Lecture 5 27.04.2018

Connection and curvature

1) M - manifold of dimension n

$(\mathcal{U}, x)$ - chart $\Rightarrow$ coordinate frame $e_\mu = \frac{\partial}{\partial x^\mu}$

dual frame $e^\mu = dx^\mu$

Take $A: \mathcal{U} \to GL(n, \mathbb{R})$

$x \mapsto A^\mu{}_\nu(x)$

This enables to produce new frames:

$$e_\mu = A^\nu{}_\mu(x) \frac{\partial}{\partial x^\nu}$$

For these, in general, $[e_\mu, e_\nu] \neq 0$.

They are called <u>anholonomic frames</u>.

$\{e_\mu\}_{\mu=1}^n$ - frame

$\{e^\mu\}_{\mu=1}^n$ - dual frame i.e. each e^μ is a 1-form such that

$$e_\nu \lrcorner e^\mu = e^\mu(e_\nu) = \delta^\mu{}_\nu.$$

If one passes from one coframe e^μ to another e'^μ then there exists a function

$$a: \mathcal{U} \to GL(n, \mathbb{R}) \qquad \text{s.t.}$$

$$e'^\mu = a^\mu{}_\nu(x) e^\nu \qquad e \mapsto e' = ae$$

Dual frame to this coframe e_μ transforms

$$e'_\mu = a^{-1\,\nu}{}_\mu(x) e_\nu$$

2) Forms of type ρ

M - manifold of dimension n

$V = \mathbb{R}^N$

$$\rho : GL(n,\mathbb{R}) \to GL(N,\mathbb{R}) \quad \text{i.e.}$$

$$\cdot\ \rho(a) \in GL(N,\mathbb{R}) \subset M_{N\times N}(\mathbb{R})$$

$$\rho(a\cdot b) = \rho(a)\cdot\rho(b).$$

E.g. $V = \mathbb{R}^{n(r+s)} \ni T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}$, $\rho = \rho^r_s$

$\Rightarrow$ V - space of tensors of type $\binom{r}{s}$.

$e = \{e^\mu\}_{\mu=1}^n$ - coframe on M

$ae = \{a^\mu{}_\nu e^\nu\}_{\mu=1}^n$ - another coframe where $a : M \to GL(n,\mathbb{R})$.

Definition

A k-form of type ρ on M is a map α asigning a field of a k-form with values in V to any coframe e such that

$$\alpha(ae) = \rho(a)\,\alpha(e)$$

$\alpha(e)$ at every point of M is an element in $V = \mathbb{R}^N$

$$\alpha(e) = \{\alpha^A(e)\}_{A=1}^N$$

E.g. in case of $V = \mathbb{R}^{n(r+s)}$, $\alpha^A(e) = \alpha^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}(e)$

What the definition wants is

$$\alpha^A(ae) = \rho^A{}_B(a)\,\alpha^B(e) = (\rho(a)\alpha(e))^A$$

E.g. in case of $V = \mathbb{R}^{n(r+s)}$

$$\alpha^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}(ae) = a^{\mu_1}{}_{\alpha_1} \dots a^{\mu_r}{}_{\alpha_r}\, \underbrace{\alpha^{\alpha_1 \dots \alpha_r}{}_{\beta_1 \dots \beta_s}(e)}_{k\text{-form.}}\, a^{-1\beta_1}{}_{\nu_1} \dots a^{-1\beta_s}{}_{\nu_s}$$

If $k=0$ ~~then~~ we have tensors.

But now we can have $k > 0$, and get for example tensor-valued forms.

3) Examples

1° Canonical 1-form on M

$$V = \mathbb{R}^n, \quad \underset{\underset{\rho^1_0}{\parallel}}{\rho} = \mathrm{id.} \quad ; \quad \rho(a) = a, \quad k=1$$

We define 1-form of type id by:

$$\theta^\mu(e) := e^\mu$$

Definition is ok, because

$$\theta^\mu(ae) := a^\mu{}_\nu e^\nu = \rho(a)^\mu{}_\nu \theta^\nu(e) = (\rho(a)\theta(e))^\mu.$$

2° A vector field X on M defines a 0-form of type id $X^\mu(e)$

$$X^\mu(e) = X \lrcorner e^\mu = e^\mu(X)$$

$$X^\mu(ae) = X \lrcorner (a^\mu{}_\nu e^\nu) = a^\mu{}_\nu X \lrcorner e^\nu = a^\mu{}_\nu X^\nu(e)$$

3° Forms of type Ad

$$V = \mathrm{End}(\mathbb{R}^n) = \mathbb{R}^{n^2} \ni (T^\mu{}_\nu) = T \qquad N = (\alpha_\beta)$$

$$\rho = \mathrm{Ad}$$

$$\rho(a)T = aTa^{-1}$$

$$(\rho(a)T)^\mu{}_\nu = a^\mu{}_\alpha T^\alpha{}_\beta a^{-1\beta}{}_\nu$$

$T^\mu{}_\nu$ – can be any k-form.

