Lecture 4, 24. 04. 2018.

Tensor fields on manifolds

1) Vector fields

M – differentiable manifold of class C^∞

$\mathcal{F}(M)$ – set of smooth functions on M

it is an algebra:

$$(f_1+f_2)(p) = f_1(p)+f_2(p)$$
$$(c\cdot f_1)(p) = c\cdot f_1(p)$$
$$(f_1\cdot f_2)(p) = f_1(p)\cdot f_2(p)$$

• Definition

X is a vector field on M iff

$$X : \mathcal{F}(M) \longrightarrow \mathcal{F}(M) \qquad \text{s.that}$$

① X is $\mathbb{R}$-linear:

$$X(c_1f_1+c_2f_2) = c_1X(f_1)+c_2X(f_2) \qquad \forall c_1,c_2\in\mathbb{R}\ \forall f_1,f_2\in\mathcal{F}(M)$$

② $X(f\cdot g) = X(f)\cdot g + f\cdot X(g) \qquad \forall f,g\in\mathcal{F}(M)$

Leibniz rule.

• One can produce new vector fields by taking commutators:

X, Y – vector field $\Rightarrow$ define $[X,Y] : \mathcal{F}(M)\to\mathcal{F}(M)$ by

$$[X,Y](f) = X(Y(f)) - Y(X(f))$$

Ex check that $[X,Y]$ is a vector field.

Define $\mathfrak{X}(M)$ – set of all vector fields on M.

It is a) Lie algebra over $\mathbb{R}$

b) module over $\mathcal{F}(M)$.

• Given $X\in\mathfrak{X}(M)$ we have a map $X_p : \mathcal{F}(M)\longrightarrow\mathbb{R}$

defined by $X_p(f) := X(f)(p)$, called a value of X at p

Obviously X_p is a) linear

b) $X_p(f\cdot g) = X_p(f)\cdot g(p) + f(p)X_p(g)$

- <u>Definition</u> A map $X_p : \mathcal{F}(M) \to \mathbb{R}$ satisfying a) and b) is called <u>tangent vector</u> to M at p.

One can add ~~and~~ tangent vectors, and multiply them by real numbers. $\Rightarrow$ the set T_pM of all tangent vectors to M at p is a <u>vector space</u>. It is called <u>tangent space to M at p.</u>

- <u>Example</u> let $]-\varepsilon, \varepsilon[\ni t \longrightarrow \gamma(t) \in M$ be a smooth <u>curve</u>, such that $\gamma(0) = p$.

Take as X_p a map defined by

$$X_p(f) = \frac{d}{dt} f \circ \gamma(t) \Big|_{t=0}.$$

It is a tangent vector at p (Ex. check it!)

- <u>Fact</u> $\dim M = n \implies \dim T_pM = n$

$p \in \mathcal{U}, x$; Let x^μ be coordinates in $\mathcal{U}$. $x^\mu : p \to x^\mu(p)$ (μ-th component of a vector in $\mathbb{R}^n$.)

$$\mathcal{U} \ni p \xrightarrow{x} \{x^\mu(p)\} \in \mathbb{R}^n$$

$\Rightarrow \frac{\partial}{\partial x^\mu}\Big|_p$ is a tangent vector at p, where $\left(\frac{\partial}{\partial x^\mu}\Big|_p\right)(f) = \frac{\partial f}{\partial x^\mu}\Big|_p$,

and $\left(\frac{\partial}{\partial x^1}\Big|_p, \dots \frac{\partial}{\partial x^n}\Big|_p\right)$ form a basis in $T_p(\mathcal{U}) = T_p(M)$

(Recall: $\frac{\partial}{\partial x^\mu}(x^\nu) = \delta^\nu_\mu$)

<u>Locally</u> <u>in</u> $(\mathcal{U}, x)$ <u>every</u> <u>vector field</u> can be written as:

$$\boxed{X = X^\mu \frac{\partial}{\partial x^\mu}} \quad \text{where } X^\mu = X(x^\mu).$$

