$.

⑥ Representation of G in V is a map:

$$\rho : G \longrightarrow GL(V) \text{ s.t.}$$

ρ is a homomorphism; $\rho(a.b) = \rho(a)\rho(b)$.

This before point ④ on page 3

⑦ Ingredients

- V - vector space,
- P(V) - set of all bases in V
- $G = GL(n, \mathbb{R})$
- $\dim V = n$
- action $\varphi : G \times P(V) \longrightarrow P(V)$
  $$(a, e) \longmapsto \varphi_a(e) = e\, a^{-1}$$
  (Check that it is free and *simply* transitive).
- $\rho : GL(n, \mathbb{R}) \longrightarrow GL(W)$
  representation of $GL(n, \mathbb{R})$ in W.

We now define an action of $G = GL(n,\mathbb{R})$ in $S = P(V)\times W$.

$$G\times S \ni (a,(e,\omega)) \xmapsto{\psi_a} (\psi_a(e), \rho(a)w) \in S$$

Exc. Check that it is an action.

Equivalence relation in S

$$(e,\omega)\sim(e',\omega') \iff \exists a\in GL(n,\mathbb{R}) \text{ s.t. } (e',\omega') = \psi_a(e,\omega)$$

Define W_ρ as set of equivalence classes.

$$W_\rho = S/\sim$$. This can be equipped with a structure of a Vectorspace by:

$$S\ni(e,\omega) \longmapsto [(e,\omega)] \in W_\rho$$

$$S\ni(e',\omega') \longmapsto [(e',\omega')] \sim [(e,\tilde\omega)] \in W_\rho$$

$$[(e,\omega)] + [(e',\omega')] := [(e,\omega+\tilde\omega)]$$

$$\alpha[(e,\omega)] := [(e,\alpha\omega)].$$

Exc. check that this definition does not depend on choices of representatives.

$$\dim(S) = \dim GL(n,\mathbb{R}) + \dim W$$

$$\dim(W_\rho) = \dim W.$$

(8) Examples of this general construction

$\boxed{\text{Tensors}}$:

$V = \mathbb{R}^n$, $W = \mathbb{R}^{n(r+s)} \ni T = \{T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}\}$

$$(\alpha T_1 + \beta T_2)^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s} = \alpha T_1{}^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s} + \beta T_2{}^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}$$

$$\rho^r_s : GL(n,\mathbb{R}) \longmapsto GL(W)$$

$$(\rho^r_s(a)T)^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s} = a^{\mu_1}{}_{\alpha_1}\dots a^{\mu_r}{}_{\alpha_r} T^{\alpha_1\dots\alpha_r}{}_{\beta_1\dots\beta_s} a^{-1\beta_1}{}_{\nu_1}\dots a^{-1\beta_s}{}_{\nu_s}$$

$S = P(\mathbb{R}^n)\times\mathbb{R}^{n(r+s)}$ $\quad \psi_a(e,T) = (ea^{-1}, \rho^r_s(a)T)$ $\quad W_{\rho^r_s}$ – $\binom{r}{s}$ tensors.

Tensor densities of weight w

$W = \mathbb{R}^{n(r+s)}$ as before, but we take different representation:

$$ {}^{w}\rho^{r}_{s} : GL(n,\mathbb{R}) \longrightarrow GL(W) $$

$$ {}^{w}\rho^{r}_{s}(a) := (\det a)^{w}\, \rho^{r}_{s}(a) $$

Easy to check that ${}^{w}\rho^{r}_{s}$ is a representation of $GL(n,\mathbb{R})$ in W.
Numerous examples, e.g.

a) <u>Levi-Civita symbol</u> it is by definition collection of numbers $\epsilon_{\mu_1 \dots \mu_n}$, $\mu_i = 1, \dots, n$, such that

- $\epsilon_{\mu_1 \dots \mu_n} = 0$ if any two indices are repeated
- $\epsilon_{\mu_1 \dots \mu_n} = \operatorname{sgn}(\pi)$ where $\pi = \begin{pmatrix} 1 & \dots & n \\ \mu_1 & \dots & \mu_n \end{pmatrix}$ if all μ_i are different.
- $\epsilon'_{\mu_1 \dots \mu_n} = \epsilon_{\mu_1 \dots \mu_n}$ in two different bases e' and e.

Note that $\epsilon_{\mu_1 \dots \mu_n}$ are totally antisymmetric i.e. interchanging any pair of μ_i's changes sign.

Calculate:

$$ a^{-1\,\nu_1}{}_{\mu_1} \dots a^{-1\,\nu_n}{}_{\mu_n}\, \epsilon_{\nu_1 \dots \nu_n} = \det(a^{-1})\, \epsilon_{\mu_1 \dots \mu_n} $$

Check it! look next page for $n=2$.

$\Rightarrow$

$$ \underbrace{\det a \cdot a^{-1\,\nu_1}{}_{\mu_1} \dots a^{-1\,\nu_n}{}_{\mu_n}\, \epsilon_{\nu_1 \dots \nu_n}}_{= \left({}^{1}\rho^{0}_{n}\, \epsilon\right)_{\mu_1 \dots \mu_n}} = \epsilon_{\mu_1 \dots \mu_n} = \epsilon'_{\mu_1 \dots \mu_n} $$

<u>-times covariant tensor density of weight +1</u> !

e.g.

$n=2$

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$(a^{-1})^{\alpha}{}_{\beta} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

$$a^{-1\,\alpha}{}_{\beta}\, a^{-1\,\gamma}{}_{\delta}\, \epsilon_{\alpha\gamma} =$$

$$= a^{-1\,\alpha}{}_{\beta}\, \epsilon_{\alpha\gamma}\, a^{-1\,\gamma}{}_{\delta}$$

$$(a^{-1})^T \cdot \epsilon \cdot a^{-1} =$$

$$= \begin{pmatrix} p & q \\ r & s \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} =$$

$$= \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} r & s \\ -p & -q \end{pmatrix} = \begin{pmatrix} 0 & ps-qr \\ qr-ps & 0 \end{pmatrix} = \underbrace{(ps-qr)}_{\det a^{-1}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

b) $\det(g_{\mu\nu})$, $g_{\mu\nu} = g_{\nu\mu}$, $\det g_{\mu\nu} \neq 0$.

$$[\det(g_{\mu\nu})]' = \det(g'_{\mu\nu}) = \det\left(a^{-1\,\mu}{}_{\rho}\, a^{-1\,\nu}{}_{\sigma}\, g_{\mu\nu}\right) =$$

$$= \det(a^{-1}) \det(a^{-1}) \det(g_{\mu\nu}) =$$

$$= (\det a)^{-2} \det(g_{\mu\nu})$$

– scalar density of weight -2

Pseudotensors

$W = \mathbb{R}^{n(r+s)}$ $\qquad \rho(a) = \operatorname{sgn}(\det(a))\, \rho^{r}_{s}(a)$.

e.g.

$$\eta_{\mu_1\ldots\mu_n} = \sqrt{|\det g_{\mu\nu}|}\; \epsilon_{\mu_1\ldots\mu_n}$$

$$\downarrow \qquad\qquad \downarrow$$

$$|\det a|^{-1} \qquad (\det a)^{+1}$$