4° Scalar forms $\quad V = \mathbb{R}, \ \rho(a) = 1.$

usual forms.

5° <u>Tensor product</u>

α_1 - k_1-form of type ρ_1

α_2 - k_2-form of type ρ_2

$$(\alpha_1 \wedge \alpha_2)^{A_1 A_2} := \alpha_1^{A_1} \wedge \alpha_2^{A_2}$$

(k_1+k_2) form of type $\rho_1 \otimes \rho_2$.

4) COVARIANT DIFFERENTIATION

X - vector field $\Rightarrow$ 0-form of type id $\quad X^\mu(e) = X \lrcorner e^\mu$.

Let us differentiate:

$$d\left[X^\mu(ae)\right] = d\,(a^\mu{}_\nu X^\nu(e)) = da^\mu{}_\nu X^\nu(e) + a^\mu{}_\nu\, dX^\nu(e)$$

$$= \underbrace{a^\mu{}_\nu\, dX^\nu(e)}_{\text{OK}} + \underbrace{da^\mu{}_\nu X^\nu(e)}_{\text{bad!}}$$

$d(X^\mu(ae))$ is NOT a 1-form of type id!

WE WANT TO DEFINE SUCH DIFFERENTIAL THAT TRANSFORMS a k-form of type ρ to a $(k+1)$-form of <u>the same type</u> ρ.

How to do this? ... Add correction terms!

Define $\boxed{DX^\mu(e) = dX^\mu(e) + \omega^\mu{}_\nu(e) \wedge X^\nu(e)}$ with some $\omega^\mu{}_\nu(e)$. How $\omega^\mu{}_\nu$ should transform,

$$\omega^\mu{}_\nu(ae) = \; ? \; \omega^\mu{}_\nu(e), \text{ for}$$

$$DX^\mu(ae) = a^\mu{}_\nu DX^\nu(e) \; ?$$

Calculate

$$DX^\mu(ae) = dX^\mu(ae) + \omega^\mu{}_\nu(ae)\wedge X^\nu(ae) =$$

$$= d\left(a^\mu{}_\nu X^\nu(e)\right) + \omega^\mu{}_\nu(ae)\wedge a^\nu{}_\rho X^\rho(e) =$$

$$= da^\mu{}_\rho \wedge X^\rho(e) + \underline{a^\mu{}_\nu\, dX^\nu(e)} + \omega^\mu{}_\nu(ae)\wedge a^\nu{}_\rho X^\rho(e)$$

$$\|$$

$$a^\mu{}_\nu DX^\nu(e) = \underline{a^\mu{}_\nu\big(dX^\nu(e)} + \omega^\nu{}_\rho(e)\wedge X^\rho(e)\big)$$

$$\left(\omega^\mu{}_\nu(ae)a^\nu{}_\rho - a^\mu{}_\nu\omega^\nu{}_\rho(e) + da^\mu{}_\rho\right)\wedge X^\rho(e) = 0.$$

$\forall X^\rho(e)$

$\Longrightarrow$

$$\omega^\mu{}_\nu(ae)a^\nu{}_\rho = a^\mu{}_\nu\omega^\nu{}_\rho(e) - da^\mu{}_\rho$$

$$\boxed{\omega^\mu{}_\nu(ae) = a^\mu{}_\alpha\,\omega^\alpha{}_\beta(e)\,a^{-1\,\beta}{}_\nu - da^\mu{}_\alpha\, a^{-1\,\alpha}{}_\nu}$$

$$0 = d(a\cdot a^{-1}) = da\cdot a^{-1} + a\,da^{-1}$$

$\boxed{\omega(e)}$ is a 1-form field with the following transformation properties:

$$\boxed{\omega(ae) = a\cdot\omega(e)a^{-1} - da\cdot a^{-1}}.$$

$$\boxed{\omega(ae) = a\cdot\omega(e)a^{-1} + a\,da^{-1}}$$

$\omega^\mu{}_\nu(e)$ – strange object. Do not transform linearly!

have this $-da\,a^{-1}$ shift.