2) Covariant tensor fields of type $\binom{0}{k}$.

$$\lambda : \underbrace{\mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M)}_{k\text{-times}} \longrightarrow \mathcal{F}(M)$$

$\mathcal{F}(M)$ linear in each argument.

one can think about λ as a smooth map

$$M \ni p \xrightarrow{\text{smooth}} \lambda_p \in (T_pM)^* \otimes \ldots \otimes (T_pM)^*$$

$$\lambda_p(X_{1p}, \ldots, X_{kp}) := (\lambda(X_1, \ldots, X_k))(p)$$

This in particular means that $\lambda_p : T_pM \times \ldots \times T_pM \longrightarrow \mathbb{R}$

$\mathbb{R}$-linear in each argument.

Examples

a) $k=1$ fields of 1-forms (1-form fields)

space of 1-form fields denoted by $\Lambda^1 M$.

In particulal, given a function $f \in \mathcal{F}(M)$ define $df \in \Lambda^1 M$ by

$$\boxed{df(X) := X(f)}$$

df – exterior differential of f.

Local basis in $\Lambda^1 M$: if (U, x) a chart around p

$\Rightarrow$ we have functions $p \longmapsto x^\mu(p) \in \mathbb{R}$

μ-th component

Take : dx^μ

This gives n 1-forms; moreover

$$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu \quad \Rightarrow \{dx^\mu\}_{\mu=1}^{n} \text{ basis in } \Lambda^1 U$$

dual to basis $\frac{\partial}{\partial x_\mu}$.

Any 1-form field in U is of the form $\lambda = \lambda_\mu dx^\mu$; $\lambda_\mu = \lambda\left(\frac{\partial}{\partial x^\mu}\right)$.

b) $k=2$.

Here a nice example is a metric, i.e.

$$g: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{F}(M)$$

$\mathcal{F}$-linear in both arguments

such that

① $g(X,Y) = g(Y,X)$ symmetric

② $(\forall X \;\; g(X,Y)=0) \Rightarrow Y=0$ nondegenerate.

A metric g on M defines a map

$$\tilde{g}: \mathfrak{X}(M) \xrightarrow{\mathcal{F}\text{linear}} \Lambda^1 M \quad \text{by} \quad X \longmapsto \tilde{g}(X) \text{ s.t.}$$

$$\underbrace{\tilde{g}(X)}_{1\text{-form}}(Y) = g(X,Y).$$

Condition ② means that $\tilde{g}$ is invertible.
(raising and lowering of an index).

$$\boxed{\begin{aligned} g &= g_{\mu\nu}\, dx^\mu \otimes dx^\nu \\ \tilde{g}(X) &= g_{\mu\nu} X^\nu dx^\mu = \\ &= X_\nu dx^\nu \end{aligned}}$$

3) Antisymmetric covariant tensor fields of type $\binom{0}{k}$ – k-forms $\Lambda^k M$

λ – tensor field of type $\binom{0}{k}$ s.t.

$$\lambda(X_1, \ldots X_i, \ldots, X_j, \ldots, X_k) = -\lambda(X_1, \ldots X_j, \ldots X_i, \ldots X_k).$$

for all pairs X_i, X_j from the set $X_1, \ldots X_k$.

They form a set (vector space over $\mathbb{R}$, module over $\mathcal{F}(M)$) $\Lambda^k M$ called k-forms on M.

How to produce them?

Wedge product $\lambda^k \in \Lambda^k M$ and $\lambda^l \in \Lambda^l M$ define $\lambda^k \wedge \lambda^l \in \Lambda^{k+l} M$

by: $$(\lambda^k \wedge \lambda^l)(X_1, \ldots, X_{k+l}) = \frac{1}{k!}\frac{1}{l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\, \lambda^k(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\, \lambda^l(X_{\sigma(k+1)}, \ldots X_{\sigma(k+l)})$$

Example (check it!)

$\lambda, \mu \in \Lambda^1 M \Rightarrow \lambda \wedge \mu = \lambda \otimes \mu - \mu \otimes \lambda$

$\lambda, \mu, \nu \in \Lambda^1 M \Rightarrow \lambda \wedge \mu \wedge \nu = \lambda \otimes \mu \otimes \nu + \nu \otimes \lambda \otimes \mu + \mu \otimes \nu \otimes \lambda - \mu \otimes \lambda \otimes \nu - \nu \otimes \mu \otimes \lambda - \lambda \otimes \nu \otimes \mu$

- **Define** $\Lambda^0 M = \mathcal{F}(M)$ and if $f \in \Lambda^0 M$ and $\lambda \in \Lambda^k M$ define $f \wedge \lambda = f \cdot \lambda$

$$\Rightarrow \boxed{\left(\Lambda M = \bigoplus_{k=0}^{n} \Lambda^k M \,,\ \wedge\right)} \text{ — Cartan algebra over } M$$

$\wedge$ product (in particular associative)

of course $\Lambda^{n+k} M = 0 \quad \forall k > 0$

Fact (check it)

$$\boxed{\overset{k}{\omega} \wedge \overset{l}{\omega} = (-1)^{kl} \overset{l}{\omega} \wedge \overset{k}{\omega}.}$$

- **Definition** Exterior differential in ΛM

$$d : \Lambda M \longrightarrow \Lambda M \quad \text{s.t.}$$

1. $d \Lambda^k M \subset \Lambda^{k+1} M$
2. d is $\mathbb{R}$-linear
3. $f \in \Lambda^0 M \Rightarrow df(X) = X(f)$
4. $d(\overset{k}{\omega} \wedge \overset{l}{\omega}) = d\overset{k}{\omega} \wedge \overset{l}{\omega} + (-1)^k \overset{k}{\omega} \wedge d\overset{l}{\omega}$
5. $d^2 \equiv 0$.

Fact these properties define d uniquely!

- **Definition** Inner product (hook operator)

if $X \in \mathfrak{X}(M)$ is a vector field define

$$X \lrcorner : \Lambda M \longrightarrow \Lambda M \quad \text{s.t.}$$

0. $X \lrcorner f = 0 \quad \forall f \in \Lambda^0 M$
1. $X \lrcorner \Lambda^k M \subset \Lambda^{k-1} M \qquad \forall k \geq 1$
2. $X \lrcorner$ is $\mathcal{F}$-linear
3. $X \lrcorner df = df(X) = X(f)$
4. $X \lrcorner (\overset{k}{\omega} \wedge \overset{l}{\omega}) = (X \lrcorner \overset{k}{\omega}) \wedge \overset{l}{\omega} + (-1)^k \overset{k}{\omega} \wedge (X \lrcorner \overset{l}{\omega})$

Check that

$$(X \lrcorner \overset{k}{\omega})(X_2, \dots X_k) = \overset{k}{\omega}(X, X_2, \dots X_k).$$

- **Derivation of degree k of algebra ΛM** is a map

$$D : \Lambda M \xrightarrow[\mathbb{R}\text{-linear}]{} \Lambda M \quad \text{such that}$$

① $D(\Lambda^j M) \subset \Lambda^{j+k} M$

② $D(\lambda^j \wedge \lambda^m) = (D\lambda^j) \wedge \lambda^m + (-1)^{jk} \lambda^j \wedge D\lambda^m$

Example d is a derivation of degree $+1$

$X \lrcorner$ is a derivation of degree -1.

- **Definition** A Lie derivative of a k-form λ with respect to the vector field X is

$$\mathcal{L}_X \lambda = (X \lrcorner \circ d + d \circ X \lrcorner)\lambda.$$

Exercise Check that $\mathcal{L}_X$ is a derivation of degree 0 of ΛM.