Affine shift! $\boxed{\text{Affine connection form}}$

Definition

An affine connection is a map assigning a field of an 1-form $\omega^\mu{}_\nu(e)$ to each coframe e such that

$$(*)\quad \boxed{\omega^\mu{}_\nu(ae) = a^\mu{}_\alpha\, \omega^\alpha{}_\beta(e)\, a^{-1\beta}{}_\nu - da^\mu{}_\alpha\, a^{-1\alpha}{}_\nu .}$$

This enables to have a differentiation that transforms objects of type ρ to objects of type ρ.

E.g.

$$DX^\mu(e) := dX^\mu(e) + \omega^\mu{}_\nu(e) \wedge X^\nu(e)$$

is a 1-form of type id, as $X^\mu(e)$ was a 0-form of type id.

5) Origins of curvature

Because of INHOMOGENEOUS transformation (*),

$$\omega(e) \mapsto a\,\omega(e)\,a^{-1} - da\, a^{-1},$$

at every point $p \in \mathcal{U}$ we can find e s.t. $\omega(e)_p = 0$.

Indeed if $\omega(e)_p \neq 0$ look at a and calculate

$$\omega(ae) = a\,\omega(e)\,a^{-1} - da \cdot a^{-1}$$

evaluate at p:

$$\omega_p(ae) = a_p\, \omega_p(e)\, a_p^{-1} - da_p\, a_p^{-1} = 0$$

$$\Rightarrow \quad da|_p = a(p)\,\omega(e)_p .$$

so taking $a(p) = \mathbb{1}$, $da|_p = \omega(e)_p$ will make $\omega_p(ae) = 0$.

Question: Well, I can always make $\omega^\mu{}_\nu$ zero at a point. When can I make it zero in the entire neighbourhood?

$$0 = \omega(ae) = a\,\omega(e)\,a^{-1} - da\cdot a^{-1}$$

$$\Rightarrow \quad a\,\omega(e) - da = 0$$

$$\Rightarrow \quad da = a\,\omega(e) \qquad \Big|\, d$$

$$\Rightarrow \quad 0 = da\wedge\omega(e) + a\,d\omega(e) =$$

$$0 = a\,\omega(e)\wedge\omega(e) + a\,d\omega(e)$$

$$\Rightarrow \quad a\big(d\omega(e) + \omega(e)\wedge\omega(e)\big) = 0 \qquad \Big|\, a^{-1}\cdot$$

$$\boxed{d\omega(e) + \omega(e)\wedge\omega(e) = 0}$$

or, with indices

$$\boxed{d\omega^\mu{}_\nu(e) + \omega^\mu{}_\alpha(e)\wedge\omega^\alpha{}_\nu(e) = 0}$$

Define

$$\boxed{\Omega(e) = d\omega(e) + \omega(e)\wedge\omega(e)}$$

or $$\boxed{\Omega^\mu{}_\nu(e) = d\omega^\mu{}_\nu(e) + \omega^\mu{}_\alpha(e)\wedge\omega^\alpha{}_\nu(e)}$$

Exercise: Check that although ω is a connection $\Omega(e)$ is a 2-form of TYPE Ad.

We created a tensorial object out of connection.

Its meaning is as follows:

$$\begin{pmatrix} \Omega(e) \equiv 0 \\ \text{vanishes everywhere} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{connection } \omega(e) \text{ can be} \\ \text{gauged to 0 everywhere} \\ \text{by choosing suitable } a, \ \omega(ae) \equiv 0 \end{pmatrix}$$

5) Covariant exterior differentiation

For tensor-valued forms, i.e. k-forms of type ρ^r_s, define

$$\boxed{\begin{aligned} DT^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}(e) &= d\left(T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}(e)\right) + \\ &+ \omega^{\mu_1}{}_{\alpha}(e) \wedge T^{\alpha \dots \mu_r}{}_{\nu_1 \dots \nu_s}(e) + \dots + \omega^{\mu_r}{}_{\alpha}(e) T^{\mu_1 \dots \alpha}{}_{\nu_1 \dots \nu_s}(e) + \\ &- \omega^{\alpha}{}_{\nu_1}(e) \wedge T^{\mu_1 \dots \mu_r}{}_{\alpha \dots \nu_s}(e) + \dots - \omega^{\alpha}{}_{\nu_s}(e) \wedge T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \alpha}(e). \end{aligned}}$$

Exercise

Check that it is a $(k+1)$-form of type ρ^r_s.