Exercise Check that on ΛM we have

a) $d \circ \mathcal{L}_X = \mathcal{L}_X \circ d.$

b) $\mathcal{L}_X \circ Y \lrcorner - Y \lrcorner \circ \mathcal{L}_X = [X, Y] \lrcorner$

c) $\mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X = \mathcal{L}_{[X,Y]}.$

- ~~This derivative can be uniquely extended to a map $\mathcal{L}_X$ from tensor fields of type $\binom{r}{s}$ to tensor fields of type $\binom{r}{s}$ such that:~~
  - ~~$\mathcal{L}_X$ is $\mathbb{R}$-linear in $\mathfrak{X}(M)^r_s$~~
  - ~~$\mathcal{L}_X(K \otimes L) = \mathcal{L}_X K \otimes L + K \otimes \mathcal{L}_X L$~~

- <u>Useful formula for the exterior derivative:</u>

$$\boxed{\begin{aligned} d\overset{k}{\omega}(X_0,\dots,X_k) &= \sum_{i=0}^{k} X_i\big(\omega(X_0,\dots\underbrace{|}_{\text{without } X_i}\dots,X_k)\big)(-1)^i + \\ &+ \sum_{0\le i<j\le k} (-1)^{i+j}\,\omega([X_i,X_j],X_0,\dots\underbrace{|\dots|}_{\text{without } X_i \text{ and without } X_j}\dots,X_k) \end{aligned}}$$

In particular:

- $k=0$

$$(df)(X_0) = X_0(f) \quad \checkmark$$

- $k=1 \qquad \lambda \in \Lambda^1 M$

$$(d\lambda)(X_0,X_1) = X_0(\lambda(X_1)) - X_1(\lambda(X_0)) - \lambda([X_0,X_1])$$

$$\boxed{(d\lambda)(X,Y) = X(\lambda(Y)) - Y(\lambda(X)) - \lambda([X,Y])} \qquad (*)$$

3) <u>Maurer-Cartan theorem</u>

<u>Local unholonomic frames</u> $\{X_\mu\}_{\mu=1}^{n}$ s.t. $X_\mu \in \mathfrak{X}(M)$ is a frame in $\mathcal{U}$ if $\{X_\mu|_p\}_{\mu=1}^{n}$ is a basis for $T_p\mathcal{U}$ $\forall p \in \mathcal{U}$.

<u>Example</u> $\left\{X_\mu = \frac{\partial}{\partial x^\mu}\right\}$ is a frame in (M,x).

↑ by definition it is HOLONOMIC FRAME.

<u>note</u>: $\left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right] = 0$.

<u>Unholonomic frames</u> such that $[X_\mu, X_\nu] \neq 0$. for some (all) μ,ν s.

e.g. $\quad X_\mu = A^\nu{}_\mu(x)\frac{\partial}{\partial x^\nu}$.

$A^\mu{}_\nu(x)$ is invertible at every point $p\in\mathcal{U}$.

$\{X_\mu\}_{\mu=1}^n$ local frame

$[X_\mu, X_\nu] = C^\rho{}_{\mu\nu} X_\rho \qquad C^\rho{}_{\mu\nu} = C^\rho{}_{\mu\nu}(x)$ s.t. $C^\rho{}_{\mu\nu} = -C^\rho{}_{\nu\mu}$,

functions. – <u>coefficients</u> of <u>anholonomy</u>.

<u>Fact</u>

- $\{X_\mu\}$ is a holonomic frame $\iff [X_\mu, X_\nu] = 0$
- $\{X_\mu\}$ – a frame, $\{\omega^\mu\}$ – dual frame (coframe), i.e.

$$\omega^\mu \in \Lambda^1 M \quad \forall \mu = 1, \dots n,$$

$$\omega^\mu(X_\nu) = \delta^\mu{}_\nu$$

$$X_\nu \lrcorner \omega^\mu$$

<u>Thm</u> <u>Maurer – Cartan</u>

$$\boxed{[X_\mu, X_\nu] = C^\rho{}_{\mu\nu} X_\rho \iff d\omega^\mu = -\tfrac{1}{2} C^\mu{}_{\rho\nu}\, \omega^\rho \wedge \omega^\nu}$$

<u>Exercise</u> Prove it, by applying ⊛ from page 7.