If not in general, check it on a form of type ρ^0_1 and form of type Ad.

6) If $T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}(e)$ 0-form of type ρ^r_s,
i.e. tensor valued form, then

$$X \lrcorner DT^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}(e) \text{ -- zero-form of type } \rho^r_s.$$

vector field.

Notation:

$$X \lrcorner DT^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} =: \nabla_X T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}$$

Covariant derivative of T
with respect to X.

of course it is $\mathcal{F}$-linear w.r.t. X.

$$\nabla_X T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s} = X^\alpha \boxed{\nabla_{e_\alpha} T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}}$$

$$\nabla_\alpha T^{\mu_1 \dots \mu_r}{}_{\nu_1 \dots \nu_s}$$

notation.

covariant derivative of T
in the direction of e_α.

7) Autoparallels

$]-\varepsilon, \varepsilon[\ni t \longrightarrow \gamma(t) \in M$ curve

Tangent vector $X(f) = \frac{d}{dt} f \circ \gamma(t)$

$$X^\mu = X(x^\mu) = \frac{d}{dt} x^\mu \circ \gamma(t) = \frac{dx^\mu}{dt} \rightsquigarrow X = X^\mu \frac{\partial}{\partial x^\mu}$$

A curve $\gamma(t)$ is autoparallel iff $\nabla_X X^\mu = 0$

$$= \frac{dx^\mu}{dt} \frac{\partial}{\partial x^\mu} = \frac{d}{dt}$$

Calculate $\qquad \omega^\mu{}_\nu = \Gamma^\mu{}_{\nu\rho}\, dx^\rho$

$$DX^\mu = dX^\mu + \omega^\mu{}_\nu X^\nu = X \lrcorner dX^\mu + X \lrcorner \omega^\mu{}_\nu X^\nu$$

$$X \lrcorner DX^\mu = X(X^\mu) + X^\rho \Gamma^\mu{}_{\nu\rho} X^\nu =$$

$$\parallel$$

$$\nabla_X X^\mu = \frac{d}{dt}\left(\frac{dx^\mu}{dt}\right) + \Gamma^\mu{}_{\nu\rho} \frac{dx^\rho}{dt}\frac{dx^\nu}{dt} = 0$$

$$\boxed{\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\rho}{dt}\frac{dx^\nu}{dt} = 0}$$

$$\boxed{\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0}$$

$$\frac{d^2x^i}{dt^2} = -\frac{\partial\varphi}{\partial x^i}$$

$$\frac{d^2x^0}{dt^2} = 0 \qquad \frac{dx^0}{dt} = 1 \qquad \Gamma$$

$$\underbrace{\frac{d^2x^0}{dt^2}}_{=0} + \Gamma^0{}_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} + \Gamma^0{}_{0j}\frac{dx^0}{dt}\frac{dx^j}{dt} + \Gamma^0{}_{j0}\frac{dx^j}{dt}\frac{dx^0}{dt} = 0$$

$$\Gamma^0{}_{ij}v^iv^j + \Gamma^0{}_{0j}v^j + \Gamma^0{}_{j0}v^j = 0$$

$$\Gamma^0{}_{ij} = 0$$
$$\Gamma^0{}_{0j} = 0$$
$$\Gamma^0{}_{j0} = 0$$

$$\frac{d^2x^i}{dt^2} + \Gamma^i{}_{00} + \cancel{\Gamma^i{}_{0j}\frac{dx^j}{dt}} + \cancel{\Gamma^i{}_{j0}\frac{dx^j}{dt}} + \Gamma^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0$$

$$\boxed{\Gamma^i{}_{00} = \frac{\partial\varphi}{\partial x^i}} \qquad \boxed{\Gamma^i{}_{jk} = 0}$$

$$\boxed{\begin{aligned} \omega^i{}_0 &= \frac{\partial\varphi}{\partial x^i}dt \\ \omega^i{}_j &= 0 \\ \omega^0{}_\mu &= 0 \end{aligned}}$$

$$\Omega = \begin{pmatrix} 0 & \frac{\partial^2}{\partial x^j\partial x^i}dx^j\wedge dt \\ 0 & 0 \end{pmatrix}$$

$$\Omega = d\omega^i{}_0 = \frac{\partial^2\varphi}{\partial x^j x^i}dx^j\wedge dt$$

$R^i{}_{jkl}$